(1a)

$$y' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} y + \begin{pmatrix} -x^2 \\ 2x \end{pmatrix}$$

$$\begin{array}{c} (+ \\ (\lambda-1) \end{array} \left(\begin{array}{cc|c} 1-\lambda & -1 & x^2 \\ 1 & 3-\lambda & -2x \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 3-\lambda & -2x \\ 0 & \underbrace{-1+(3-\lambda)(\lambda-1)}_{-\lambda^2+4\lambda-4} & \underbrace{x^2-2x(\lambda-1)}_{\underbrace{x^2+2x-\lambda\cdot(2x)}_{x^2+2x-2}} \end{array}\right)$$

y_2: $\lambda^2-4\lambda+4=0$

$(\lambda-2)^2=0$

$\lambda=2$

$y_{2,H} = c_1 e^{2x} + c_2 x e^{2x}$

$0+0i$ není kořen

$\to y_{2p} = Ax^2+Bx+C$

$-y_{2p}'' + 4y_{2p}' - 4y_{2p} = x^2+2x-2$

$-4Ax^2 + x(8A-4B) + 4B-4C-2A = 1x^2+2x-2$

$\underline{A=-1/4}$ $\quad -2-4B=2 \quad \underline{B=-1}$

$-4-4C+\frac{1}{2} = -2 \quad -4C=\frac{3}{2}$

$\underline{C=-\frac{3}{8}}$

$$y_2 = \underline{\underline{c_1e^{2x}+c_2xe^{2x}-\frac{x^2}{4}-x-\frac{3}{8}}}$$

$y_1 = -2x - (3-\lambda)y_2$

$y_1 = -2x - 3y_2 + y_2'$

$$y_1 = \underline{\underline{-c_1e^{2x} - c_2e^{2x}\cdot x + c_2e^{2x} + \frac{3}{4}x^2 + \frac{x}{2} + \frac{1}{8}}}$$

$x, c_{1,2} \in \mathbb{R}$

Nez.

Od druhého řádku odečteme třetí řádek a poté od třetího řádku odečteme $(\lambda - 4)$-násobek druhého řádku

$$\sim \begin{pmatrix} \lambda - 2 & -1 & 1 \\ \lambda^2 + 2 & 1 & 0 \\ -\lambda^2 - \lambda - 7 & \lambda - 4 & 0 \end{pmatrix} \sim \begin{pmatrix} \lambda - 2 & -1 & 1 \\ \lambda^2 + 2 & 1 & 0 \\ -\lambda^3 + 3\lambda^2 - 3\lambda + 1 & 0 & 0 \end{pmatrix}$$

V posledním řádku máme nyní rovnici $-x''' + 3x'' - 3x' + x = 0$, v matici je tedy rovnou charakteristický polynom této rovnice. Dostáváme tedy řešení

$$x(t) = c_1 e^t + c_2 t e^t + c_3 t^2 e^t \,, \quad t \in \mathbb{R}\,, c_1, c_2, c_3 \in \mathbb{R}\,.$$

Z druhého řádku matice dopočítáme y:

$$y(t) = -x'' - 2x = -e^t(3c_1 + 2c_2 + 2c_3 + t(3c_2 + 4c_3) + t^2 \cdot 3c_3)\,, \quad t \in \mathbb{R}\,.$$

Z první rovnice dopočítáme z:

$$z(t) = e^t(-2c_1 - 3c_2 - 2c_3 + t(-2c_2 - 6c_3) + t^2 \cdot (-2c_3))\,, \quad t \in \mathbb{R}\,.$$

Příklad 3. Řešte soustavu

(1b)

$$\begin{aligned} 2y'' + 3z'' - 7y - 6z &= t + 1 \\ 4y'' + 3z'' - 4y - 3z &= 2t. \end{aligned}$$

Řešení. Napíšeme si λ-matici s pravou stranou:

$$\left(\begin{array}{cc|c} 2\lambda^2 - 7 & 3\lambda^2 - 6 & t+1 \\ 4\lambda^2 - 4 & 3\lambda^2 - 3 & 2t \end{array}\right) \sim \left(\begin{array}{cc|c} 2\lambda^2 - 7 & 3\lambda^2 - 6 & t+1 \\ 2\lambda^2 + 3 & 3 & t-1 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 2\lambda^2 - 7 - (\lambda^2 - 2)(2\lambda^2 + 3) & 0 & t + 1 - (t-1)'' + 2(t-1) \\ 2\lambda^2 + 3 & 3 & t - 1 \end{array}\right)$$

$$= \left(\begin{array}{cc|c} -2\lambda^4 + 3\lambda^2 - 1 & 0 & 3t - 1 \\ 2\lambda^2 + 3 & 3 & t - 1 \end{array}\right).$$

Řešení charakteristického polynomu v prvním řádku jsou $\lambda_{1,2} = \pm 1$, $\lambda_{3,4} = \pm\sqrt{2}/2$. Tedy řešení homogenní rovnice

$$-2y^{(4)} + 3y'' - y = 0$$

jsou

$$ae^t + be^{-t} + ce^{\sqrt{2}/2t} + de^{-\sqrt{2}/2t}.$$

Partikulární řešení nehomogenní rovnice bude ve tvaru $y_p(t) = rt + s$, což po dosazení do rovnice dává $y_p(t) = -3t + 1$. Tedy

$$y(t) = ae^t + be^{-t} + ce^{\sqrt{2}/2t} + de^{-\sqrt{2}/2t} - 3t + 1\,.$$

Z druhé rovnice pak máme

$$\begin{aligned} 3z(t) &= t - 1 - 2y'' - 3y = t - 1 - 2(ae^t + be^{-t} + \frac{1}{2}ce^{\sqrt{2}/2t} + \frac{1}{2}de^{-\sqrt{2}/2t}) \\ &\quad -3(ae^t + be^{-t} + ce^{\sqrt{2}/2t} + de^{-\sqrt{2}/2t} - 3t + 1) \\ &= 10t - 4 - 5ae^t - 5be^{-t} - 4ce^{\sqrt{2}/2t} - 4de^{-\sqrt{2}/2t}\,. \end{aligned}$$

(2a)

$y_1' = y_1 - y_2 - x$ $\qquad$ $y(0) = (0,0)^T$

$y_2' = y_1 + y_2$

$$\begin{array}{c} \\ \cdot(\lambda-1) \end{array}\left(\begin{array}{cc|c} 1-\lambda & -1 & +x \\ 1 & 1-\lambda & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 0 & -1+(1-\lambda)(\lambda-1) & x \\ 1 & 1-\lambda & 0 \end{array}\right)$$

$$\begin{array}{cc} y_1 & y_2 \\ \end{array}$$
$$\left(\begin{array}{cc|c} 0 & -\lambda^2+2\lambda-1-1 & x \\ 1 & 1-\lambda & 0 \end{array}\right)$$

$\rightarrow$ $-y_2'' + 2y_2' - 2y_2 = x$ $\qquad$ $y_{2h} = c_1 e^x \sin x + c_2 e^x \cos x$

$-\lambda^2 + 2\lambda - 2 = 0$ $\qquad$ spec. PS.

$\lambda^2 - 2\lambda + 2 = 0$ $\qquad$ $x = e^{0x}(x\cos 0x + 0 \sin 0x)$

$\lambda = 1 \pm i$ $\qquad$ $0+0i$ není kořen

hledáme řešení tvaru $\qquad$ $y_{2p} = Ax + B$

$y_{2p}' = A$

$2A - 2Ax - 2B = 1\cdot x$ $\qquad$ $y_{2p}'' = 0$

$A = -1/2$ $\qquad$ $-1-2B=0$

$B = -1/2$ $\qquad$ $y_2 = \underline{c_1 e^x \sin x + c_2 e^x \cos x} - \frac{x}{2} - \frac{1}{2}$

$y_1 + (1-\lambda)y_2 = 0$ $\qquad$ Podm.: $0 = c_2 - 1/2 \rightarrow c_2 = 1/2$

$0 = c_1$

$y_1 = y_2' - y_2$

$\underline{y_1 = c_1 e^x \cos x - c_2 e^x \sin x + \frac{x}{2}}$

celkem $\quad \underline{y_1 = -\frac{1}{2} e^x \sin x + \frac{x}{2}}$ $\qquad \underline{y_2 = \frac{1}{2} e^x \cos x - \frac{x}{2} - \frac{1}{2}}$ $\qquad x \in \mathbb{R}$

(2b) $y_1' = y_2 + \sin x$

$y_2' = -y_1 + \cos x$

$$\begin{array}{c} \\ {}^{+} \\ \cdot(-\lambda) \end{array}\left(\begin{array}{cc|c} \lambda & -1 & \sin x \\ 1 & \lambda & \cos x \end{array}\right) \sim \left(\begin{array}{cc|c} 0 & -1-\lambda^2 & \sin x - \lambda \cos x \\ 1 & \lambda & \cos x \end{array}\right)$$

$-y_2 - y_2'' = \underbrace{\sin x - (\cos x)'}_{2 \sin x}$ $\quad y_{2H} = c_1 \cos x + c_2 \sin x$

$-1 - \lambda^2 = 0 \qquad \lambda = \pm i$

$-1 = \lambda^2$

PS: $\quad 2 \sin x = e^{0x} (0 \cos 1x + 2 \sin 1x)$

i je kořen

$y_{2p} = x (A \cos x + B \sin x)$

dosazení

$2A \sin x - 2B \cos x = 2 \sin x$ $\qquad y_2 = c_1 \cos x + c_2 \sin x + x \cos x$

$B = 0 \qquad A = 1$

$y_1 = \cos x - y_2'$

$y_1 = x \sin x + c_1 \sin x - c_2 \cos x$

$c_1, c_2, x \in \mathbb{R}$

(2c)

$$y_1' = y_1 + 2y_2 + \sin \cancel{x} \qquad y(0) = (0,0)$$
$$y_2' = -y_1 - y_2$$

$$\begin{pmatrix} 1-\lambda & 2 & -\sin x \\ -1 & \underbrace{-1-\lambda}_{-(1+\lambda)} & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 2+(-1)(1+\lambda)(1-\lambda) & -\sin x \\ 1 & 1+\lambda & 0 \end{pmatrix}$$

$1-\lambda$

$$\begin{pmatrix} 0 & \lambda^2+1 & -\sin x \\ 1 & 1+\lambda & 0 \end{pmatrix}$$

y_2: $\quad \lambda^2 + 1 = 0 \qquad y_{2h} = c_1 \cos x + c_2 \sin x$

$\lambda = \pm i$

Spec. PS: $\quad -\sin x = e^{0x}(0 \cos 1x - 1 \sin 1x)$

$0 + 1i$ je 1-nas. koren

$$y_p = x^1 (A \cos x + B \sin x)$$

$$y_p'' + y_p = -\sin x \qquad y_2 = c_1 \cos x + c_2 \sin x + \tfrac{1}{2} x \cos x$$

$$-2A \sin x + 2B \cos x = -\sin x$$

$$\to B = 0 \quad A = \tfrac{1}{2}$$

paž $\quad y_1 = -y_2 - y_2'$

$$y_1 = (-c_2 + c_1)\sin x + (-c_1 - c_2)\cos x + \tfrac{-1}{2}(x\cos x + \cos x - x \sin x)$$

Podm. $\quad 0 = c_1 \qquad c_2 = -\tfrac{1}{2}$

$$0 = -c_2 - \tfrac{1}{2}$$

Celkem $\quad y_1 = \tfrac{1}{2}\sin x + \tfrac{1}{2}\cos x - \tfrac{1}{2}(x\cos x + \cos x - x\sin x)$

$$y_2 = -\tfrac{1}{2}\sin x + \tfrac{1}{2} x \cos x$$

$x \in \mathbb{R}$

(2d)

$$u' = 4u + 3v - 3w$$

$$v' = -3u - 2v + 3w$$

$$w' = 3u + 3v - 2w + 2e^{-x}$$

$$\begin{array}{c c} & \begin{array}{ccc} u & v & w \end{array} \\ \begin{array}{c} 3\cdot \\ +(\\ (4-\lambda)\cdot \end{array} & \left(\begin{array}{ccc|c} 4-\lambda & 3 & -3 & 0 \\ -3 & -2-\lambda & 3 & 0 \\ 3 & 3 & -2-\lambda & -2e^{-x} \end{array}\right) \sim \end{array}$$

$$\sim \left(\begin{array}{ccc|c} -3 & -2-\lambda & 3 & 0 \\ 0 & 9-(2+\lambda)(4-\lambda) & -9+12-3\lambda & 0 \\ 0 & 1-\lambda & 1-\lambda & -2e^{-x} \end{array}\right) \begin{array}{c} \sim \\ + \\ \cdot(-3) \end{array}$$

$$\sim \left(\begin{array}{ccc|c} -3 & -2-\lambda & 3 & 0 \\ 0 & (\lambda-1)^2 & 3(1-\lambda) & 0 \\ 0 & (\lambda-1)^2-3(1-\lambda) & 0 & 6e^{-x} \end{array}\right) \sim \left(\begin{array}{ccc|c} -3 & -2-\lambda & 3 & 0 \\ 0 & (\lambda-1)^2 & 3(1-\lambda) & 0 \\ 0 & \underbrace{(\lambda-1)(\lambda+2)}_{\lambda^2+\lambda-2} & 0 & 6e^{-x} \end{array}\right)$$

v: $(\lambda-1)(\lambda+2)=0$

$\lambda_1 = 1 \quad \lambda_2 = -2$

$v_H = c_1 e^{x} + c_2 e^{-2x}$

PS: $6e^{-x} = e^{-x}(6\cos 0x + 0 \sin 0x)$

$-1+0i$ není kořen

$v_P = A e^{-x}$

$v'' + v' - 2v = 6e^{-x}$

$-2Ae^{-x} = 6e^{-x} \qquad A = -3$

$\underline{v = c_1e^{x} + c_2e^{-2x}} - 3e^{-x}$

w: $3w - 3w' = -v'' + 2v' - v$

$-w' + w = \frac{1}{3}(12e^{-x} - 9c_2e^{-2x})$

variace konstant $\quad w_H = k e^{x}$

$-k'e^{x} - ke^{x} + ke^{x} = 4e^{-x} - 3c_2e^{-2x}$

$k' = -e^{-x}(4e^{-x} - 3c_2e^{-2x})$

$k = e^{-3x}(2e^{x} - c_2) + c_3$

$w = \underline{2e^{-x} - c_2e^{-2x} + c_3e^{x}}$

u:
$$-3u = (2+\lambda)v - 3w$$
$$u = -\frac{1}{3}(2v + v' - 3w)$$
$$\underline{\underline{u = -e^{x}(c_1 - c_3) + e^{2x}(-c_2) + 3e^{-x}}}$$

$x, c_{1-3} \in \mathbb{R}$

(2e)

$$u' = u + v + z + w$$
$$v' = u + v + z + w + 1$$
$$z' = u + v + z + w + 2$$
$$w' = u + v + z + w + 3$$

$$\left(\begin{array}{cccc|c} 1-\lambda & 1 & 1 & 1 & 0 \\ 1 & 1-\lambda & 1 & 1 & -1 \\ 1 & 1 & 1-\lambda & 1 & -2 \\ 1 & 1 & 1 & 1-\lambda & -3 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1-\lambda & 1 & 1 & -1 \\ 0 & (1-\lambda)(\lambda-1)+1 & (\lambda-1)+1 & (\lambda-1)+1 & -(\lambda-1) \\ 0 & \lambda & -\lambda & 0 & -1 \\ 0 & 0 & \lambda & -\lambda & -1 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1-\lambda & 1 & 1 & -1 \\ 0 & -\lambda^2+2\lambda & \lambda & \lambda & 1-\lambda \\ 0 & 0 & -\lambda^2+2\lambda+\lambda & -\lambda & -\lambda+1 \; -\lambda+2 \\ 0 & 0 & \lambda & -\lambda & -1 \end{array}\right)$$

$$\begin{array}{c} \quad u \qquad\quad v \qquad\qquad z \qquad\qquad w \\ \sim \left(\begin{array}{cccc|c} 1 & 1-\lambda & 1 & 1 & -1 \\ 0 & -\lambda^2+2\lambda & \lambda & \lambda & 1-\lambda \\ 0 & 0 & -\lambda^2+2\lambda+\lambda & \lambda & -2\lambda+3 \\ 0 & 0 & -\lambda^2+4\lambda & 0 & -2\lambda+1 \end{array}\right) \end{array}$$

dostávame

$$-z'' + 4z' = +1 \qquad z_h = c_1 + c_2 e^{4x}$$
$$+\lambda(-\lambda+4) = 0$$
$$\lambda_1 = 0 \quad \lambda_2 = 4 \qquad -1 = e^{0x}(-1\cos 0x + 0 \sin 0x)$$
$0+0i$ je koreň

$$z_p = Ax$$
$$z_p' = A \qquad 4A = +1 \quad A = \frac{1}{2} \qquad z = c_1 + c_2 e^{4x} + \frac{x}{2}$$
$$z_p'' = 0$$

$\lambda z - \lambda w = -1$

$4c_2 e^{4x} + \frac{1}{2} + 1 = w'$

$w' = 4c_2 e^{4x} + \frac{3}{2}$

$\underline{\underline{w = c_2 e^{4x} + \frac{3}{2}x + c_3}}$

$\lambda v - \lambda z = -1$

$v' = -1 + z'$

$v' = 4c_2 e^{4x} - \frac{1}{2}$

$\underline{\underline{v = c_2 e^{4x} - \frac{1}{2}x + c_4}}$

$u - u' = -v - z - w$

$-u' + u = -\cancel{c_2 e^{4x}} + \cancel{\frac{1}{2}x} - c_4 - c_1 - \underline{c_2} e^{4x} - \cancel{\frac{x}{2}} - \cancel{c_2 e^{4x}} - \underline{\frac{3}{2}} x - c_3$

$-u' + u = -3c_2 e^{4x} - \frac{3}{2}x - (c_1 + c_3 + c_4)$

integr. faktor

$u' - u = c_2 e^{4x} + \frac{3x}{2} + (c_1 + c_3 + c_4) \qquad | \cdot e^{-x}$

$(u e^{-x})' = 3c_2 e^{3x} + \frac{3}{2} x e^{-x} + (c_4 + c_1 + c_3) e^{-x}$

$u e^{-x} = \underline{c_2} e^{3x} - (c_1 + c_3 + c_4) e^{-x} - \frac{3}{2} \underline{e}^{-x} x - \underline{\frac{3}{2}} e^{-x}$

$\underline{\underline{u = c_2 e^{4x} - \frac{3}{2}x + \frac{3}{2} - c_1 - c_3 - c_4}}$

$$y' = \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix} y + \begin{pmatrix} 0 \\ \frac{e^{3x}}{e^{2x}+1} \end{pmatrix}$$

$$\left(\begin{array}{cc|c} \lambda+1 & -2 & 0 \\ 3 & \lambda-4 & \frac{e^{3x}}{e^{2x}+1} \end{array}\right)_{(\cdot 2)} \sim \left(\begin{array}{cc|c} \lambda+1 & -2 & 0 \\ 6 & 2(\lambda-4) & \frac{2e^{3x}}{e^{2x}+1} \end{array}\right) \begin{array}{l} \cdot(\lambda-4) \\ + \end{array} \sim$$

$$\sim \left(\begin{array}{cc|c} \lambda+1 & -2 & 0 \\ 6+(\lambda+1)(\lambda-4) & 0 & \frac{2e^{3x}}{e^{2x}+1} \end{array}\right) \sim \left(\begin{array}{cc|c} \lambda+1 & -2 & 0 \\ \lambda^2-3\lambda+2 & 0 & \frac{2e^{3x}}{e^{2x}+1} \end{array}\right)$$

y_1: $\quad y_1'' - 3y_1' + 2y_1 = \frac{2e^{3x}}{e^{2x}+1}$

$\lambda^2 - 3\lambda + 2 = 0$ $\qquad y_{1H} = c_1 e^x + c_2 e^{2x}$

$(\lambda-2)(\lambda-1) = 0$

$\lambda_1 = 1 \quad \lambda_2 = 2$

variace konstant:

$$y_{1H}' = c_1 e^x + c_1' e^x + 2c_2 e^{2x} + c_2' e^{2x}$$

položíme $\quad c_1' e^x + c_2' e^{2x} = 0$

$$y_{1H}'' = c_1 e^x + c_1' e^x + 4c_2 e^{2x} + 2c_2' e^{2x}$$

dosadíme

$$c_1 e^x + c_1' e^x + 4c_2 e^{2x} + 2c_2' e^{2x} - 3(c_1 e^x + 2c_2 e^{2x}) + 2c_1 e^x + 2c_2 e^{2x} = \frac{2e^{3x}}{e^{2x}+1}$$

soustava $\quad c_1' e^x + 2c_2' e^{2x} = \frac{2e^{3x}}{e^{2x}+1}$

(1)-(2) $\quad c_1' e^x + c_2' e^{2x} = 0$

$$c_2' e^{2x} = \frac{2e^{3x}}{e^{2x}+1} \rightarrow c_2' = \frac{2e^x}{e^{2x}+1}$$

$$\int \frac{2e^x}{e^{2x}+1}\,dx = \int \frac{2}{1+y^2}\,dy = 2\arctan y + k_2 = 2\arctan e^x + k_1$$

$y = e^x$
$dy = e^x\,dx$

$$c_2 = 2\arctan e^x + k_1$$

$$c_1' e^x = -\frac{2e^x \cdot e^{2x}}{e^{2x}+1} \qquad c_1' = \frac{-2e^{2x}}{e^{2x}+1}$$

$$\int \frac{e^{2x}\cdot(-2)}{e^{2x}+1}\,dx = \int \frac{-2y}{1+y^2}\,dy = -\log(1+y^2) + k_1 = -\log(1+e^x) + k_1$$

$y = e^x$

$dy = e^x\,dx$

$$c_1 = -\log(1+e^x) + k_1$$

dohromady

$$y_1 = k_1 e^x - e^x \cdot \log(1+e^x) + k_2 e^{2x} + e^{2x}\cdot 2\arctan e^x$$

y_2 :

$$2y_2 = y_1' + y_1$$

$$y_2 = \frac{1}{2}\Big(k_1 e^x - e^x \log(1+e^x) - e^x \cdot \frac{1}{1+e^x}\cdot e^x + 2k_2 e^{2x}$$
$$+ 4e^{2x}\arctan e^x + 2e^{2x}\cdot\frac{1}{1+e^{2x}}\cdot e^x$$
$$+ k_1 e^x - e^x \log(1+e^x) + k_2 e^{2x} + e^{2x}\cdot 2\arctan e^x\Big)$$

$$y_2 = k_1 e^x - e^x \log(1+e^x) + \frac{3}{2}k_2 e^{2x} + e^{2x}\arctan e^x \cdot 3$$
$$- \frac{1}{2}\frac{e^{2x}}{1+e^x} + \frac{e^{3x}}{1+e^{2x}}$$

$k_{1,2}, x \in \mathbb{R}$

$$y' = \begin{pmatrix} -4 & 0 & 6 \\ -7 & 2 & 8 \\ -3 & 0 & 5 \end{pmatrix} y + \begin{pmatrix} 2e^x \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} \lambda+4 & 0 & -6 & 2e^x \\ 7 & \lambda-2 & -8 & 0 \\ 3 & 0 & \lambda-5 & 0 \end{array}\right) (-3) \sim \left(\begin{array}{ccc|c} 7 & \lambda-2 & -8 & 0 \\ -3(\lambda+4) & 0 & 18 & -6e^x \\ 3 & 0 & \lambda-5 & 0 \end{array}\right) (\lambda$$

$$\sim \left(\begin{array}{ccc|c} 7 & \lambda-2 & -8 & 0 \\ 0 & 0 & 18+(\lambda+4)\cdot(\lambda-5) & -6e^x \\ 3 & 0 & \lambda-5 & 0 \end{array}\right) \sim \quad 0 \quad 0 \quad \lambda^2-\lambda-2 \,\big|\, -6e^x$$

y_3: $\quad y_3'' - y_3' - 2y_3 = -6e^x \qquad y_{3H} = c_1 e^{-x} + c_2 e^{2x}$

$(\lambda^2-\lambda-2) = 0$

$(\lambda-2)(\lambda+1) = 0$

$\lambda_1 = -1 \quad \lambda_2 = 2$

PS: $\quad -6e^x \qquad 1+0i$ není kořen

$y_{3P} = Ae^x$

$Ae^x - Ae^x - 2Ae^x = -6e^x \qquad A = 3$

$$y_3 = c_1 e^{-x} + c_2 e^{2x} + 3e^x$$

y_1: $\quad 3y_1 + y_3' - 5y_3 = 0$

$$y_1 = \frac{1}{3}(5y_3 - y_3') = \frac{1}{3}\left(5c_1e^{-x} + 5c_2e^{2x} + 15e^x + c_1e^{-x} - 2c_2e^{2x} - 3e^x\right)$$

$$= 2c_1e^{-x} + c_2e^{2x} + 4e^x$$

y_2:

$$7y_1 + y_2' - 2y_2 - 8y_3 = 0$$

$$y_2' - 2y_2 = 8y_3 - 7y_1$$

$$-\text{''}- = 8c_1 e^{-x} + 8c_2 e^{2x} + 24e^{x} - 14c_1 e^{-x} - 7c_2 e^{2x} - 28e^{x}$$

$$-\text{''}- = -6c_1 e^{-x} + c_2 e^{2x} - 4e^{x}$$

$$(\lambda - 2) = 0 \qquad \lambda = 2$$

$$y_{2H} = C_3 e^{2x}$$

variace konstant:
(možno i spec PS)

$$c_3' e^{2x} + \cancel{2c_3 e^{2x}} - \cancel{2c_3 e^{2x}} = -6c_1 e^{-x} + c_2 e^{2x} - 4e^{x}$$

$$c_3' = -6c_1 e^{-3x} + c_2 - 4e^{-x}$$

$$c_3 = 2c_1 e^{-3x} + c_2 x + 4e^{-x} + k$$

tedy $\quad y_2 = \underline{\underline{2c_1 e^{-x} + c_2 x e^{2x} + 4e^{x} + c_3 e^{2x}}}$

$\underline{\underline{c_{1,2,3}, x \in \mathbb{R}}}$

$$y' = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix} y + \begin{pmatrix} e^{-x} \\ 0 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} \lambda-1 & 1 & -1 & e^{-x} \\ -1 & \lambda-1 & 1 & 0 \\ 0 & 1 & \lambda-2 & 1 \end{array}\right) (\lambda-1) \sim \left(\begin{array}{ccc|c} 0 & 1+(\lambda-1)^2 & -1+(\lambda-1) & e^{-x} \\ -1 & \lambda-1 & 1 & 0 \\ 0 & 1 & \lambda-2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} -1 & \lambda-1 & 1 & 0 \\ 0 & \lambda^2-2\lambda+2 & \lambda-2 & e^{-x} \\ 0 & 1 & \lambda-2 & 1 \end{array}\right) (-1) \sim \left(\begin{array}{ccc|c} -1 & \lambda-1 & 1 & 0 \\ 0 & \lambda^2-2\lambda+1 & 0 & e^{-x}-1 \\ 0 & 1 & \lambda-2 & 1 \end{array}\right)$$

y_2:

$$y''-2y'+y = e^{-x}-1$$

$$(\lambda^2-2\lambda+1)=0 \qquad y_{2H} = c_1 e^{1x} + c_2 x e^{1x}$$

$$(\lambda-1)^2 = 0$$

$$\lambda_{12} = 1$$

y_{2P} PS: spec ps: e^{-x} $\quad -1+0i$ není kořen

tedy bude tvaru $y_{2P}^1 = Ae^{-x}$

$$y_{2P}^{1\,\prime} = -Ae^{-x}$$

$$y_{2P}^{1\,\prime\prime} = Ae^{-x}$$

dosazení $Ae^{-x} - 2\cdot(-Ae^{-x}) + Ae^{-x} = e^{-x}$

$$4Ae^{-x} = e^{-x} \qquad A = 1/4$$

spec PS $\quad -1 \quad$ $0+0i$ není kořen

$$y_{2P}^2 = B$$

dosazení $0-0+B=-1$

celkem $y_2 = c_1 e^{x} + c_2 x e^{x} + \frac{1}{4}e^{-x} + (-1)$

y_3:

$$y_2 + y_3' - 2y_3 = 1$$

$$y_3' - 2y_3 = 1 - c_1 e^x - c_2 x e^x - \frac{1}{4} e^{-x} + 1$$

$$\lambda - 2 = 0$$
$$\lambda = 2 \qquad \underline{\underline{y_{3H} = c_3 e^{2x}}}$$

y_{3P} PS:

(možno řešit i spec PS)

$$c_3 \cdot \cancel{2e^{2x}} + c_3' e^{2x} - \cancel{2c_3 e^{2x}} = 2 - c_1 e^x - c_2 x e^x - \frac{1}{4} e^{-x}$$

$$c_3' = 2e^{-2x} - c_1 e^{-x} - c_2 x e^{-x} - \frac{1}{4} e^{-3x}$$

$$c_3 = -e^{-2x} + c_1 e^{-x} + \frac{1}{12} e^{-3x} \underbrace{+ c_2 x e^{-x} + c_2 e^{-x}}_{\text{z per partes}}$$

tedy $\underline{\underline{y_3 = -1 + c_1 e^x + \frac{1}{12} e^x + c_2 x e^x + c_2 e^x + c_3 e^{2x}}}$

y_1:

$$-y_1 + y_2' - y_2 + y_3 = 0$$

$$y_1 = y_2' - y_2 + y_3$$

$$y_1 = c_1 e^x + c_2 e^x + c_2 x e^x - \frac{1}{4} e^{-x} - 1 - \left(c_1 e^x + c_2 x e^x + \frac{1}{4} e^{-x} - 1 \right)$$

$$+ -1 + c_1 e^x + \frac{1}{12} e^{-x} + c_2 x e^x + c_2 e^x + c_3 e^{2x}$$

$$= \underline{\underline{c_3 e^{2x} - \frac{7}{12} e^{-x} + e^x (c_1 + 2c_2)}} + c_2 x e^x$$

$\underline{\underline{c_{1,2,3}, x \in \mathbb{R}}}$