(1a)

$$y' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} y + \begin{pmatrix} -x^2 \\ 2x \end{pmatrix}$$

$$\begin{matrix} (+ \\ (\lambda-1) \end{matrix} \left(\begin{array}{cc|c} 1-\lambda & -1 & x^2 \\ 1 & 3-\lambda & -2x \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 3-\lambda & -2x \\ 0 & \underbrace{-1+(3-\lambda)(\lambda-1)}_{-\lambda^2+4\lambda-4} & \underbrace{x^2-2x(\lambda-1)}_{\underbrace{x^2+2x-\lambda\cdot(2x)}_{x^2+2x-2}} \end{array}\right)$$

$y_2:$ $\lambda^2-4\lambda+4=0$

$(\lambda-2)^2=0$

$\lambda=2$

$y_{2,H} = c_1 e^{2x} + c_2 x e^{2x}$

$0+0i$ není kořen

$\to y_{2p} = Ax^2+Bx+C$

$-y_{2p}'' + 4y_{2p}' - 4y_{2p} = x^2+2x-2$

$-4Ax^2 + x(8A-4B) + 4B-4C-2A = 1x^2+2x-2$

$\underline{A=-1/4}$ $\quad -2-4B=2 \quad \underline{B=-1}$

$-4-4C+\frac{1}{2} = -2 \quad -4C=\frac{3}{2}$

$\underline{C=-\frac{3}{8}}$

$$y_2 = \underline{\underline{c_1 e^{2x} + c_2 x e^{2x} - \frac{x^2}{4}}} - x - \frac{3}{8}$$

$y_1 = -2x - (3-\lambda)y_2$

$y_1 = -2x - 3y_2 + y_2'$

$$y_1 = \underline{\underline{-c_1 e^{2x} - c_2 e^{2x}\cdot x + c_2 e^{2x} + \frac{3}{4}x^2 + \frac{x}{2} + \frac{1}{8}}}$$

$x, c_{1,2} \in \mathbb{R}$

Nebr.

Od druhého řádku odečteme třetí řádek a poté od třetího řádku odečteme $(\lambda-4)$-násobek druhého řádku

$$\sim\begin{pmatrix} \lambda-2 & -1 & 1 \\ \lambda^2+2 & 1 & 0 \\ -\lambda^2-\lambda-7 & \lambda-4 & 0 \end{pmatrix} \sim \begin{pmatrix} \lambda-2 & -1 & 1 \\ \lambda^2+2 & 1 & 0 \\ -\lambda^3+3\lambda^2-3\lambda+1 & 0 & 0 \end{pmatrix}$$

V posledním řádku máme nyní rovnici $-x'''+3x''-3x'+x=0$, v matici je tedy rovnou charakteristický polynom této rovnice. Dostáváme tedy řešení

$$x(t)=c_1e^t+c_2te^t+c_3t^2e^t\,,\quad t\in\mathbb{R}\,,c_1,c_2,c_3\in\mathbb{R}\,.$$

Z druhého řádku matice dopočítáme y:

$$y(t)=-x''-2x=-e^t(3c_1+2c_2+2c_3+t(3c_2+4c_3)+t^2\cdot 3c_3)\,,\quad t\in\mathbb{R}\,.$$

Z první rovnice dopočítáme z:

$$z(t)=e^t(-2c_1-3c_2-2c_3+t(-2c_2-6c_3)+t^2\cdot(-2c_3))\,,\quad t\in\mathbb{R}\,.$$

Příklad 3. Řešte soustavu

(1b)

$$\begin{aligned} 2y''+3z''-7y-6z &= t+1 \\ 4y''+3z''-4y-3z &= 2t. \end{aligned}$$

Řešení. Napíšeme si λ-matici s pravou stranou:

$$\left(\begin{array}{cc|c} 2\lambda^2-7 & 3\lambda^2-6 & t+1 \\ 4\lambda^2-4 & 3\lambda^2-3 & 2t \end{array}\right) \sim \left(\begin{array}{cc|c} 2\lambda^2-7 & 3\lambda^2-6 & t+1 \\ 2\lambda^2+3 & 3 & t-1 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 2\lambda^2-7-(\lambda^2-2)(2\lambda^2+3) & 0 & t+1-(t-1)''+2(t-1) \\ 2\lambda^2+3 & 3 & t-1 \end{array}\right)$$

$$= \left(\begin{array}{cc|c} -2\lambda^4+3\lambda^2-1 & 0 & 3t-1 \\ 2\lambda^2+3 & 3 & t-1 \end{array}\right).$$

Řešení charakteristického polynomu v prvním řádku jsou $\lambda_{1,2}=\pm 1$, $\lambda_{3,4}=\pm\sqrt{2}/2$. Tedy řešení homogenní rovnice

$$-2y^{(4)}+3y''-y=0$$

jsou

$$ae^t+be^{-t}+ce^{\sqrt{2}/2t}+de^{-\sqrt{2}/2t}.$$

Partikulární řešení nehomogenní rovnice bude ve tvaru $y_p(t) = rt + s$, což po dosazení do rovnice dává $y_p(t) = -3t + 1$. Tedy

$$y(t) = ae^t + be^{-t} + ce^{\sqrt{2}/2t} + de^{-\sqrt{2}/2t} - 3t + 1\,.$$

Z druhé rovnice pak máme

$$\begin{aligned} 3z(t) &= t - 1 - 2y'' - 3y = t - 1 - 2(ae^t + be^{-t} + \frac{1}{2}ce^{\sqrt{2}/2t} + \frac{1}{2}de^{-\sqrt{2}/2t}) \\ &\quad -3(ae^t + be^{-t} + ce^{\sqrt{2}/2t} + de^{-\sqrt{2}/2t} - 3t + 1) \\ &= 10t - 4 - 5ae^t - 5be^{-t} - 4ce^{\sqrt{2}/2t} - 4de^{-\sqrt{2}/2t}\,. \end{aligned}$$

(2a)

$y_1' = y_1 - y_2 - x$ $\quad\quad y(0) = (0,0)^T$

$y_2' = y_1 + y_2$

$$\begin{pmatrix} 1-\lambda & -1 & \Big| & +x \\ 1 & 1-\lambda & \Big| & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & -1+(1-\lambda)(\lambda-1) & \Big| & x \\ 1 & 1-\lambda & \Big| & 0 \end{pmatrix}$$

$$\begin{array}{cc} y_1 & y_2 \end{array}$$
$$\begin{pmatrix} 0 & -\lambda^2+2\lambda-1-1 & \Big| & x \\ 1 & 1-\lambda & \Big| & 0 \end{pmatrix}$$

$\to \quad -y_2'' + 2y_2' - 2y_2 = x$ $\quad\quad y_{2h} = c_1 e^x \sin x + c_2 e^x \cos x$

$-\lambda^2 + 2\lambda - 2 = 0$ $\quad\quad$ Spec. PS.

$\lambda^2 - 2\lambda + 2 = 0$ $\quad\quad x = e^{0x}(x\cos 0x + 0 \sin 0x)$

$\lambda = 1 \pm i$ $\quad\quad 0+0i$ není kořen

hledáme řešení tvaru $\quad y_{2p} = Ax + B$

$y_{2p}' = A$

$2A - 2Ax - 2B = 1 \cdot x$ $\quad\quad y_{2p}'' = 0$

$A = -1/2$ $\quad\quad -1 - 2B = 0$

$B = -1/2$

$y_2 = c_1 e^x \sin x + c_2 e^x \cos x - \frac{x}{2} - \frac{1}{2}$

Podm.: $0 = c_2 - 1/2 \to c_2 = 1/2$

$0 = c_1$

$y_1 + (1-\lambda) y_2 = 0$

$y_1 = y_2' - y_2$

$y_1 = c_1 e^x \cos x - c_2 e^x \sin x + \frac{x}{2}$

celkem $\quad y_1 = -\frac{1}{2} e^x \sin x + \frac{x}{2}$ $\quad\quad y_2 = \frac{1}{2} e^x \cos x - \frac{x}{2} - \frac{1}{2}$ $\quad\quad x \in \mathbb{R}$

(2b) $y_1' = y_2 + \sin x$

$y_2' = -y_1 + \cos x$

$$\begin{array}{l} \\ +\\ \cdot(-\lambda) \end{array}\left(\begin{array}{cc|c} \lambda & -1 & \sin x \\ 1 & \lambda & \cos x \end{array}\right) \sim \left(\begin{array}{cc|c} 0 & -1-\lambda^2 & \sin x - \lambda \cos x \\ 1 & \lambda & \cos x \end{array}\right)$$

$-y_2 - y_2'' = \underbrace{\sin x - (\cos x)'}_{2\sin x}$ $\quad y_{2H} = c_1 \cos x + c_2 \sin x$

$-1-\lambda^2 = 0 \qquad \lambda = \pm i$

$-1 = \lambda^2$

PS: $\quad 2\sin x = e^{0x}(0 \cos 1x + 2 \sin 1x)$

i je kořen

$y_{2p} = x(A\cos x + B \sin x)$

dosazení

$2A \sin x - 2B\cos x = 2 \sin x$ $\qquad y_2 = c_1 \cos x + c_2 \sin x + x\cos x$

$B = 0 \qquad A = 1$

$y_1 = \cos x - y_2'$

$y_1 = x \sin x + c_1 \sin x - c_2 \cos x$

$c_1, c_2, x \in \mathbb{R}$

(2c)

$$y_1' = y_1 + 2y_2 + \sin \cancel{x} \qquad y(0) = (0,0)$$
$$y_2' = -y_1 - y_2$$

$$\begin{array}{c} \\ 1-\lambda \end{array} \begin{pmatrix} 1-\lambda & 2 & | & -\sin x \\ -1 & \underbrace{-1-\lambda}_{-(1+\lambda)} & | & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 2+(-1)(1+\lambda)(1-\lambda) & | & -\sin x \\ 1 & 1+\lambda & | & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \lambda^2+1 & | & -\sin x \\ 1 & 1+\lambda & | & 0 \end{pmatrix}$$

y_2: $\quad \lambda^2+1=0 \qquad y_{2h} = c_1 \cos x + c_2 \sin x$

$\lambda = \pm i$

Spec. PS: $\quad -\sin x = e^{0x}(0 \cos 1x - 1 \sin 1x)$

$0+1i$ je 1-nas. koren

$$y_p = x^1(A\cos x + B \sin x)$$

$$y_p'' + y_p = -\sin x \qquad\qquad y_2 = c_1 \cos x + c_2 \sin x + \tfrac{1}{2} x \cos x$$

$$-2A \sin x + 2B\cos x = -\sin x$$

$$\to B=0 \quad A = 1/2$$

pak $\quad y_1 = -y_2 - y_2'$

$$y_1 = (-c_2 + c_1)\sin x + (-c_1 - c_2)\cos x + \tfrac{-1}{2}(x\cos x + \cos x - x \sin x)$$

Podm. $\quad 0 = c_1 \qquad c_2 = -1/2$

$\qquad 0 = -c_2 - 1/2$

Celkem $\quad y_1 = \frac{1}{2}\sin x + \frac{1}{2}\cos x - \frac{1}{2}(x\cos x + \cos x - x\sin x)$

$\qquad y_2 = -\frac{1}{2}\sin x + \frac{1}{2}x\cos x$

$x \in \mathbb{R}$

(2d)

$$u' = 4u + 3v - 3w$$

$$v' = -3u - 2v + 3w$$

$$w' = 3u + 3v - 2w + 2e^{-x}$$

$$\begin{array}{c|c} & \begin{array}{ccc} u & v & w \end{array} \\ \begin{array}{c} 3\cdot \\ +(\\ (4-\lambda)\cdot \end{array} & \left(\begin{array}{ccc|c} 4-\lambda & 3 & -3 & 0 \\ -3 & -2-\lambda & 3 & 0 \\ 3 & 3 & -2-\lambda & -2e^{-x} \end{array}\right) \sim \end{array}$$

$$\sim \left(\begin{array}{ccc|c} -3 & -2-\lambda & 3 & 0 \\ 0 & 9-(2+\lambda)(4-\lambda) & -9+12-3\lambda & 0 \\ 0 & 1-\lambda & 1-\lambda & -2e^{-x} \end{array}\right) \begin{array}{c} \sim \\ + \\ \cdot(-3) \end{array}$$

$$\sim \left(\begin{array}{ccc|c} -3 & -2-\lambda & 3 & 0 \\ 0 & (\lambda-1)^2 & 3(1-\lambda) & 0 \\ 0 & (\lambda-1)^2-3(1-\lambda) & 0 & 6e^{-x} \end{array}\right) \sim \left(\begin{array}{ccc|c} -3 & -2-\lambda & 3 & 0 \\ 0 & (\lambda-1)^2 & 3(1-\lambda) & 0 \\ 0 & \underbrace{(\lambda-1)(\lambda+2)}_{\lambda^2+\lambda-2} & 0 & 6e^{-x} \end{array}\right)$$

v: $(\lambda-1)(\lambda+2)=0$ $\quad\lambda_1=1 \quad \lambda_2=-2$

$v_H = c_1 e^{x} + c_2 e^{-2x}$

PS: $6e^{-x} = e^{-x}(6\cos 0x + 0\sin 0x)$

$-1+0i$ není kořen

$v_P = Ae^{-x}$

$v'' + v' - 2v = 6e^{-x}$

$-2Ae^{-x} = 6e^{-x} \qquad A=-3$

$$\underline{v = c_1 e^{x} + c_2 e^{-2x}} - 3e^{-x}$$

w: $3w - 3w' = -v'' + 2v' - v$

$-w' + w = \frac{1}{3}(12e^{-x} - 9c_2 e^{-2x})$

variace konstant $\quad w_H = k e^{x}$

$-k'e^{x} - ke^{x} + ke^{x} = 4e^{-x} - 3c_2 e^{-2x}$

$k' = -e^{-x}(4e^{-x} - 3c_2 e^{-2x})$

$k = e^{-3x}(2e^{x} - c_2) + c_3$

$$w = \underline{2e^{-x} - c_2 e^{-2x} + c_3 e^{x}}$$

$u:\quad -3u = (2+\lambda)v - 3w$

$$u = -\frac{1}{3}(2v + v' - 3w)$$

$$\underline{\underline{u = -e^{x}(c_1 - c_3) + e^{2x}(-c_2) + 3e^{-\lambda}}}$$

$x, c_{1-3} \in \mathbb{R}$

(2e)

$$u' = u + v + z + w$$
$$v' = u + v + z + w + 1$$
$$z' = u + v + z + w + 2$$
$$w' = u + v + z + w + 3$$

$$\cdot(\lambda-1)\left(\begin{array}{cccc|c} 1-\lambda & 1 & 1 & 1 & 0 \\ 1 & 1-\lambda & 1 & 1 & -1 \\ 1 & 1 & 1-\lambda & 1 & -2 \\ 1 & 1 & 1 & 1-\lambda & -3 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1-\lambda & 1 & 1 & -1 \\ 0 & (1-\lambda)(\lambda-1)+1 & (\lambda-1)+1 & (\lambda-1)+1 & -(\lambda-1) \\ 0 & \lambda & -\lambda & 0 & -1 \\ 0 & 0 & \lambda & -\lambda & -1 \end{array}\right) \cdot(\lambda-2)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1-\lambda & 1 & 1 & -1 \\ 0 & -\lambda^2+2\lambda & \lambda & \lambda & 1-\lambda \\ 0 & 0 & -\lambda^2+2\lambda+\lambda & -\lambda & -\lambda+1-\lambda+2 \\ 0 & 0 & \lambda & -\lambda & -1 \end{array}\right)$$

$$\sim \begin{array}{c} \begin{array}{cccc} u & v & z & w \end{array} \\ \left(\begin{array}{cccc|c} 1 & 1-\lambda & 1 & 1 & -1 \\ 0 & -\lambda^2+2\lambda & \lambda & \lambda & 1-\lambda \\ 0 & 0 & -\lambda^2+2\lambda+\lambda & \lambda & -2\lambda+3 \\ 0 & 0 & -\lambda^2+4\lambda & 0 & -2\lambda+2 \end{array}\right) \end{array}$$

dostaneme

$$-z''+4z' = +2$$
$$+\lambda(-\lambda+4)=0$$
$$\lambda_1=0 \quad \lambda_2=4$$

$$z_h = c_1 + c_2 e^{4x}$$

$$-1 = e^{0x}(-1\cos 0x + 0 \sin 0x)$$
$0+0i$ je koren

$$z_p = Ax$$
$$z_p' = A$$
$$z_p'' = 0$$

$$4A = +2 \quad A = \frac{1}{2}$$

$$z = c_1 + c_2 e^{4x} + \frac{x}{2}$$

$$\lambda z - \lambda w = -1 \qquad w' = 4c_2e^{4x} + \frac{3}{2}$$

$$4c_2e^{4x} + \frac{1}{2} + 1 = w' \qquad \underline{\underline{w = c_2e^{4x} + \frac{3}{2}x + c_3}}$$

$$\lambda v - \lambda z = -1$$

$$v' = -1 + z' \qquad \underline{\underline{v = c_2e^{4x} - \frac{1}{2}x + c_4}}$$

$$v' = 4c_2e^{4x} - \frac{1}{2}$$

$$u - u' = -v - z - w$$

$$-u' + u = \cancel{-c_2e^{4x}} + \cancel{\frac{1}{2}x} - c_4 - c_1 - \underline{c_2}e^{4x} - \cancel{\frac{x}{2}} - \cancel{c_2e^{4x}} - \underline{\frac{3}{2}}x - c_3$$

$$-u' + u = -3c_2e^{4x} - \frac{3}{2}x - (c_1 + c_3 + c_4)$$

integr. faktor

$$u' - u = c_2e^{4x} + \frac{3x}{2} + (c_1 + c_3 + c_4) \qquad | \cdot e^{-x}$$

$$(ue^{-x})' = 3c_2e^{3x} + \frac{3}{2}xe^{-x} + (c_4 + c_1 + c_3)e^{-x}$$

$$ue^{-x} = \underline{c_2}e^{3x} - (c_1 + c_3 + c_4)e^{-x} - \frac{3}{2}\underline{e}^{-x}x - \underline{\frac{3}{2}}e^{-x}$$

$$\underline{\underline{u = c_2e^{4x} - \frac{3}{2}x + \frac{3}{2} - c_1 - c_3 - c_4}}$$

$$y' = \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix} y + \begin{pmatrix} 0 \\ \dfrac{e^{3x}}{e^{2x}+1} \end{pmatrix}$$

$$\left(\begin{array}{cc|c} \lambda+1 & -2 & 0 \\ 3 & \lambda-4 & \dfrac{e^{3x}}{e^{2x}+1} \end{array}\right) \begin{array}{l} /\cdot(-3) \\ + \\ /(\lambda+1) \end{array} \sim$$

$$\sim \left(\begin{array}{cc|c} -3(\lambda+1)+3(\lambda+1) & \overbrace{6+(\lambda+1)(\lambda-4)}^{\lambda^2-3\lambda+2} & (\lambda+1)\dfrac{e^{3x}}{e^{2x}+1} \\ 3 & \lambda-4 & \dfrac{e^{3x}}{e^{2x}+1} \end{array}\right)$$

lze přepsat jako

$$(\lambda^2-3\lambda+2)y_2 = (\lambda+1)\frac{e^{3x}}{e^{2x}+1}$$

$$y_2'' - 3y_2' + 2y_2 \qquad \downarrow$$

$$\left(\frac{e^{3x}}{e^{2x}+1}\right)' + \frac{e^{3x}}{e^{2x}+1} = \frac{e^{3x}(e^{2x}+3)}{(e^{2x}+1)^2} + \frac{e^{3x}}{e^{2x}+1} = \frac{2e^{3x}(e^{2x}+2)}{(e^{2x}+1)^2}$$

Řešíme prve homogenní ři:

$$y_2'' - 3y_2' + 2y_2 = 0 \qquad y_{2p} = c_1 e^{2x} + c_2 e^{x}$$

$$(\lambda^2 - 3\lambda + 2) = 0$$

$$(\lambda-2)(\lambda-1) = 0$$

$$\lambda_1 = 1 \quad \lambda_2 = 2$$

Spravou stranou: aplikujeme variaci konstant

$$y_{2p}' = \underline{c_1'} e^{2x} + c_1 \cdot 2e^{2x} + \underline{c_2' e^{x}} + c_2 e^{x}$$

položíme $\boxed{c_1' e^{2x} + c_2' e^{x} = 0}$

$$y_{2p}'' = 2c_1' e^{2x} + 4c_1 e^{2x} + c_2' e^{x} + c_2 e^{x}$$

po dosazení:

$$2c_1'e^{2x} + 4c_1e^{2x} + c_2'e^x + c_2e^x - 3\left(2c_1e^{2x} + c_2e^x\right)$$
$$+ 2\left(c_1e^{2x} + c_2e^x\right) = \frac{2e^{3x}(e^{2x}+2)}{(e^{2x}+1)^2}$$

dohromady

$$c_1'e^{2x} + c_2'e^x = 0 \qquad |\cdot(-1)$$
$$2c_1'e^{2x} + c_2'e^x = \frac{2e^{3x}(e^{2x}+2)}{(e^{2x}+1)^2}$$

$$c_1'e^{2x} = \frac{2e^{3x}(e^{2x}+2)}{(e^{2x}+1)^2}$$

$$c_1' = \frac{2e^{x}(e^{2x}+2)}{(e^{2x}+1)^2}$$

integrace

$$\int \frac{2e^x(e^{2x}+2)}{(e^{2x}+1)^2}\,dx = 2\int \frac{u^2+2}{(u^2+1)^2}\,du$$

$$u = e^x \quad du = e^x dx \qquad = 2\int \frac{1}{u^2+1} + \frac{1}{(u^2+1)^2}\,du$$

← triky z Lovaka

$$\stackrel{c}{=} 2\arctan u + 2\left(\frac{1}{2}\left(\frac{u}{1+u^2} + \arctan u\right)\right)$$
$$= \frac{u}{1+u^2} + 3\arctan u$$
$$= \frac{e^x}{1+e^{2x}} + 3\arctan e^{2x}$$

tedy $\displaystyle c_1 = \frac{e^x}{1+e^{2x}} + 3\arctan(e^{2x}) + k_1$

pak

$$c_2' = -\frac{c_1'e^{2x}}{e^x} = -e^xc_1'$$

$$c_2' = -e^x \frac{2e^x(e^{2x}+2)}{(e^{2x}+1)^2}$$

$$-2\int \frac{e^x e^x (e^{2x}+2)}{(e^{2x}+1)^2}\,dx = \int \frac{u(u^2+2)}{(u^2+1)^2}\,du =$$

$u = e^x \quad du = e^x dx$

$$= -2\int \frac{u}{u^2+1} + \frac{u}{(u^2+1)^2}\,du =$$

$$\stackrel{e}{=} -\ln(u^2+1) + \frac{1}{u^2+1} = -\ln(e^{2x}+1) + \frac{1}{e^{2x}+1}$$

tedy $\quad c_2 = -\ln(e^{2x}+1) + \frac{1}{e^{2x}+1} + k_2$

Celé řešení dohromady

$$y_2 = e^{2x}\left(\frac{e^x}{1+e^{2x}} + 3\arctan e^{2x} + k_1\right) + e^x\left(-\ln(e^{2x}+1) + \frac{1}{e^{2x}+1} + k_2\right)$$

pro y_1 máme $\qquad y_2' = -3y_1 + 4y_2 + \frac{e^{3x}}{e^{2x}+1}$

$$y_1 = \frac{1}{3}\left(-y_2' + 4y_2 + \frac{e^{3x}}{e^{2x}+1}\right) =$$

$$= \frac{1}{3}\Big[-\Big(2e^{2x}\left(\frac{e^x}{1+e^{2x}} + 3\arctan e^{2x} + k_1\right) + e^{2x}\left(\frac{e^x - e^{3x}}{(1+e^{2x})^2} + \frac{3}{1+e^{4x}}\cdot 2e^{2x}\right)$$

$$+ e^x\left(-\ln(e^{2x}+1) + \frac{1}{1+e^{2x}} + k_2\right) + e^x\left(\frac{-2e^{2x}}{1+e^{2x}} + \frac{-1}{(1+e^{2x})^2}\cdot 2e^{2x}\right)\Big)$$

$$+ 4\left(e^{2x}\left(\frac{e^x}{1+e^{2x}} + 3\arctan e^{2x} + k_1\right) + e^x\left(-\ln(e^{2x}+1) + \frac{1}{e^{2x}+1} + k_2\right)\right)$$

$$+ \frac{e^{3x}}{e^{2x}+1}\Big]$$