① $\lim\limits_{n\to\infty} n\left(\left(1+\frac{1}{n}\right)^n - e\right) \overset{\text{Heine}}{=} -\frac{e}{2}$

Heine $x_n = n \quad n\to\infty \quad n\neq\infty \quad \forall n\in\mathbb{N}$

$$\lim_{x\to\infty} x\left(e^{x\log\left(1+\frac{1}{x}\right)} - e\right) = \lim_{x\to\infty} \frac{e^{x\log(1+1/x)} - e}{\frac{1}{x}} =$$

$$\overset{LH}{\underset{0/0}{=}} \lim_{x\to\infty} \frac{\left(1+\frac{1}{x}\right)^x \cdot \left(\log\left(1+\frac{1}{x}\right) + x\,\frac{1}{1+1/x}\cdot\frac{-1}{x^2}\right) - 0}{-\frac{1}{x^2}} =$$

$$= \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x \cdot \frac{\log\left(1+\frac{1}{x}\right) + x\cdot\frac{1}{\frac{1+x}{x}}\cdot\frac{-1}{x^2}}{-\frac{1}{x^2}} =$$

$$= \lim_{x\to\infty} \underbrace{\left(1+\frac{1}{x}\right)^x}_{\downarrow\ e} \cdot \underbrace{\frac{\log\left(1+\frac{1}{x}\right) + \frac{-1}{1+x}}{-\frac{1}{x^2}}}_{\downarrow} \overset{AL}{=} e\cdot -\frac{1}{2}$$

$$\overset{LH}{\underset{0/0}{=}} \lim_{x\to\infty} \frac{\frac{1}{1+1/x}\cdot\frac{-1}{x^2} + \frac{1}{(1+x)^2}}{2\cdot\frac{1}{x^3}} = \lim_{x\to\infty} \frac{1}{2}x^3\left(\frac{-1/x}{x+1} + \frac{1}{(1+x)^2}\right)$$

$$= \lim_{x\to\infty} \frac{1}{2}\,\frac{x^3(-1-x+x)}{x(1+x)^2} = \lim_{x\to\infty} \frac{1}{2}\,\frac{-\cancel{x^3}}{\cancel{x^3}(1/x+1)^2} \overset{AL}{=} -\frac{1}{2(1+0)(1+0)}$$

(2)

$$f(x) = \begin{cases} \sin x \cdot \arctan \frac{1}{\sin x} & x \neq k\pi \\ 0 & x = k\pi \end{cases}$$

- $D_f = \mathbb{R}$

- $x \neq k\pi$:

$$f'(x) = \cos x \cdot \arctan \frac{1}{\sin x} + \sin x \, \frac{1}{1+\left(\frac{1}{\sin x}\right)^2} \cdot \frac{-1}{\sin^2 x} \cdot \cos x$$

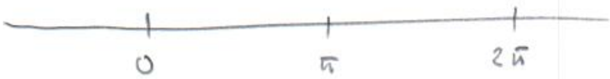
$$= \cos x \left(\arctan \frac{1}{\sin x} + \frac{-1}{\sin x} \cdot \frac{1}{\sin^2 x + 1} \cdot \sin^2 x \right)$$

$$= \cos x \left(\arctan \frac{1}{\sin x} - \frac{\sin x}{1+\sin^2 x} \right)$$

- f je spoj. v $k\pi$:

$$\lim_{x \to k\pi} \underbrace{\sin x}_{\downarrow \, 0} \cdot \underbrace{\arctan \frac{1}{\sin x}}_{\downarrow \, \text{omezená } |\arctan y| \leq \pi/2} = 0$$

- body $k\pi$:



f je 2π-period, stačí tedy vyšetřit jen body $0, \pi$.
(dokonce π-period)

Protože f je spojitá, tak lze užít větu o lim derivaci:

$$f'_+(0) = \lim_{x \to 0+} \cos x \left(\arctan \frac{1}{\sin x} - \frac{\sin x}{1+\sin^2 x} \right) \overset{AL}{=} 1 \cdot \left(\frac{\pi}{2} - \frac{0}{1+0} \right) = \frac{\pi}{2}$$

VOLSF: $f(y) = \arctan y$ $\quad \lim_{y \to \infty} \arctan y = \pi/2$ $\quad$ (P) $\frac{1}{\sin x} \neq \infty$ na $P^+(0, \frac{\pi}{4})$

$g(x) = \frac{1}{\sin x}$ $\quad \lim_{x \to 0+} \frac{1}{\sin x} = \infty$

$\downarrow$

"$\frac{1}{0+}$" $\quad \sin x > 0$ na $(0, \frac{\pi}{4})$

dle věty o kovo. lim podílu

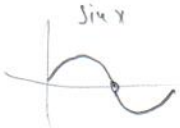
analogicky

$$f'_-(0) = \lim_{x \to 0-} \cos x \left(\arctan \frac{1}{\sin x} - \frac{\sin x}{1+\sin^2 x}\right) \overset{AL}{=} 1\left(-\frac{\pi}{2} - 0\right) = -\pi/2$$

$\downarrow$

$\frac{1}{0-}$ $\sin x < 0$ na $(-\pi/4, 0)$

$$f'_-(\pi) = \lim_{x \to \pi-} \cos x \left(\arctan \frac{1}{\sin x} - \frac{\sin x}{1+\sin^2 x}\right) = -1 \cdot \left(\frac{\pi}{2} - 0\right) = -\frac{\pi}{2}$$



$\downarrow$

$\frac{1}{0+}$ $\sin x > 0$ na $(\frac{3}{4}\pi, \pi)$

$$f'_+(\pi) = \lim_{x \to \pi+} \cos x \left(\arctan \frac{1}{\sin x} - \frac{\sin x}{1+\sin^2 x}\right) = -1\left(-\pi/2 - 0\right) = \pi/2$$

$\downarrow$

$\frac{1}{0-}$ $\sin x < 0$ na $(\pi, 5/4\,\pi)$

Závěr:

$$f'(x) = \cos x \left(\arctan \frac{1}{\sin x} - \frac{\sin x}{1+\sin^2 x}\right) \quad x \in (0+k\pi, \pi+k\pi)$$

$$\left.\begin{array}{l} f'_+(0+k\pi) = \frac{\pi}{2} \\ f'_-(0+k\pi) = -\frac{\pi}{2} \end{array}\right\} f'(k\pi) \nexists$$

(3) $$\lim_{x\to 0} \frac{2\sqrt{1+x} - 3\sqrt[3]{1+x} + \cos x}{x^2}$$

$$\overset{LH}{\underset{0/0}{=}} \lim_{x\to 0} \frac{\frac{1}{\sqrt{1+x}} - \frac{1}{\sqrt[3]{(1+x)^2}} - \sin x}{2x} \overset{LH}{\underset{0/0}{=}} \lim_{x\to 0} \frac{1}{2}\left(-\frac{1}{2}\frac{1}{\sqrt{(1+x)^3}} + \frac{2}{3}\frac{1}{\sqrt[3]{(1+x)^5}} - \cos x\right)$$

$$\overset{AL}{=} \frac{1}{2}\left(-\frac{1}{2} + \frac{2}{3} - 1\right) = -\underline{\underline{\frac{5}{12}}}$$

VOLSF 1: $f_1(y) = \sqrt{y}$ $\quad \lim_{y\to 1} \sqrt{y} = 1$ $\quad$ (S) $\sqrt{y}$ spoj. v 1

$g_1(x) = (1+x)^3$ $\quad \lim_{x\to 0} (1+x)^3 = 1$

spojitost

VOLSF 2: $f_2(y) = \sqrt[3]{y}$ $\quad \lim_{y\to 1} \sqrt[3]{y} = 1$ $\quad$ (S) $\sqrt[3]{y}$ spoj. v 1

$g_2(x) = (1+x)^5$ $\quad \lim_{x\to 0} (1+x)^5 = 1$

spojitost

(4)

$$f(x) = \begin{cases} \exp \dfrac{-1}{\sin^2 x} & x \neq k\pi \\ 0 & x = k\pi \end{cases}$$

- f je π-period, protože $\sin^2 x$ je π-period

(a) D_f: $\sin x = 0 \iff x = k\pi$

$\to D_f = \mathbb{R}$

(b) na $(0+k\pi, \pi+k\pi)$ jde o složení a podíl spoj. funkcí → f spojitá

v $0+k\pi$: $f(0+k\pi) = 0$

$$\lim_{x\to 0+k\pi} e^{-1/\sin^2 x} = 0$$ } f spojitá v $0+k\pi$

→ f spoj. na $\mathbb{R}$

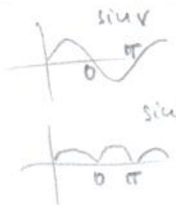
↓

$f(y) = e^{-y}$ $\quad \lim_{y\to\infty} e^{-y} = 0$ $\quad$ (P) $g(x) \neq \infty$ na $P(0+k\pi, \pi/4)$

$g(x) = \dfrac{1}{\sin^2 x}$ $\quad \lim_{x\to 0+k\pi} \dfrac{1}{\sin^2 x} = \infty$

↓ $\dfrac{1}{0+}$ $\quad \sin^2 x > 0$ na $P(0+k\pi, \pi/4)$

V o nevl. lim. podílu



(c) $\lim_{x\to\pm\infty} f(x) \nexists$

Heine: uvažujme $a_n = n\cdot\pi$, pak $f(n\pi) = 0$ $\quad \lim_{n\to\infty} f(n\pi) = 0$

$b_n = \frac{\pi}{2} + 2n\pi$ $\quad f(\frac{\pi}{2}+2n\pi) = 1$ $\quad \lim_{n\to\infty} f(2n\pi + \pi/2) = 1$

Spor s ∃ limity

pro $-\infty$ analogicky, $a_n = -n\pi$, $b_n = \frac{\pi}{2} - 2n\pi$.

(d) $x \in (0+k\pi, \pi+k\pi)$:

$$f'(x) = e^{\frac{-1}{\sin^2 x}} \cdot (-1)\cdot(-2)\frac{1}{\sin^3 x}\cdot\cos x = 2\, e^{\frac{-1}{\sin^2 x}} \cdot \frac{\cos x}{\sin^3 x}$$

v $x = 0+k\pi$, f spoj., věta o lim. der.:

$$f'(0+k\pi) = \lim_{x\to 0+k\pi} 2\, e^{-\frac{1}{\sin^2 x}} \frac{\cos x}{\sin^3 x} = \lim_{x\to 0+k\pi} \cos x \cdot \frac{2e^{-1/\sin^2 x}}{\sin^3 x} \overset{AL}{=} 1\cdot 0$$

protože

- $\lim\limits_{y \to \pm\infty} 2e^{-y^2} \cdot y^3 = 0$

  $\uparrow$ škála

nebo LH: $\lim\limits_{y \to \pm\infty} \dfrac{2y^3}{e^{y^2}} \underset{\frac{\infty}{\infty}}{\overset{LH}{=}} \lim\limits_{y \to \pm\infty} \dfrac{6y^2}{e^{y^2} \cdot 2y} =$

$= \lim\limits_{y \to \pm\infty} \dfrac{3y}{e^{y^2}} \underset{\frac{\infty}{\infty}}{\overset{LH}{=}} \lim\limits_{y \to \pm\infty} \dfrac{3}{e^{y^2} \cdot 2y} \overset{AL}{=} \dfrac{3}{\infty} = 0$

navíc $\lim\limits_{x \to (0+k\pi)^{\pm}} \dfrac{1}{\sin x} = \pm\infty$ (Věta o nevl. lim. podílu)

$g(x) = \dfrac{1}{\sin x}$ $\quad f(y) = 2e^{-y^2} y^3$ $\qquad$ (P) $\dfrac{1}{\sin x} \neq \pm\infty$ na $P(0+k\pi, \pi/4)$

Závěr: $f'(x) = \begin{cases} 2e^{-\frac{1}{\sin^2 x}} \cdot \dfrac{\cos x}{\sin^3 x} & x \in (0+k\pi, \pi+k\pi) \\ 0 & x = k\pi \end{cases}$

(e) monotonie:

$x \neq k\pi$: $f' = \underbrace{2e^{-\frac{1}{\sin^2 x}}}_{>0} \cdot \cos x \cdot \dfrac{1}{\sin^3 x}$

f': 0 + 0 − 0 + 0 − 0

$-\pi \quad -\pi/2 \quad 0 \quad \pi/2 \quad \pi$

f: ↗ ↘ ↗ ↘

→ f je ostře rostoucí na $(0+k\pi, \frac{\pi}{2}+k\pi)$

klesající $(\frac{\pi}{2}+k\pi, \pi+k\pi)$ $\quad k \in \mathbb{Z}$

(f) f má lok. maximum v $\frac{\pi}{2}+k\pi$

lok. minimum $0+k\pi$ $\quad k \in \mathbb{Z}$

glob extrémy nemá (periodicita)

(g) $f(0) = 0$

$f(\frac{\pi}{2}) = e^{-1}$ $\Big\}$ $H_f = [0, 1/e]$

(h) $x \in (0+2\pi,\ \pi+2\pi)$:

(i)

$$f'' = 2e^{-\frac{1}{\sin^2 x}} \cdot 2\,\frac{1}{\sin^3 x} \cdot \cos x \cdot \frac{\cos x}{\sin^3 x} + 2e^{-\frac{1}{\sin^2 x}} \cdot \frac{-\sin^4 x - \cos x\, 3\sin^2 x \cos x}{\sin^6 x}$$

$$= \underbrace{2e^{-\frac{1}{\sin^2 x}}}_{>0} \left(\frac{2\cos^2 x - \sin^4 x - 3\cos^2 x \sin^2 x}{\underbrace{\sin^6 x}_{\geq 0}} \right)$$

$$\cos^2 x\,(2 - 3\sin^2 x) - \sin^4 x =$$

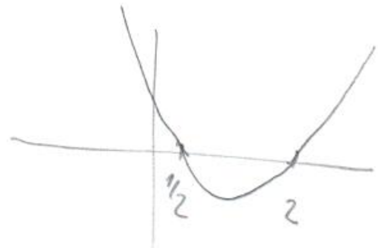
$$= (1-\sin^2 x)(2-3\sin^2 x) - \sin^4 x$$

$$u = \sin^2 x$$

$$(1-u)(2-3u) - u^2 = 0$$

$$2 + 2u^2 - 5u = 0$$

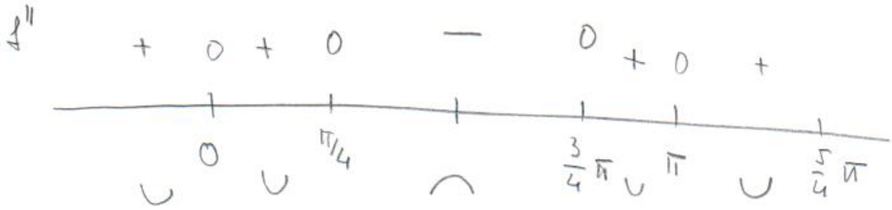
$$u_{1,2} = \frac{5 \pm \sqrt{25-16}}{4} = \begin{cases} 2 \\ 1/2 \end{cases}$$



$$\rightarrow \quad \sin^2 x = 1/2$$

$$\sin x = \pm \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$x_c = \pm \pi/4 + k\pi$$



f'' v $x = 0 + k\pi$

f' je spojitá v $0+2\pi$ $\left(\lim_{x \to 0+2\pi} f' = f'(0+2\pi)\right)$

tedy 3 věty

$$f''(0) = \lim_{x \to 0} 2\underbrace{\left(2\cos^2 x - \sin^4 x - 3\cos^2 x \sin^2 x\right)}_{2-0-0} \cdot e^{-\frac{1}{\sin^2 x}} \cdot \frac{1}{\sin^6 x} \overset{AL}{=} 2 \cdot 2 \cdot 0 = 0$$

$$\lim_{x \to 0} e^{-\frac{1}{\sin^2 x}} \frac{1}{\sin^6 x} = 0 \qquad \text{analogicky jako v (d)}$$

Závěr: f je konkávní na $[\pi/4, \frac{3\pi}{4}] + k\pi$

konvexní $[\frac{3}{4}\pi, \frac{5}{4}\pi] + k\pi$

(j) f' je spoj. na $\mathbb{R}$, navíc

$$\begin{array}{ccccc} f'' + & 0 & - & 0 & + \\ \hline & \pi/4 & & \frac{3\pi}{4} & \end{array}$$

→ inflexní bod je v $\pi/4 + k\pi$, $\frac{3}{4}\pi + k\pi$

(Věta postač. podm. pro inflexi)

(k) asymptoty

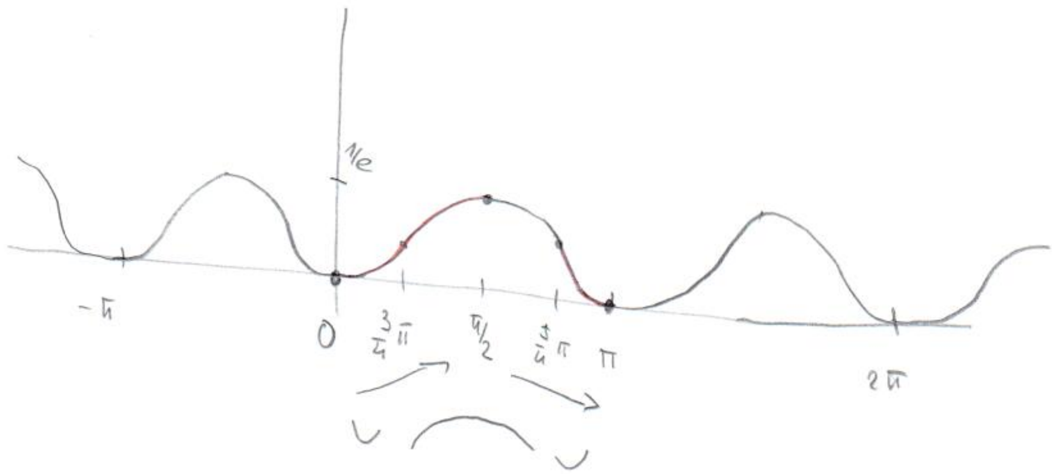
$$a = \lim_{x \to \pm\infty} \frac{f(x)}{x} = 0$$

$\frac{1}{x} \to 0$ $\quad$ $f(x)$ je omezená (víme z (g))

$$b = \lim_{x \to \pm\infty} f(x) - 0 \cdot x \quad \nexists \quad \text{(víme z (c))}$$

→ asymptoty $\nexists$

(l)



$$F(\alpha) = \int_0^\infty \frac{\overbrace{e^{-\alpha x^2} - 1}^{f}}{x^2}\, dx \qquad \alpha \geq 0$$

Derivace

$\alpha \in I = (0,\infty)$ $\qquad X = (0,\infty)$

- pro $\forall \alpha$ je $x \mapsto f(x,\alpha)$ spoj → měř
- $x \in X\ \alpha \in I$

$$\frac{\partial f}{\partial \alpha} = \frac{e^{-\alpha x^2}}{x^2} \cdot (-x^2) = -e^{-\alpha x^2}$$

- $\alpha_0 = 1 \quad \int_0^\infty \frac{e^{-x^2} - 1}{x^2}$ konv.
- majoranta $\quad \alpha \in \tilde{I} = (p,\infty)$

$1 > p > 0$

$$|-e^{-\alpha x^2}| \leq e^{-px^2}$$

Výpočet

$$F'(\alpha) = \int_0^\infty -e^{-\alpha x^2}\, dx = -\frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\to F(\alpha) = -\sqrt{\pi\alpha} + c$$

$$F(0) = 0 \to c = 0$$

Závěr: $\quad F(\alpha) = -\sqrt{\pi\alpha} \qquad \alpha \geq 0$

Spojitost

$\alpha \in T = [0,\infty) \qquad X = (0,\infty)$

- $\forall \alpha \in T \quad x \mapsto f$ spoj. měř
- $\forall x \in X \quad \alpha \mapsto f$ spoj.
- majoranta $\quad \alpha \in [0,p)$

$$\left|\frac{e^{-\alpha x^2} - 1}{x^2}\right| \leq \begin{cases} \frac{|e^{-px^2} - 1|}{x^2} \\ \frac{2}{x^2} & x \geq 1 \end{cases}$$

$$|e^{-\alpha x^2} - 1| \leq |e^{-px^2} - 1|$$

$$-e^{-\alpha x^2} + 1 \leq -e^{-px^2} + 1$$

$$e^{-px^2} \leq e^{-\alpha x^2}$$

$$e^{\alpha x^2} \leq e^{px^2} \quad ✓$$

→ spoj na $[0,p)$:)

Pak F je spoj na $[0,\infty)$