(1) $\displaystyle\sum_{n=1}^{\infty} \frac{3+\sin n\frac{\pi}{6}}{2^n} x^n$

$$\limsup_{n\to\infty} \sqrt[n]{\frac{3+\sin n\frac{\pi}{6}}{2^n}} = \lim_{n\to\infty} \sqrt[n]{\frac{3+1}{2^n}} = \frac{1}{2}$$

$\underline{\underline{R = 2}}$

(2) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{n(2n+1)}$

$$\limsup \sqrt[n]{\frac{|(-1)^n|}{n(2n+1)}} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n(2n+1)}} = 1 \quad \to R = 1$$

Pracujeme na intervalu $I = (-1, 1)$

Označme $\displaystyle f(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{n(2n+1)}$

na $(-1,1)$ je (věta odci)

$$f'(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{n}$$

tady sčítáme na $I \setminus \{0\}$

$$f''(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-1}}{n} \cdot 2n = \sum_{n=1}^{\infty} 2\,(-1)^n\, x^{2n-1} = 2 \sum_{n=1}^{\infty} \frac{1}{x} (-x^2)^n$$

$$= 2 \cdot \frac{1}{x} \sum_{n=1}^{\infty} (-x^2)^n = 2 \cdot \frac{1}{x} \frac{-x^2}{1+x^2} = \frac{-2x}{1+x^2}$$

$\to$ výsledek platí i pro $x = 0$ (dosazení)

pak $\displaystyle f' = \int \frac{-2x}{1+x^2}\,dx = -\log(1+x^2) + c_1$

$f'(0) = 0 \quad \to \quad c_1 = 0$

$$f = \int -\log(1+x^2)\,dx = -x\log(1+x^2) + \underbrace{\int \frac{2x^2}{1+x^2}\,dx}_{\int \frac{x^2+1-1}{1+x^2} = \int 1 - \frac{1}{1+x^2}}$$

$u' = -1 \qquad v = \log(1+x^2)$

$u = -x \qquad v' = \frac{1}{1+x^2} \cdot 2x$

$$= -x\log(1+x^2) + 2x - 2\arctan x + c_2$$

$f(0) = 0 \quad \to c_2 = 0$

Závěr: $\quad f(x) = -x \log(1+x^2) + 2x - 2\arctan x \quad$ na $(-1,1)$

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2n^2+n}$

$\sum$ konv. z LSK $\left(b_n = \frac{1}{n^2} \right)$

tedy z Abelovy věty

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n^2+n} = \lim_{x \to 1} f(x) = -1 \cdot \log 2 + 2 - 2\arctan 1.$$

$$= -\log 2 + 2 - \frac{\pi}{2}$$