11. cvičení – Cylindrické a sférické souřadnice

https://www2.karlin.mff.cuni.cz/~kuncova/vyuka.php, kuncova@karlin.mff.cuni.cz

Příklady

Zdroje příkladů a řešení:
http://mat.fsv.cvut.cz/Sibrava/Vyuka/vic_int.pdf
https://math.fme.vutbr.cz/download.aspx?id_file=602492416
https://fix.prf.jcu.cz/~eisner/lock/UMB-566-materialy/matematika-sbirka-III-Krivkovy_integral.pdf
https://homel.vsb.cz/~bou10/archiv/ip2.pdf
https://is.muni.cz/el/1433/jaro2009/MB102/7448541/skripta4.pdf
https://math.fel.cvut.cz/en/people/habala/teaching/veci-ma2/ma2r4.pdf
http://www.matematika-lucerna.cz/matalyza/resene-matika3.pdf

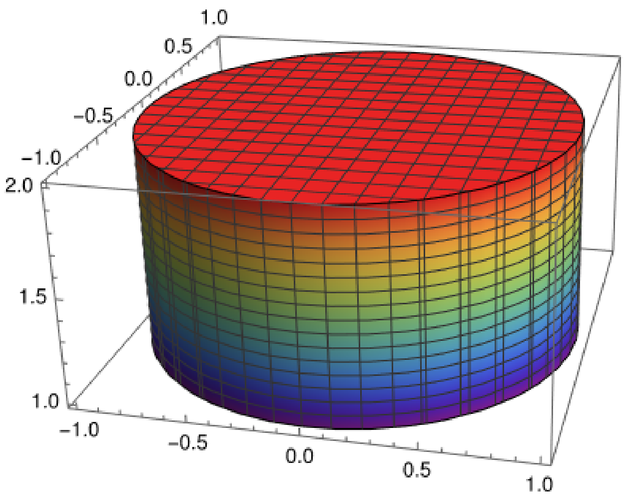
1. Za pomoci substitucí spočtěte integrály

   (a) $\int_M x^2 + y^2 \,\mathrm{d}\lambda$, kde $M := \{[x, y, z] \in \mathbb{R}^3; 1 \le z \le 2; x^2 + y^2 \le 1\}$

   Řešení: Válcové souřadnice.

$$x(r, \alpha, z) := r\,\cos\alpha$$
$$y(r, \alpha, z) := r\,\sin\alpha,$$
$$z(r, \alpha, z) := z$$

   Navíc $z \in (1, 2)$, $r \in (0, 1)$, $\alpha \in (-\pi, \pi)$.



   Fubinka a věta o substituci.

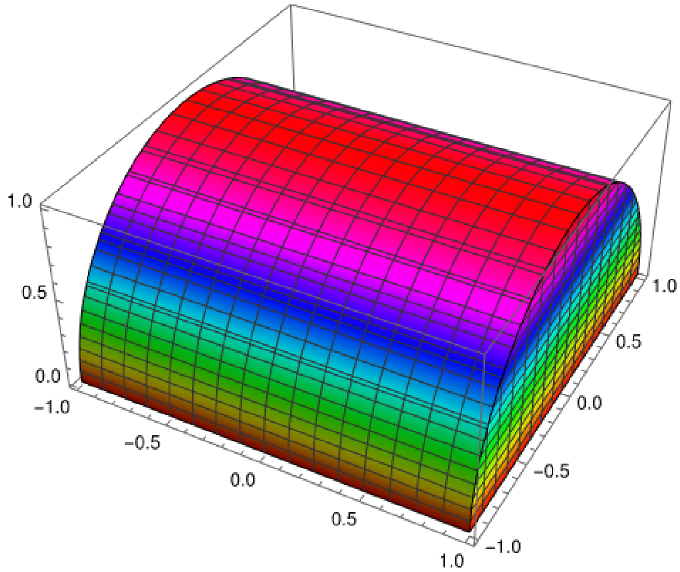
$$\int_0^1 \int_{-\pi}^{\pi} \int_1^2 (r^2 \cos^2\alpha + r^2 \sin^2\alpha) r \,\mathrm{d}z \,\mathrm{d}\alpha \,\mathrm{d}r = \int_0^1 \int_{-\pi}^{\pi} \int_1^2 r^3 \,\mathrm{d}z \,\mathrm{d}\alpha \,\mathrm{d}r$$
$$= \int_0^1 \int_{-\pi}^{\pi} 1r^3 \,\mathrm{d}\alpha \,\mathrm{d}r = \int_0^1 2\pi r^3 \,\mathrm{d}r$$
$$= \left[\frac{2\pi r^4}{4}\right]_0^1 = \frac{\pi}{2}$$

(b) Spočtěte objem množiny M, kde $M := \{[x,y,z] \in \mathbb{R}^3; -1 < x < 1, z > 0, y^2 + z^2 \leq 1\}$

Řešení: Proházené válcové souřadnice.

$$x(r,\alpha,z) := x$$
$$y(r,\alpha,z) := r\,\cos\alpha$$
$$z(r,\alpha,z) := r\,\sin\alpha,$$

Navíc $x \in (-1,1)$, $r \in (0,1)$, $\alpha \in (0,\pi)$.



Fubinka a věta o substituci.

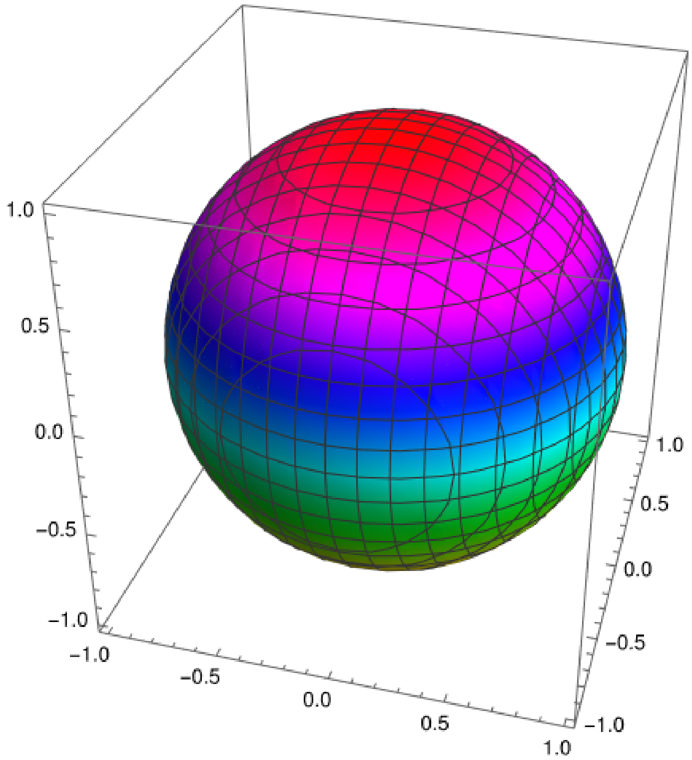
$$\int_0^1 \int_0^\pi \int_{-1}^1 r \,\mathrm{d}z\,\mathrm{d}\alpha\,\mathrm{d}r = \int_0^1 \int_{-\pi}^\pi 2r\,\mathrm{d}\alpha\,\mathrm{d}r = \int_0^1 2\pi r\,\mathrm{d}r$$
$$= \left[\frac{2\pi r^2}{2}\right]_0^1 = \pi$$

(c) $\displaystyle\int_M 1\,\mathrm{d}\lambda$, kde $M := \{[x,y,z] \in \mathbb{R}^3; x^2 + y^2 + z^2 \leq 1\}$

Řešení: Sférické souřadnice.

$$x(r,\beta,\gamma) := r\,\cos\gamma\,\cos\beta,$$
$$y(r,\beta,\gamma) := r\,\cos\gamma\,\sin\beta,$$
$$z(r,\beta,\gamma) := r\,\sin\gamma$$

$$1 > r > 0,$$
$$-\pi < \beta < \pi,$$
$$-\frac{\pi}{2} < \gamma < \frac{\pi}{2},$$



Fubinka a věta o substituci:

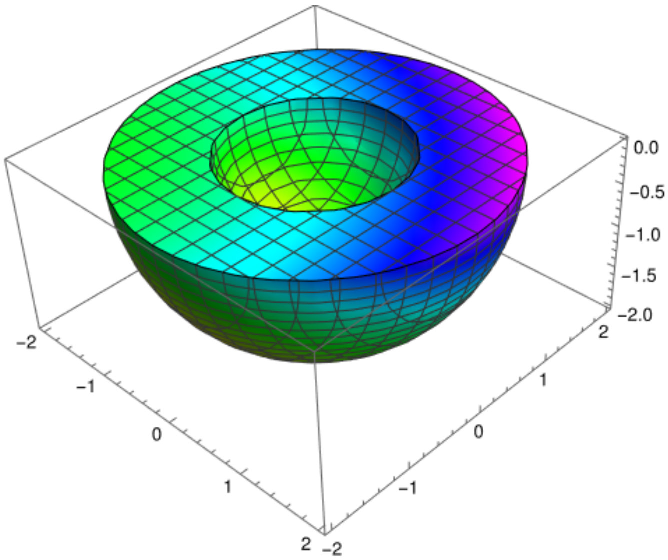
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\pi}^{\pi} \int_0^1 r^2 \cos\gamma \, \mathrm{d}r \, \mathrm{d}\beta \, \mathrm{d}\gamma = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\pi}^{\pi} \cos\gamma \left[\frac{r^3}{3}\right]_0^1 \mathrm{d}\beta \, \mathrm{d}\gamma = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{3} \cdot \cos\gamma \cdot 2\pi \, \mathrm{d}\gamma$$
$$= \frac{2\pi}{3} [\sin\gamma]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{4\pi}{3}$$

(d) $\displaystyle\int_M \frac{1}{(x^2+y^2+z^2)^3} \, \mathrm{d}\lambda$, kde $M = \{[x,y,z] \in \mathbb{R}^3; 1 \leq x^2+y^2+z^2 \leq 4, z \leq 0\}$

Řešení: Sférické souřadnice.

$$x(r,\beta,\gamma) := r \, \cos\gamma \, \cos\beta,$$
$$y(r,\beta,\gamma) := r \, \cos\gamma \, \sin\beta,$$
$$z(r,\beta,\gamma) := r \, \sin\gamma$$

kde $r \in (1,2)$, $\beta \in (-\pi,\pi)$, $\gamma \in (-\frac{\pi}{2}, 0)$.



Fubinka a věta o substituci:

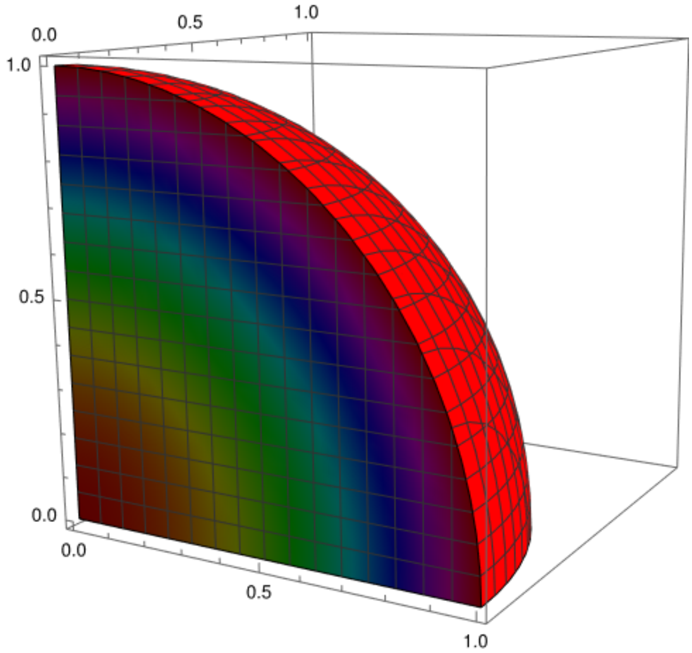
$$\begin{aligned}\int_{-\frac{\pi}{2}}^{0}\int_{-\pi}^{\pi}\int_{1}^{2}\frac{1}{(r^2)^3}r^2\cos\gamma\,\mathrm{d}r\,\mathrm{d}\beta\,\mathrm{d}\gamma &= \int_{-\frac{\pi}{2}}^{0}\int_{-\pi}^{\pi}\int_{1}^{2} r^{-4}\cos\gamma\,\mathrm{d}r\,\mathrm{d}\beta\,\mathrm{d}\gamma\\ &= \int_{-\frac{\pi}{2}}^{0}\int_{-\pi}^{\pi}\cos\gamma\left[\frac{r^{-3}}{-3}\right]_1^2\mathrm{d}\beta\,\mathrm{d}\gamma\\ &= \int_{-\frac{\pi}{2}}^{0}\int_{-\pi}^{\pi}\frac{7}{24}\cos\gamma\,\mathrm{d}\beta\,\mathrm{d}\gamma\\ &= \int_{-\frac{\pi}{2}}^{0}\frac{7\cdot 2\pi}{24}\cos\gamma\,\mathrm{d}\gamma\\ &= \frac{7\pi}{12}[\sin\gamma]_{-\frac{\pi}{2}}^{0} = \frac{7\pi}{12}\end{aligned}$$

(e) $\int_M \sqrt{x^2+y^2+z^2}\,\mathrm{d}\lambda$, kde $M = \{[x,y,z] \in \mathbb{R}^3; x,y,z \geq 0; x^2+y^2+z^2 \leq 1\}$

Řešení: Sférické souřadnice.

$$\begin{aligned}x(r,\beta,\gamma) &:= r\ \cos\gamma\ \cos\beta,\\ y(r,\beta,\gamma) &:= r\ \cos\gamma\ \sin\beta,\\ z(r,\beta,\gamma) &:= r\ \sin\gamma\end{aligned}$$

kde $r \in (0,1)$, $\beta \in (0,\frac{\pi}{2})$, $\gamma \in (0,\frac{\pi}{2})$.



Fubinka a věta o substituci:

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 \sqrt{r^2}r^2\cos\gamma\,\mathrm{d}r\,\mathrm{d}\beta\,\mathrm{d}\gamma = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 r^3\cos\gamma\,\mathrm{d}r\,\mathrm{d}\beta\,\mathrm{d}\gamma$$
$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\cos\gamma\left[\frac{r^4}{4}\right]_0^1\mathrm{d}\beta\,\mathrm{d}\gamma$$
$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{1}{4}\cos\gamma\,\mathrm{d}\beta\,\mathrm{d}\gamma$$
$$= \int_0^{\frac{\pi}{2}}\frac{\pi}{8}\cos\gamma\,\mathrm{d}\gamma$$
$$= \frac{\pi}{8}[\sin\gamma]_0^{\frac{\pi}{2}} = \frac{\pi}{8}$$

(f) Spočtěte objem tělesa, $M = \{[x,y,z]\in\mathbb{R}^3; x^2+4y^2+z^2\leq 4\}$

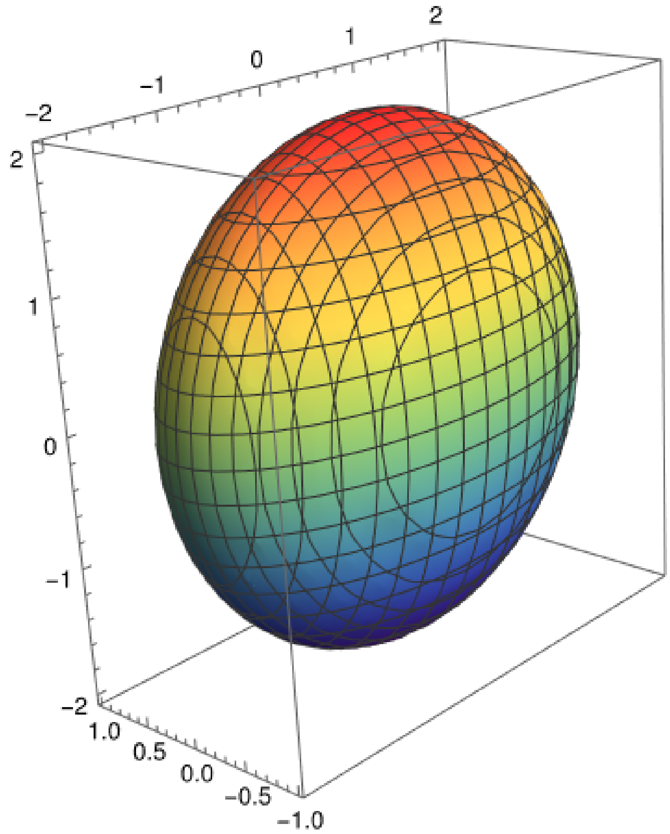
Řešení: Zobecněné sférické souřadnice.

$$x(r,\beta,\gamma) := r\ \cos\gamma\ \cos\beta,$$
$$y(r,\beta,\gamma) := \frac{1}{2}r\ \cos\gamma\ \sin\beta,$$
$$z(r,\beta,\gamma) := r\ \sin\gamma$$

kde $r\in(0,2)$, $\beta\in(-\pi,\pi)$, $\gamma\in(-\frac{\pi}{2},\frac{\pi}{2})$.

Fubinka a věta o substituci (pozor na Jakobián):

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\pi}^{\pi}\int_0^2\frac{1}{2}r^2\cos\gamma\,\mathrm{d}r\,\mathrm{d}\beta\,\mathrm{d}\gamma = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\pi}^{\pi}\cos\gamma\left[\frac{r^3}{6}\right]_0^1\mathrm{d}\beta\,\mathrm{d}\gamma = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\pi}^{\pi}\frac{8}{6}\cos\gamma\,\mathrm{d}\beta\,\mathrm{d}\gamma$$
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{8\pi}{3}\cos\gamma\,\mathrm{d}\gamma = \frac{8\pi}{3}[\sin\gamma]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{16}{3}\pi$$



(g) $\int_M z \, d\lambda$, kde $M = \{[x, y, z] \in \mathbb{R}^3; x^2 + y^2 \le z^2 \le 1, z \ge 0\}$

Řešení: Jde o kužel. Válcové souřadnice.

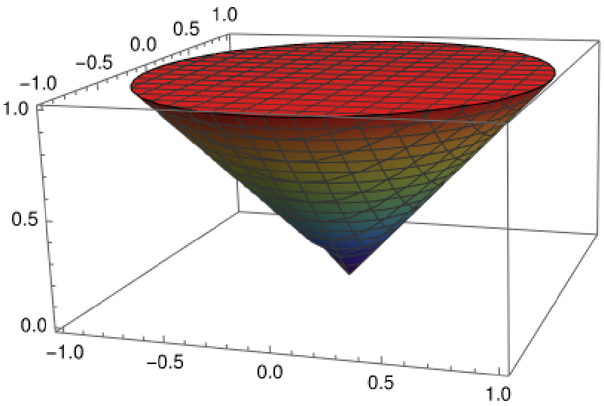
$$x(r, \alpha, z) := r \cos \alpha$$
$$y(r, \alpha, z) := r \sin \alpha,$$
$$z(r, \alpha, z) := z$$

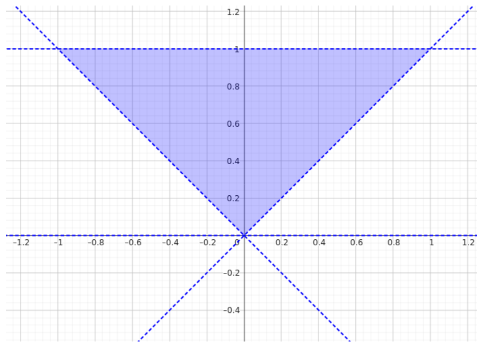
Po dosazení do rovnic dostáváme

$$r^2 \le z^2 \le 1$$

Tedy $z \in (0, 1)$, $r \in (0, z)$, $\alpha \in (-\pi, \pi)$.
Fubinka a věta o substituci.

$$\int_0^1 \int_{-\pi}^{\pi} \int_0^z zr \, dr \, d\alpha \, dz = \int_0^1 \int_{-\pi}^{\pi} z \left[\frac{r^2}{2}\right]_0^z d\alpha \, dz = \int_0^1 \int_{-\pi}^{\pi} \frac{z^3}{2} \, d\alpha \, dz = \int_0^1 \pi z^3 \, dz$$
$$= \pi \left[\frac{z^4}{4}\right]_0^1 = \frac{\pi}{4}$$





(h) $\displaystyle\int_M \sqrt{x^2+y^2}\,\mathrm{d}\lambda$, kde $M = \{[x,y,z] \in \mathbb{R}^3; x^2+y^2 \leq z \leq 1\}$

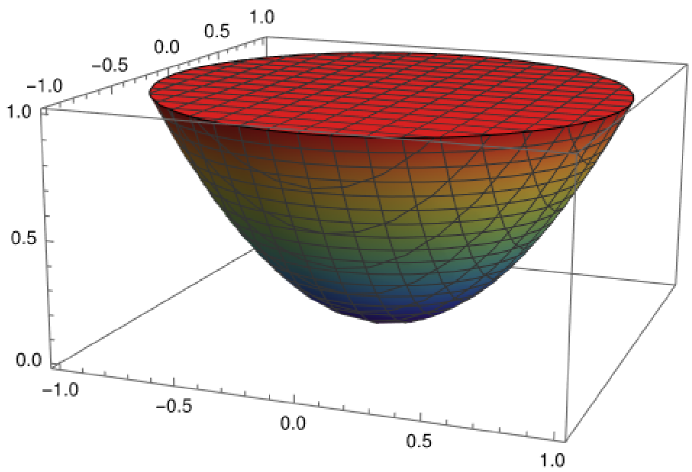
Řešení: Parabolický kužel. Válcové souřadnice.

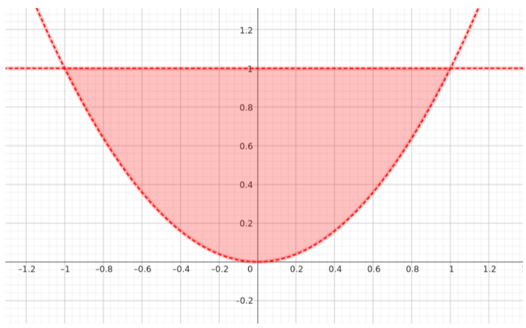
$$\begin{aligned} x(r,\alpha,z) &:= r\,\cos\alpha \\ y(r,\alpha,z) &:= r\,\sin\alpha, \\ z(r,\alpha,z) &:= z \end{aligned}$$

Po dosazení do rovnic dostáváme

$$0 \leq r^2 \leq z \leq 1$$

Tedy $z \in (0,1)$, $r \in (0,\sqrt{z})$, $\alpha \in (-\pi,\pi)$.





Fubinka a věta o substituci.

$$\int_0^1 \int_{-\pi}^{\pi} \int_0^{\sqrt{z}} r \cdot r \,\mathrm{d}r\,\mathrm{d}\alpha\,\mathrm{d}z = \int_0^1 \int_{-\pi}^{\pi} \left[\frac{r^3}{3}\right]_0^{\sqrt{z}} \mathrm{d}\alpha\,\mathrm{d}z = \int_0^1 \int_{-\pi}^{\pi} \frac{z^{\frac{3}{2}}}{3}\,\mathrm{d}\alpha\,\mathrm{d}z$$
$$= \int_0^1 2\pi \frac{z^{\frac{3}{2}}}{3}\,\mathrm{d}z = \frac{2\pi}{3}\left[\frac{2z^{\frac{5}{2}}}{5}\right]_0^1 = \frac{4\pi}{15}$$

(i) $\int_M z \mathrm{d}A$, kde $M = \{[x,y,z] \in \mathbb{R}^3; 0 \le z \le 4 - 2\sqrt{x^2+y^2}\}$

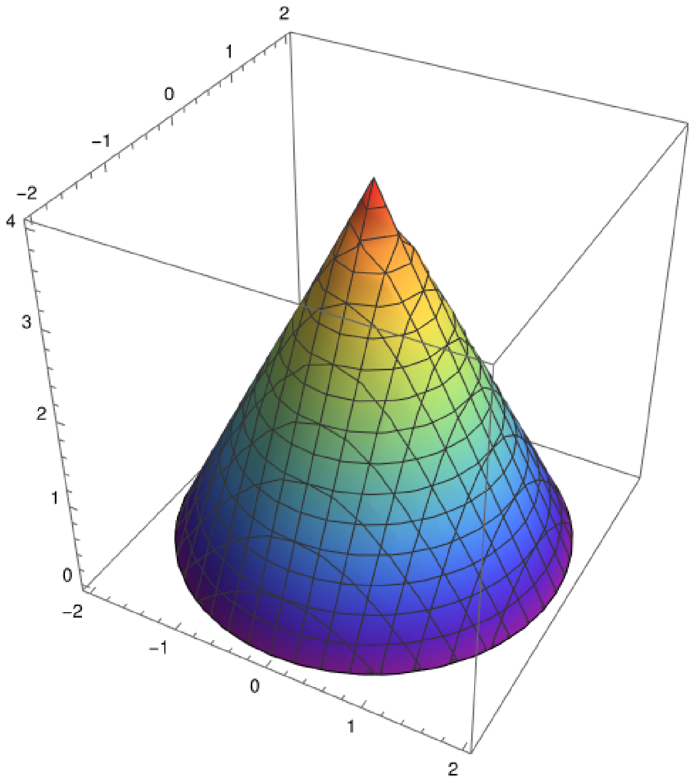
Řešení: Otočený kužel. Válcové souřadnice.

$$x(r,\alpha,z) := r\,\cos\alpha$$
$$y(r,\alpha,z) := r\,\sin\alpha,$$
$$z(r,\alpha,z) := z$$

Po dosazení do rovnic dostáváme

$$0 \le z \le 4 - 2\sqrt{r^2} = 4 - 2r$$

Tedy $z \in (0, 4-2r)$, $r \in (0,2)$, $\alpha \in (-\pi,\pi)$.



Fubinka a věta o substituci.

$$\begin{aligned}\int_{-\pi}^{\pi}\int_0^2\int_0^{4-2r} r\cdot z \,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\alpha &= \int_{-\pi}^{\pi}\int_0^2 r\left[\frac{z^2}{2}\right]_0^{4-2r}\mathrm{d}r\,\mathrm{d}\alpha\\ &= \int_{-\pi}^{\pi}\int_0^2 \frac{1}{2}r(4-2r)^2\,\mathrm{d}r\,\mathrm{d}\alpha\\ &= \int_{-\pi}^{\pi}\int_0^2 2r^3 - 8r^2 + 8r\,\mathrm{d}r\,\mathrm{d}\alpha\\ &= \int_{-\pi}^{\pi}\left[\frac{r^4}{2} - \frac{8r^3}{3} + 4r^2\right]_0^2\mathrm{d}\alpha\\ &= \int_{-\pi}^{\pi}\frac{8}{3}\,\mathrm{d}\alpha = \frac{16\pi}{3}\end{aligned}$$

(j) $\displaystyle\int_M (x^2+y^2)z \,\mathrm{d}\lambda$, kde $M := \{[x,y,z] \in \mathbb{R}^3; 1 \le x^2+y^2+z^2 \le 4, x^2+y^2 \le z^2, z \ge 0\}$

Řešení: Kužel v kouli. Sférické souřadnice.

$$\begin{aligned} x(r,\beta,\gamma) &:= r\,\cos\gamma\,\cos\beta,\\ y(r,\beta,\gamma) &:= r\,\cos\gamma\,\sin\beta,\\ z(r,\beta,\gamma) &:= r\,\sin\gamma \end{aligned}$$

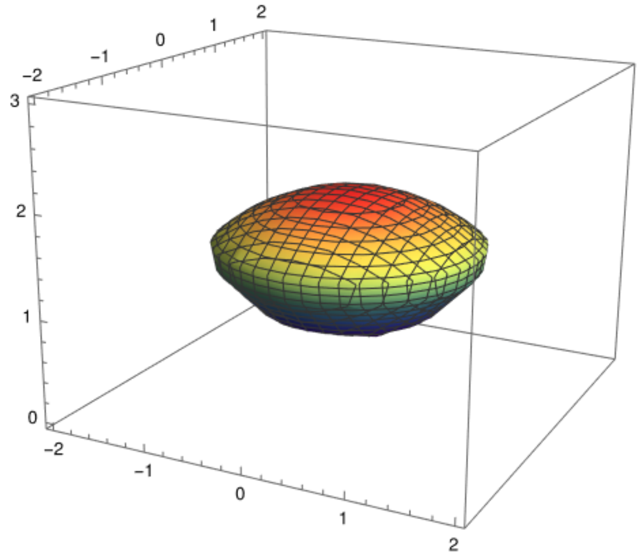
Po dosazení do rovnic dostáváme

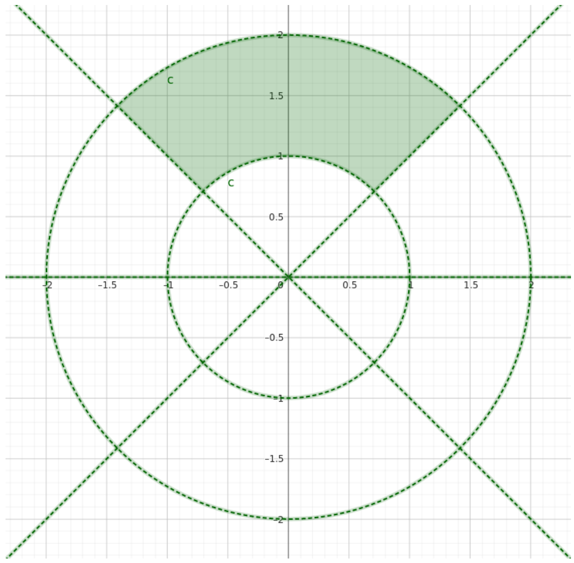
$$\begin{aligned} 1 &\le r^2 \le 4,\\ r^2\cos^2\gamma &\le r^2\sin^2\gamma,\\ 0 &\le r\sin\gamma. \end{aligned}$$

Neboli

$$\begin{aligned} 1 &\le r \le 2,\\ \tan^2\gamma &\le 1,\\ 0 &\le \sin\gamma. \end{aligned}$$

Tedy $r \in (1,2)$, $\beta \in (-\pi,\pi)$ a $\gamma \in (0,\frac{\pi}{4})$.





Fubinka a věta o substituci.

$$\begin{aligned}\int_{-\pi}^{\pi}\int_1^2\int_0^{\frac{\pi}{4}}(r^2\cos^2\gamma)r\sin\gamma\; r^2\cos\gamma\,\mathrm{d}\gamma\,\mathrm{d}r\,\mathrm{d}\beta &= \int_{-\pi}^{\pi}\int_1^2\int_0^{\frac{\pi}{4}} r^5\cos^3\gamma\sin\gamma\,\mathrm{d}\gamma\,\mathrm{d}r\,\mathrm{d}\beta\\
&= \int_{-\pi}^{\pi}\int_1^2 r^5\left[-\frac{1}{4}\cos^4\gamma\right]_0^{\frac{\pi}{4}}\,\mathrm{d}r\,\mathrm{d}\beta\\
&= \int_{-\pi}^{\pi}\int_1^2 r^5\frac{1}{4}\left(-\frac{4}{16}+1\right)\,\mathrm{d}r\,\mathrm{d}\beta\\
&= \frac{3}{16}\int_{-\pi}^{\pi}\int_1^2 r^5\,\mathrm{d}r\,\mathrm{d}\beta\\
&= \frac{3}{6\cdot 16}\int_{-\pi}^{\pi}\left[r^6\right]_1^2\,\mathrm{d}\beta\\
&= \frac{1}{32}\int_{-\pi}^{\pi} 63\,\mathrm{d}\beta\\
&= \frac{63}{16}\pi.\end{aligned}$$

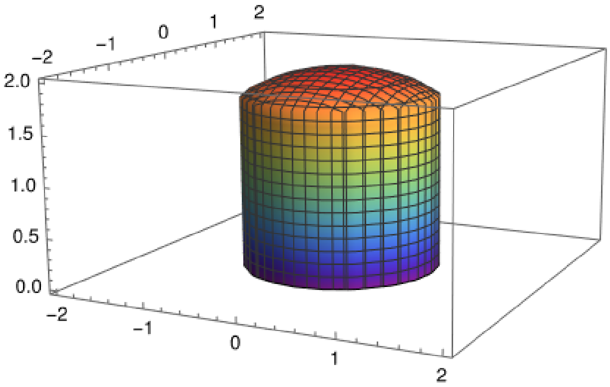
(k) $\int_M (x^4+y^4)z\,\mathrm{d}\lambda$,
kde $M := \{[x,y,z]\in\mathbb{R}^3; x^2+y^2\le 1, z\ge 0, x^2+y^2+z^2\le 4\}$
Řešení: Válec v kouli. Válcové souřadnice.

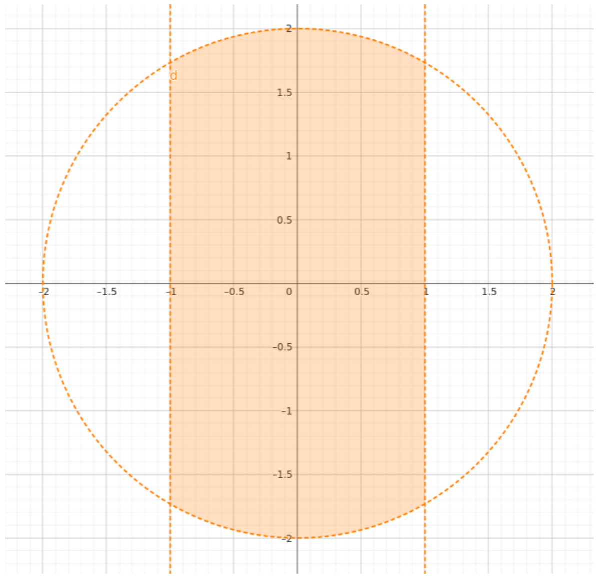
$$\begin{aligned}x(r,\alpha,z) &:= r\,\cos\alpha\\ y(r,\alpha,z) &:= r\,\sin\alpha,\\ z(r,\alpha,z) &:= z\end{aligned}$$

Po dosazení do rovnic dostáváme

$$\begin{aligned}r^2 &\le 1,\\ r^2+z^2 &\le 4,\end{aligned}$$

Tedy $z\in(0,\sqrt{4-r^2})$, $r\in(0,1)$, $\alpha\in(-\pi,\pi)$.





Fubinka a věta o substituci.

$$
\begin{aligned}
&\int_{-\pi}^{\pi}\int_0^1\int_0^{\sqrt{4-r^2}} (r^4\cos^4\alpha + r^4\sin^4\alpha) zr \,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\alpha \\
&\quad = \int_{-\pi}^{\pi}\int_0^1 (r^4\cos^4\alpha + r^4\sin^4\alpha) r \left[\frac{z^2}{2}\right]_0^{\sqrt{4-r^2}} \mathrm{d}r\,\mathrm{d}\alpha \\
&\quad = \frac{1}{2}\int_{-\pi}^{\pi}\int_0^1 (r^4\cos^4\alpha + r^4\sin^4\alpha) r(4-r^2)\,\mathrm{d}r\,\mathrm{d}\alpha \\
&\quad = \frac{1}{2}\int_{-\pi}^{\pi}\int_0^1 (\cos^4\alpha + \sin^4\alpha)(4r^5-r^7)\,\mathrm{d}r\,\mathrm{d}\alpha \\
&\quad = \frac{1}{2}\int_{-\pi}^{\pi} (\cos^4\alpha + \sin^4\alpha)\left[\frac{2r^6}{3}-\frac{r^8}{8}\right]_1^2 \mathrm{d}\alpha \\
&\quad = \frac{13}{48}\int_{-\pi}^{\pi} (\cos^4\alpha + \sin^4\alpha)\,\mathrm{d}\alpha \\
&\quad = \frac{13}{48}\int_{-\pi}^{\pi} \frac{1}{4}(1+\cos 2\alpha)^2 + \frac{1}{4}(1-\cos 2\alpha)^2\,\mathrm{d}\alpha \\
&\quad = \frac{13}{4\cdot 48}\int_{-\pi}^{\pi} 2(\cos^2\alpha + 1)\,\mathrm{d}\alpha \\
&\quad = \frac{13}{4\cdot 48}\int_{-\pi}^{\pi} (\cos 4\alpha + 3)\,\mathrm{d}\alpha \\
&\quad = \frac{13}{4\cdot 48}\left[\frac{\sin 4\alpha}{4} + 3\alpha\right]_{-\pi}^{\pi} \\
&\quad = \frac{13}{4\cdot 48}\cdot 6\pi \\
&\quad = \frac{13}{32}\pi
\end{aligned}
$$

(l) $\displaystyle\int_M z \,\mathrm{d}\lambda,$

kde $M := \{[x,y,z] \in \mathbb{R}^3; \frac{x^2}{4} + \frac{y^2}{9} + z^2 \leq 2z\}$

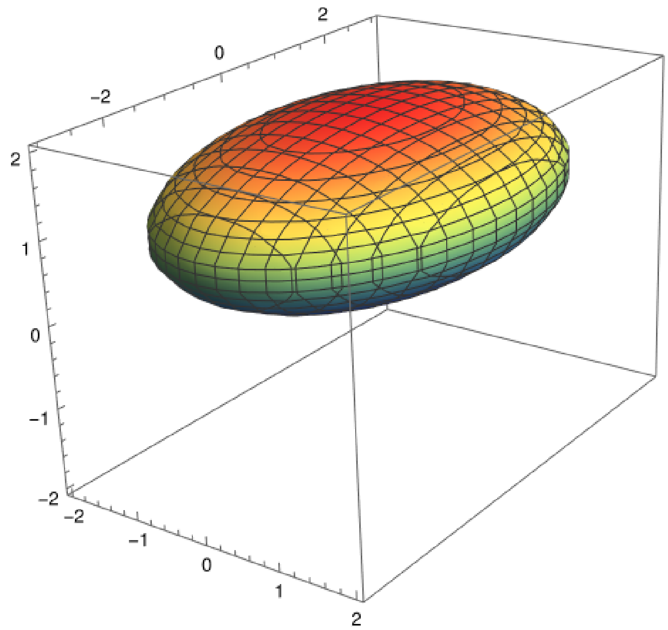
Řešení: Posunutý elipsoid. Nerovnici lze vyjádřit jako

$$\frac{x^2}{4} + \frac{y^2}{9} + (z-z)^2 \leq 1$$

Zobecněné sférické souřadnice.

$$\begin{aligned} x(r,\beta,\gamma) &:= 2r\,\cos\gamma\,\cos\beta, \\ y(r,\beta,\gamma) &:= 3r\,\cos\gamma\,\sin\beta, \\ z(r,\beta,\gamma) &:= 1 + r\,\sin\gamma \end{aligned}$$

kde $r \in (0,1)$, $\beta \in (-\pi,\pi)$, $\gamma \in (-\frac{\pi}{2}, \frac{\pi}{2})$.



Fubinka a věta o substituci (pozor na Jakobián):

$$\begin{aligned} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\pi}^{\pi} \int_0^1 (1 + r\sin\gamma) 6r^2 \cos\gamma \,\mathrm{d}r\,\mathrm{d}\beta\,\mathrm{d}\gamma &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\pi}^{\pi} \int_0^1 6r^2\cos\gamma + 6r^3 \cos\gamma\sin\gamma \,\mathrm{d}r\,\mathrm{d}\beta\,\mathrm{d}\gamma \\ &= 6\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\pi}^{\pi} \left[\cos\gamma\frac{r^3}{3} + \cos\gamma\sin\gamma\frac{r^4}{4}\right]_0^1 \mathrm{d}\beta\,\mathrm{d}\gamma \\ &= 6\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\pi}^{\pi} \frac{1}{3}\cos\gamma + \frac{1}{4}\cos\gamma\sin\gamma \,\mathrm{d}\beta\,\mathrm{d}\gamma \\ &= 12\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{3}\cos\gamma + \frac{1}{4}\cos\gamma\sin\gamma \,\mathrm{d}\gamma \\ &= 12\pi \left[\frac{1}{3}\sin\gamma + \frac{1}{8}\sin^2\gamma\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}\gamma \\ &= 8\pi \end{aligned}$$

(m) Spočtěte objem tělesa (anuloid - torus) určeného $M := \{[x,y,z] \in \mathbb{R}^3; (\sqrt{x^2+y^2} - a)^2 + z^2 \leq b^2\}$, $0 < b < a$.

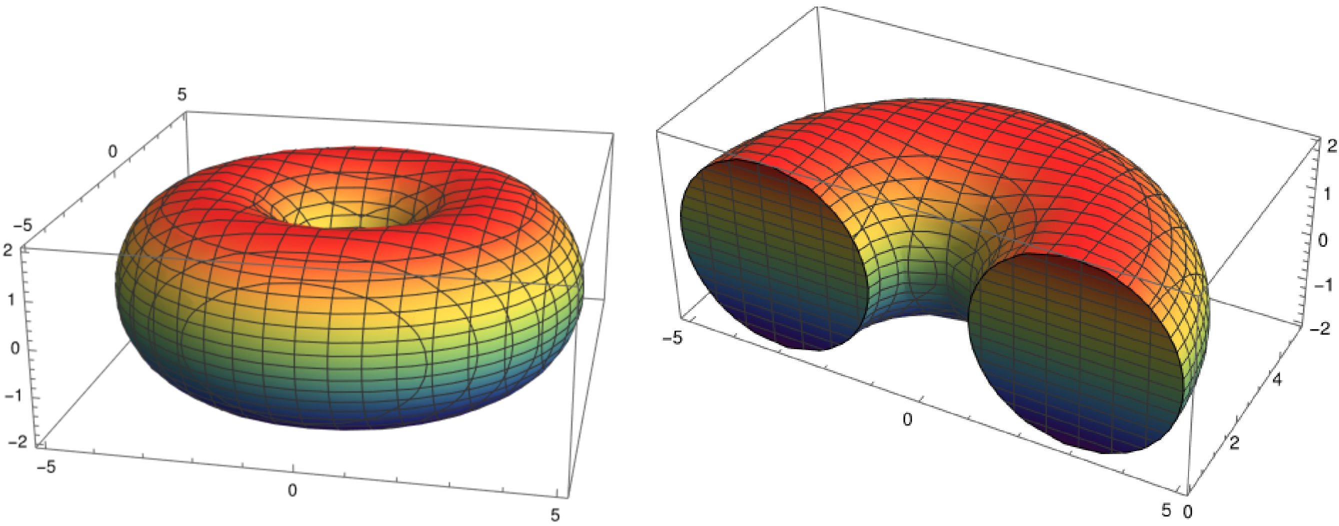
Řešení: Donut. Válcové souřadnice.

$$\begin{aligned} x(r,\alpha,z) &:= r\,\cos\alpha \\ y(r,\alpha,z) &:= r\,\sin\alpha, \\ z(r,\alpha,z) &:= z \end{aligned}$$

Po dosazení do nerovnic dostáváme

$$(\sqrt{r^2} - a)^2 + z^2 \leq b^2$$

Tedy $z \in (-\sqrt{b^2-(r-a)^2}, \sqrt{b^2-(r-a)^2})$, $r \in (a-b, a+b)$, $\alpha \in (-\pi, \pi)$.



Fubinka a věta o substituci.

$$\begin{aligned} \int_{-\pi}^{\pi}\int_{a-b}^{a+b}\int_{-\sqrt{b^2-(r-a)^2}}^{\sqrt{b^2-(r-a)^2}} r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\alpha &= \int_{-\pi}^{\pi}\int_{a-b}^{a+b} r\,[z]_{-\sqrt{b^2-(r-a)^2}}^{\sqrt{b^2-(r-a)^2}}\,\mathrm{d}r\,\mathrm{d}\alpha \\ &= \int_{-\pi}^{\pi}\int_{a-b}^{a+b} 2r\sqrt{b^2-(r-a)^2}\,\mathrm{d}r\,\mathrm{d}\alpha \end{aligned}$$

Provedeme substituci $s = r - a,\ \mathrm{d}s = \mathrm{d}r$,

$$\begin{aligned} \int_{-\pi}^{\pi}\int_{a-b}^{a+b} 2r\sqrt{b^2-(r-a)^2}\,\mathrm{d}r\,\mathrm{d}\alpha &= 2\int_{-\pi}^{\pi}\int_{-b}^{b}(s+a)\sqrt{b^2-s^2}\,\mathrm{d}s\,\mathrm{d}\alpha \\ &= 2\int_{-\pi}^{\pi}\int_{-b}^{b} s\sqrt{b^2-s^2} + a\sqrt{b^2-s^2}\,\mathrm{d}s\,\mathrm{d}\alpha \end{aligned}$$

První část integrálu je lichá v s, tedy integrál je nulový. Na druhou (sudou) část aplikujeme substituci $s = b\sin t$.

$$\begin{aligned}
2\int_{-\pi}^{\pi}\int_{-b}^{b} a\sqrt{b^2-s^2}\,\mathrm{d}s\,\mathrm{d}\alpha &= 4\int_{-\pi}^{\pi}\int_{0}^{b} a\sqrt{b^2-s^2}\,\mathrm{d}s\,\mathrm{d}\alpha \\
&= 4a\int_{-\pi}^{\pi}\int_{0}^{\frac{\pi}{2}} \sqrt{b^2-b^2\sin^2 t}\; b\cos t\,\mathrm{d}t\,\mathrm{d}\alpha \\
&= 4ab^2\int_{-\pi}^{\pi}\int_{0}^{\frac{\pi}{2}} \cos^2 t\,\mathrm{d}t\,\mathrm{d}\alpha \\
&= 4ab^2\int_{-\pi}^{\pi}\int_{0}^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 2t)\,\mathrm{d}t\,\mathrm{d}\alpha \\
&= 2ab^2\int_{-\pi}^{\pi} \left[t+\frac{1}{2}\sin 2t\right]_0^{\frac{\pi}{2}}\,\mathrm{d}\alpha \\
&= 2ab^2\pi^2
\end{aligned}$$

(n) Spočtěte objem tělesa určeného vztahy $M := \{[x,y,z] \in \mathbb{R}^3; x^2+y^2+z^2 \leq 16; x^2+y^2 \leq 4y\}$

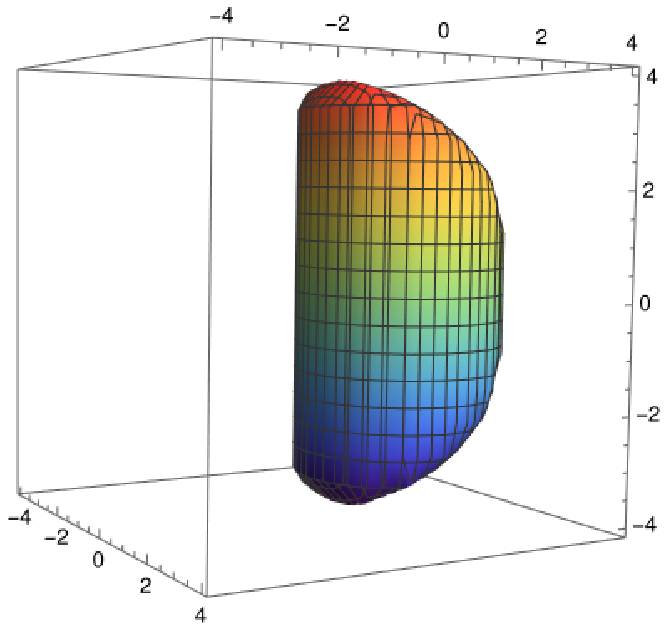
Řešení: Vivianiho okénko. Válcové souřadnice.

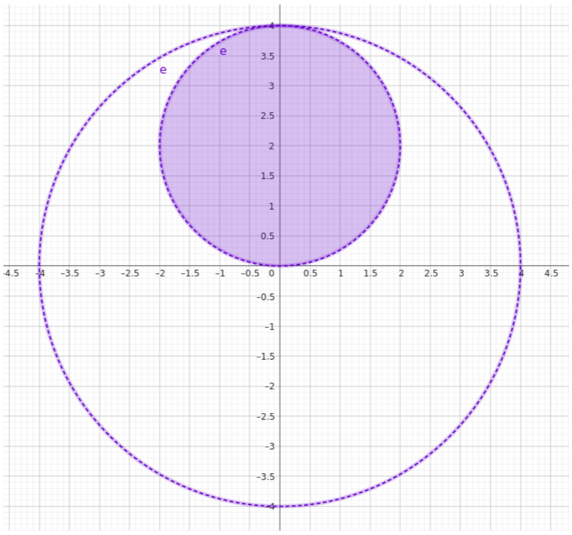
$$\begin{aligned}
x(r,\alpha,z) &:= r\,\cos\alpha \\
y(r,\alpha,z) &:= r\,\sin\alpha, \\
z(r,\alpha,z) &:= z
\end{aligned}$$

Po dosazení do rovnic dostáváme

$$\begin{aligned}
r^2+z^2 &\leq 16, \\
r^2 &\leq 4r\sin\alpha \\
0 \leq r &\leq 4\sin\alpha
\end{aligned}$$

Tedy $z \in (-\sqrt{16-r^2}, \sqrt{16-r^2})$, $r \in (0, 4\sin\alpha)$, $\alpha \in (0,\pi)$.





Fubinka a věta o substituci.

$$\begin{aligned}\int_0^\pi \int_0^{4\sin\alpha} \int_{-\sqrt{16-r^2}}^{\sqrt{16-r^2}} r \,\mathrm{d}z \,\mathrm{d}r \,\mathrm{d}\alpha &= \int_0^\pi \int_0^{4\sin\alpha} 2r\sqrt{16-r^2} \,\mathrm{d}r \,\mathrm{d}\alpha \\
&= \int_0^\pi \left[-\frac{2}{3}(16-r^2)^{\frac{3}{2}}\right]_0^{4\sin\alpha} \mathrm{d}\alpha \\
&= -\frac{2}{3}\int_0^\pi (\sqrt{16-16\sin^2\alpha})^3 - 16^{\frac{3}{2}} \,\mathrm{d}\alpha \\
&= -\frac{2}{3}\int_0^\pi (4\sqrt{\cos^2\alpha})^3 - 64 \,\mathrm{d}\alpha \\
&= -\frac{2}{3}\cdot 64 \int_0^\pi |\cos^3\alpha| - 1 \,\mathrm{d}\alpha \\
&= -\frac{2}{3}\cdot 64 \cdot 2 \int_0^{\frac{\pi}{2}} \cos^3\alpha - 1 \,\mathrm{d}\alpha \\
&= -\frac{256}{3}\int_0^{\frac{\pi}{2}} (1-\sin^2\alpha)\cos\alpha - 1 \,\mathrm{d}\alpha \\
&= -\frac{256}{3}\left[\sin\alpha - \frac{1}{3}\sin^3\alpha - \alpha\right]_0^{\frac{\pi}{2}} \\
&= -\frac{256}{3}\left(\frac{2}{3} - \frac{\pi}{2}\right)\end{aligned}$$

2. Přiřaďte rovnici obrázku.

(a) 2 $\{[x,y,z] \in \mathbb{R}^3; 1 \le z \le 2; x^2 + y^2 \le 1\}$

(b) 12 $\{[x,y,z] \in \mathbb{R}^3; x^2 + y^2 \le z^2; 1 \le x^2 + y^2 + z^2 \le 4; z \ge 0\}$

(c) 4 $\{[x,y,z] \in \mathbb{R}^3; x^2 + y^2 \le z \le 1\}$

(d) 6 Ohraničeno plochami $z = 0$, $z = 3$, $x^2 + y^2 - 2x = 0$ a navíc $y \ge 0$

(e) 7 $\{[x,y,z] \in \mathbb{R}^3; x^2 + y^2 + z^2 \le 1; z \ge 0\}$

(f) 5 $\{[x,y,z] \in \mathbb{R}^3; 0 \le z \le 4 - 2\sqrt{x^2 + y^2}\}$

(g) 9 $\{[x,y,z] \in \mathbb{R}^3; 1 \le x^2 + y^2 + z^2 \le 4, z \le 0\}$

(h) 10 $\{[x,y,z] \in \mathbb{R}^3; x^2 + 4y^2 + z^2 \le 4\}$

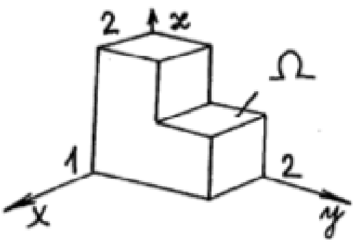
(i) 11 $\{[x,y,z] \in \mathbb{R}^3; x^2 + y^2 + z^2 \le z\}$

(j) 8 $\{[x,y,z] \in \mathbb{R}^3; \sqrt{x^2 + y^2} \le z \le 6 - (x^2 + y^2)\}$

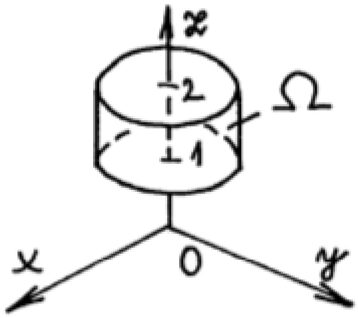
(k) 1 $M = M_1 \cup M_2$, kde $M_1 = [0,1] \times [0,1] \times [0,2]$ a $M_2 = [0,1] \times [1,2] \times [0,1]$

(l) 3 $\{[x,y,z] \in \mathbb{R}^3; -1 \le x \le 1; z \ge 0; y^2 + z^2 \le 1\}$

Zdroj: https://math.fme.vutbr.cz/download.aspx?id_file=602492416



1k



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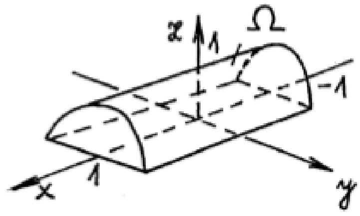
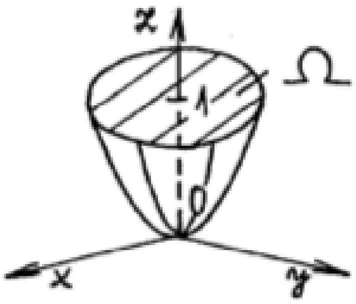


3l



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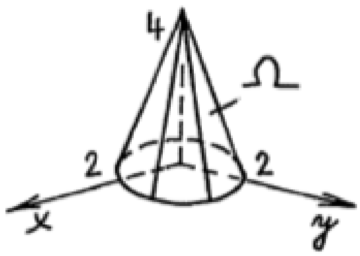

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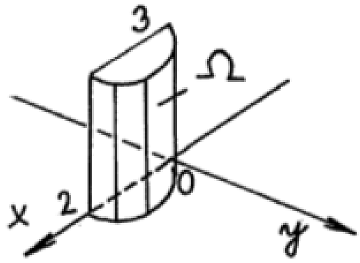

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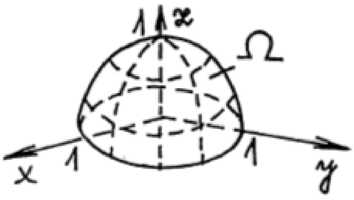

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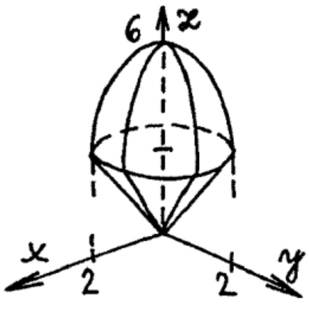

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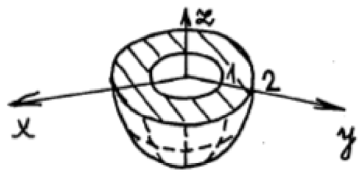
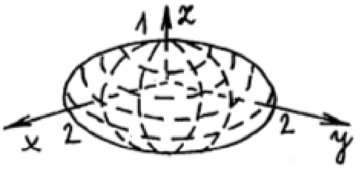
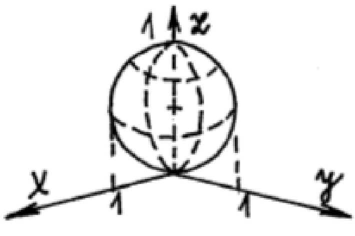


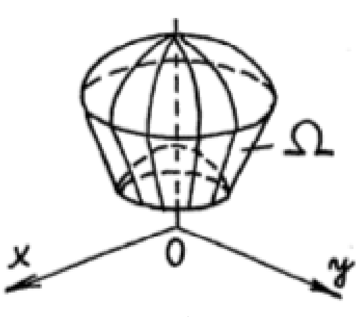
9g



10h



11i



12b