(1) $\sum_{n=1}^{\infty} \frac{n}{n^2+1} (\arctan x)^n \qquad x \in \mathbb{R}, \ K \ i \ AK$

Položme $b = (\arctan x)$, pak $b \in (-\pi/2, \pi/2)$

$$\sum_{n=1}^{\infty} \underbrace{\frac{n}{n^2+1} b^n}_{a_n}$$

- $b = 0 \quad \sum 0 \quad AK \ i \ K$

- $AK: \quad \sum \underbrace{\left| \frac{n}{n^2+1} b^n \right|}_{\geq 0}$

d'Al: $\lim_{n\to\infty} \frac{\frac{n+1}{(n+1)^2+1} |b|^{n+1}}{\frac{n}{n^2+1} |b|^n} = \lim_{n\to\infty} |b| \cdot \frac{n+1}{n} \cdot \frac{n^2+1}{n^2+2n+2} \stackrel{AL}{=} |b| \cdot 1 \cdot 1 = |b|$

$\frac{n}{n} \cdot \frac{1/n+1}{1} \qquad \cdot \frac{n^2}{n^2} \cdot \frac{1+1/n^2}{1+\frac{2}{n}+2/n^2}$

Pro $|b| < 1 \quad \sum AK$ a tedy i konverguje (Věta o vztahu AK a K).

Pro $|b| > 1 \quad \sum \left| \frac{n}{n^2+1} b^n \right|$ diverguje

- K: pro $b \geq 0 \quad AK$ a K splývá

pro $b \in (-\pi/2, -1)$ spočteme NP

n sudé

$\lim_{n\to\infty} \frac{n}{n^2+1} |b|^n = \infty$

n liché

$\lim_{n\to\infty} \frac{n}{n^2+1} (-1) \cdot |b|^n = -\infty$

$\left. \right\} \lim_{n\to\infty} \neq 0$ tedy $\sum$ nesplňuje NP a diverguje

plyne ze škály nebo: Heine $y_n = n$, $y_n \to \infty$, $y_n \neq \infty$

$\lim_{y\to\infty} \frac{y \cdot |b|^y}{y^2+1} \stackrel{LH}{\underset{\frac{\infty}{\infty}}{=}} \lim_{y\to\infty} \frac{|b|^y + y|b|^y \log|b|}{2y} \stackrel{LH}{\underset{\frac{\infty}{\infty}}{=}} \lim_{y\to\infty} \frac{1}{2} \left(2|b|^y \log|b| + y \log^2|b| \cdot |b|^y \right)$

$= \infty$

• $b = 1$

$$\sum \underbrace{\frac{n}{n^2+1}}_{c_n \geq 0}$$

LSK s $d_n = \frac{1}{n} \geq 0$ $\qquad \sum \frac{1}{n}$ Div.

$$\lim_{n\to\infty} \frac{\frac{n}{n^2+1}}{\frac{1}{n}} = \lim \frac{n^2}{n^2+1} = 1 \in (0,\infty)$$

Tedy $\sum c_n$ k $\iff$ $\sum d_n$ k. Odtud $\sum c_n$ Div.

• $b = -1$ $\qquad$ Až jako pro $b = 1$

k: $\sum \frac{n}{n^2+1} (-1)^n$

<u>Leibniz</u>: $\lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{1}{n+1/n} = 0$.

monotonie:

$$\frac{n}{n^2+1} \geq \frac{n+1}{(n+1)^2+1}$$

$$n(n^2+2n+2) \geq (n+1)(n^2+1)$$

$$\cancel{n^3} + \underline{2n^2} + \underline{2n} \geq \cancel{n^3} + \underline{n} + \underline{n^2} + 1$$

Tedy $\sum$ konv. $\qquad n^2 + n \geq 1 \qquad$ ($n \geq 1$, tedy zjevně platí)

<u>Závěr</u>

Pro $b \in (-1,1)$, tedy $x \in (\tan -1, \tan 1)$ $\sum \frac{n}{n^2+1} (\arctan x)^n$

absolutně konverguje, tedy i konverguje

Pro $b \in (-\infty,-1) \cup (1,\infty)$ tedy $x \in (-\infty, \tan -1) \cup (\tan 1, \infty)$ $\sum$ diverguje

Pro $b = 1$ tedy $x = \tan 1$ $\sum$ diverguje.

$b = -1$ $\qquad x = \tan -1$ $\sum$ konverguje neabsolutně

(2)

$$\int \underbrace{\frac{\sin x + \cos x}{1+\sin x}}_{g(x)}\, dx \qquad \text{na} \quad \left(-\tfrac{\pi}{2}, \tfrac{3\pi}{2}\right)$$

$g(x)$ je spoj na $\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right) \to$ má tam PF

Substituce

$$t = \tan\frac{x}{2} \qquad \sin x = \frac{2t}{1+t^2}$$

$$dx = \frac{2}{1+t^2}\, dt \qquad \cos x = \frac{1-t^2}{1+t^2}$$

$$\int \frac{\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}}{1 + \frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}\, dt = \int \frac{\frac{1-t^2+2t}{\cancel{1+t^2}}}{\frac{1+t^2+2t}{\cancel{1+t^2}}} \cdot \frac{2}{1+t^2}\, dt$$

$$= 2\int \frac{-t^2+2t+1}{(1+t^2)(1+t)^2}\, dt = 2\int \overset{A}{\frac{1}{1+t}} + \overset{B}{\frac{-1}{(1+t)^2}} + \overset{Ct+D}{\frac{-t+1}{1+t^2}}\, dt = (*)$$

$$-t^2+2t+1 = A\cdot(1+t)(1+t^2) + B(1+t^2) + (Ct+D)(1+t)^2$$

$t=-1 \qquad -2 = 2B \qquad \boxed{B=-1}$

$t=0 \qquad 1 = A - 1 + D$

$t=1 \qquad 2 = 4A - 2 + 4C + 4D$

$t=2 \qquad 1 = A\cdot 15 - 5 + 9D + 18C$

$$2 = A + D$$
$$1 = A + C + D$$
$$2 = 5A + 6C + 3D$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 5 & 6 & 3 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & -3 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 0 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 \end{array}\right)$$

$C = -1 \quad A = 1$

$D = 1$

$$2\int \frac{1}{1+t}\,dt \overset{c}{=} 2\log|1+t|$$

$$+2\int \frac{-1}{(1+t)^2}\,dt \overset{c}{=} \frac{2}{1+t}$$

$$\int \frac{-2t+2}{1+t^2}\,dt = -\int \frac{2t}{t^2+1}\,dt + \int \frac{2}{1+t^2}\,dt \overset{c}{=} -\log|1+t^2| + 2\arctan t$$

celkem

$$(*) \overset{c}{=} 2\log|1+t| + \frac{2}{1+t} - \log(1+t^2) + 2\arctan t$$

$$t\in(-\infty,-1),\ (-1,\infty)$$

$$\int \frac{\sin x+\cos x}{1+\sin x}\,dx \overset{c}{=} 2\log\left|1+\tan\frac{x}{2}\right| + \frac{2}{1+\tan\frac{x}{2}} - \log\left(1+\tan^2\frac{x}{2}\right) + 2\arctan\left(\tan\frac{x}{2}\right)$$

$$=: G(x)$$

Substituce

1\. VoS $\quad \varphi(x) = \tan\frac{x}{2}$ $\qquad (\alpha,\beta) = (-\frac{\pi}{2},\pi)$ $\qquad f(t) = 2\frac{-t^2+2t+1}{(1+t^2)(1+t)^2}$

$\varphi'(x) = \frac{1}{\cos^2\frac{x}{2}}\cdot\frac{1}{2}$ $\qquad (\tilde\alpha,\tilde\beta) = (\pi,\frac{3\pi}{2})$

$\exists$ na (α,β) (resp. $(\tilde\alpha,\tilde\beta)$) $\qquad (a,b) = (-1,\infty)$

$(\tilde a,\tilde b) = (-\infty,-1)$

$\varphi(\alpha,\beta) = (-1,\infty) \subseteq (a,b)$

$\varphi(\tilde\alpha,\tilde\beta) = (-\infty,-1) \subseteq (\tilde a,\tilde b)$ $\qquad$ tedy jsou splněny podm. 1VoS

a $\displaystyle\int \frac{\sin x+\cos x}{1+\sin x}\,dx \overset{c}{=} G(x)$ na $(-\frac{\pi}{2},\pi)$ a $(\pi,\frac{3\pi}{2})$

$\qquad\qquad\qquad\qquad c_1 \nearrow \qquad\qquad c_2 \nearrow$

Lepení v π

$$\lim_{x\to\pi-} \log\left(\frac{\left|1+\tan\frac{x}{2}\right|^2}{1+\tan^2\frac{x}{2}}\right) + \frac{2}{1+\tan\frac{x}{2}} + 2\arctan\left(\tan\frac{x}{2}\right) + c_1$$

$$= 0 + 0 + 2\cdot\frac{\pi}{2} + c_1$$

$$\lim_{x\to\pi+} G(x) \qquad + c_2 = 2\cdot\frac{-\pi}{2} + c_2$$

$$\pi + C_1 = -\pi + C_2$$

$$2\pi + C_1 = C_2$$

$$\int \frac{\sin x + \cos x}{1 + \sin x}\, dx = \begin{cases} G(x) + C_1 & x \in \left(-\frac{\pi}{2}, \pi\right) \\ \pi + C_1 & x = \pi \\ G(x) + 2\pi + C_1 & x \in \left(\pi, \frac{3\pi}{2}\right) \end{cases}$$

(3) $\displaystyle\int_0^\infty \underbrace{\frac{x^2+1}{x+\sqrt{x}} (\operatorname{arccot} x)^\alpha}_{} \, dx \qquad \alpha \in \mathbb{R}$

$f \geq 0$, spoj na $(0,\infty)$

$\int_0^\infty f = \int_0^1 + \int_1^\infty$

- $\int_0^1 f$

LSK s $g(x) = \frac{1}{\sqrt{x}}$ $\qquad \int_0^1 \frac{1}{\sqrt{x}} = \left[2\sqrt{x}\right]_0^1 = 2$ (konv.)

$f, g \geq 0$, spoj na $(0,1]$

$$\lim_{x\to 0+} \frac{f}{g} = \lim_{x\to 0+} \frac{\frac{x^2+1}{\cancel{\sqrt{x}}(\sqrt{x}+1)}(\operatorname{arccot} x)^\alpha}{\frac{1}{\cancel{\sqrt{x}}}} = \frac{0+1}{0+1} \cdot \left(\frac{\pi}{2}\right)^\alpha = \left(\frac{\pi}{2}\right)^\alpha \in (0,\infty)$$

spojitost

Tedy $\int_0^1 f$ k $\iff \int_0^1 g$ k

Tedy $\int_0^1 f$ k.

- $\int_1^\infty f$

LSK s $g(x) = \frac{x^2}{x} \cdot \frac{1}{x^\alpha} = \frac{1}{x^{\alpha-1}}$ $\qquad \int_1^\infty x^{1-\alpha}\,dx$ k $\iff 1-\alpha < -1$

$\iff 2 < \alpha$

$f, g \geq 0$, spoj na $[1,\infty)$

$$\lim_{x\to\infty} \frac{f}{g} = \lim_{x\to\infty} \frac{\frac{x^2+1}{x+\sqrt{x}}(\operatorname{arccot} x)^\alpha}{\frac{x^2}{x} \cdot \frac{1}{x^\alpha}} = \lim \frac{x^2+1}{x^2} \cdot \frac{x}{x+\sqrt{x}} \cdot (x \operatorname{arccot} x)^\alpha$$

$$= \lim \left(1+\frac{1}{x^2}\right) \cdot \frac{1}{1+1/\sqrt{x}} \cdot (x \operatorname{arccot} x)^\alpha \overset{AL}{=} (1+0) \cdot \frac{1}{1+0} \cdot 1^\alpha = 1 \in (0,\infty)$$

Tedy $\int_1^\infty f$ k $\iff \int^\infty g \iff 2 < \alpha$

Závěr: $\int_0^\infty f$ k $\iff 2 < \alpha$ $\qquad$ (Tedy $\int_0^\infty f$ D $\iff \alpha \leq 2$)

(4) $F: \mathbb{R}^2 \to \mathbb{R}^2$

$$F(x,y) = \left(\log(1+2x^2+3y^2)(|x|+|y|),\ \operatorname{sgn}(x-y)\sqrt{3x^2+2y^2}\right)$$

(a) $F'(1,3)$?

(b) $F_1'(0,0)$?

(c) $\frac{\partial F_2}{\partial y}(0,0)$?

(a)
$$\begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{1}{1+2x^2+3y^2}\cdot 4x\,(|x|+|y|) + \log(1+2x^2+3y^2)\cdot 1 & \frac{6y\cdot(x+y)}{1+2x^2+3y^2} + \log(1+2x^2+3y^2)\cdot 1 \\ -\frac{1}{2}\frac{1}{\sqrt{3x^2+2y^2}}\cdot 6x & \cdot -\frac{1}{2}\frac{4y}{\sqrt{3x^2+2y^2}} \end{pmatrix}$$

Na $M = B((1,3), 1/4)$ je $|x| = x$, $|y| = y$

a $\operatorname{sgn}(x-y) = -1$

$\Downarrow$

$\forall$ parc. der. jsou na M spojité

$\Rightarrow \exists\ F'(1,3)$ a je tvaru

$$\begin{pmatrix} \frac{1}{1+2+27}\cdot 4\,(1+3) + \log(1+2+27) & \frac{6\cdot 3(1+3)}{1+2+27} + \log 30 \\ -\frac{1}{2}\frac{1}{\sqrt{3+18}}\cdot 6 & -\frac{1}{2}\frac{4\cdot 3}{\sqrt{3+2\cdot 9}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{8}{15} + \log 30 & \frac{12}{5} \\ \frac{-3}{\sqrt{21}} & \frac{-6}{\sqrt{21}} \end{pmatrix}$$

(b) $F_1 = \log(1+2x^2+3y^2)(|x|+|y|)$

$$\frac{\partial F_1(0,0)}{\partial x} = \lim_{t\to 0} \frac{\log(1+2t^2+0)(|0+t|+0) - 0}{t}$$

$$= \lim_{t\to 0} \frac{|t|\log(1+2t^2)}{t} = \lim_{t\to 0} \frac{\log(1+2t^2)}{2t^2} \cdot \frac{2t^2 \cdot |t|}{t}$$

$$= \lim_{t\to 0} \frac{\log(1+2t^2)}{2t^2} \cdot 2t|t| = 1 \cdot 0 = 0$$

$$\frac{\partial F_1}{\partial y}(0,0) = \lim_{t\to 0} \frac{\log(1+0+3t^2)(0+|t|)}{t} = \lim \frac{\log(1+3t^2)}{3t^2} \cdot |t| \cdot t \cdot 3$$

$$= 0$$

kandidát na tot. dif. $\quad F_1'(0,0)(h) = 0h_1 + 0h_2$

$$\lim_{h\to 0} \frac{\log(1+2h_1^2+3h_2^2)(|h_1|+|h_2|) - (0h_1+0h_2)}{|h_1|+|h_2|}$$

sčítací norma (normy jsou ekviv.)

$$= \lim_{h\to 0} \log(1+2h_1^2+3h_2^2) = \log 1 = 0$$

$\Rightarrow \quad F_1'(0,0) \ \exists$ a rovná se $\underline{F_1'(0,0) = 0h_1 + 0h_2}$,

(c)

$$\frac{\partial F_2}{\partial y}(0,0) = \lim_{t\to 0} \frac{\operatorname{sgn}(0-t)\sqrt{0+2t^2} - 0}{t}$$

$$= \lim_{t\to 0} \frac{\operatorname{sgn}(-t)\sqrt{2}\,|t|}{t} = \underline{\underline{-\sqrt{2}}}$$

$$\lim_{t\to 0+} \frac{-\sqrt{2}\cdot t}{t} = -\sqrt{2}$$

$$\lim_{t\to 0-} \frac{1\cdot\sqrt{2}\cdot(-t)}{t} = -\sqrt{2}$$

(5) $f, g : (0,1) \to [0, \infty)$, spojité

(a) $\int_0^1 f$ K $\quad \int_0^1 g$ K $\quad \overset{?}{\Rightarrow} \quad \int_0^1 \max\{f,g\}$ K

(b) $\int_0^1 f$ D $\quad \int_0^1 g$ D $\quad \overset{?}{\Rightarrow} \quad \int_0^1 \max\{f,g\}$ D

(c) $\int_0^1 f$ K $\quad \int_0^1 g$ D $\quad \overset{?}{\Rightarrow} \quad \int_0^1 \max\{f,g\}$ D

$$\max\{f,g\} = \frac{f+g+|f-g|}{2}$$

spoj. → má na $(0,1)$ PF, označme ji H
≥ 0 → H je neklesající → $\exists \lim_{x \to b-} H$ a $\lim_{x \to a+} H$. A jejich rozdíl je definován

(a) platí

$$\int_0^1 \max\{f,g\} = \frac{1}{2}\left(\int_0^1 f + \int_0^1 g + \int_0^1 |f-g|\right)$$

z linearity $\int$ a (N)$\int$ a uspořádání

$$\leq \frac{1}{2}\left(\int_0^1 f + \int_0^1 g + \int_0^1 \underbrace{|f|}_{=f} + \int_0^1 \underbrace{|g|}_{=g}\right) =$$

$$= \frac{1}{2}\left(2\int_0^1 f + 2\int_0^1 g\right) < \infty$$

konv + konv = konv.

(b) platí

$$\frac{1}{2}\left(\int_0^1 f + \int_0^1 g + \int_0^1 \underbrace{|f-g|}_{\geq 0}\right) \geq \frac{1}{2}\left(\int_0^1 f + \int_0^1 0 + \int_0^1 0\right) = \frac{1}{2}\int_0^1 f = \infty$$

$g \geq 0$

linearita a uspořádání

($\int_0^1 f$ divergentní $\Rightarrow$ $\int_0^1 f = \infty$, analog. Dk. z věty o $\int$ a abs. hodnotě)

(c) platí

$$\frac{1}{2}\left(\underbrace{\int_0^1 f}_{\geq 0} + \int_0^1 g + \underbrace{\int_0^1 |f-g|}_{\geq 0}\right) \geq \frac{1}{2}\int_0^1 g = \infty$$

(analogicky k (b))