$x^n e^{-x^2/n}$ na $(-1,1)$

- fix x: $\lim_{n\to\infty} \frac{x^n}{e^{x^2/n}} = \lim \frac{x^n \to 0}{\underbrace{\sqrt[n]{e^{x^2}}}_{\to 1}} = 0$ $\qquad f_n \to 0$

- f_n sudé / liché pro sudá / lichá n

- fix n: $r_n = \sup_{x\in(-1,1)} \left|\frac{x^n}{e^{x^2/n}}\right|$

$$f_n' = \frac{n x^{n-1} e^{x^2/n} - x^n e^{x^2/n}\cdot\frac{1}{n}\,2x}{\left(e^{x^2/n}\right)^2} \qquad x^{n-1} e^{x^2/n}\left(n - x\cdot\frac{1}{n}\cdot 2x\right)$$

$x = 0 \quad \vee \quad n = 2x^2\cdot\frac{1}{n}$

$\frac{n^2}{2} = x^2 \qquad x = \pm\frac{n}{\sqrt{2}}$

pro $n \geq 2$ je $\frac{n}{\sqrt{2}} \notin (-1,1)$

na $[0,1)$

z parity je: $\to$

$\to$ pro $n \geq 2$ je $r_n = f_n(1) = \frac{1}{\sqrt[n]{e}}$

- $\lim_{n\to\infty} r_n = \lim \frac{1}{\sqrt[n]{e}} = 1 \not\to 0 \quad \to \quad f_n \not\rightrightarrows$ na $(-1,1)$

- fix $[-r,r]$, $0<r<1$

$$r_n = \sup_{x\in[-r,r]} \left|\frac{x^n}{e^{x^2/n}}\right| = \frac{r^n}{e^{r^2/n}} \qquad \lim_{n\to\infty} \frac{r^n}{e^{r^2/n}} = 0$$

$\to f_n \rightrightarrows$ na $[-r,r]$

$\to f_n \overset{loc}{\rightrightarrows}$ na $(-1,1)$

$\frac{n^2-x^2}{n^2+x^2}$ na $\mathbb{R}$

• $D_{f_n} = \mathbb{R}$ ✓ $\qquad$ f_n sudé

fix x: $\lim\limits_{n\to\infty} \frac{n^2-x^2}{n^2+x^2} = 1$

• fix n: $\quad r_n = \sup\limits_{x\in\mathbb{R}} \left| \underbrace{\frac{n^2-x^2}{n^2+x^2} - 1} \right|$

$$g_n(x) = \frac{n^2-x^2-n^2-x^2}{n^2+x^2} = \frac{-2x^2}{n^2+x^2}$$

$\lim\limits_{x\to\infty} g_n(x) = -2$

$\lim r_n = \lim -2 \neq 0 \quad \to \quad f_n \not\rightrightarrows$

• na $[-r,r]$: $r_n = \sup\limits_{x\in\mathbb{R}} \left| \frac{-2x^2}{n^2+x^2} \right|$

$$g_n'(x) = \frac{-4x(n^2+x^2) + 2x^2 \cdot 2x}{(n^2+x^2)^2} \qquad 4x\left[-n^2-x^2+x^2\right] = 4x\cdot(-n^2)$$

$\lim\limits_{x\to\infty} g_n(x) = -2 \qquad g_n(0) = 0$

$r_n = |g_n(r)| = \frac{2r^2}{n^2+r^2}$

• $\lim\limits_{n\to\infty} r_n = \lim\limits_{n\to\infty} \frac{2r^2}{n^2+r^2} = 0$ ✓

$f_n \rightrightarrows$ na $[-r,r]$

$\to \quad f_n \overset{loc}{\rightrightarrows}$ na $\mathbb{R}$

$n \sin \frac{x}{n}$ $\quad \mathbb{R}$

- fix x: $x \neq 0$ $\quad \lim\limits_{n\to\infty} n \sin\frac{x}{n} = \lim n \cdot \underbrace{\frac{\sin\frac{x}{n}}{\frac{x}{n}}}_{\to 1 \atop \frac{x}{n}\to 0} \cdot \frac{x}{n} = x$

$x = 0 \rightarrow f(0) = 0$

$f_n \to x$

- fix n: $\quad \sigma_n = \sup\limits_{x\in\mathbb{R}} |\underbrace{n \sin\frac{x}{n} - x}_{g_n}|$

$g_n'(x) = n \cos\frac{x}{n} \cdot \frac{1}{n} - 1 = \cos\frac{x}{n} - 1 \rightarrow \cos\frac{x}{n} = 1$

$\frac{x}{n} = 0 + 2k\pi$

$x = 2k\pi n$

$g_n(2k\pi n) = n \underbrace{\sin(2k\pi)}_{0} - 2k\pi n$

$= -2k\pi n$

$\rightarrow \sigma_n \geq 2k\pi n \quad \rightarrow \lim\limits_{n\to\infty} \sigma_n = \infty \qquad f_n \not\rightrightarrows$ na $\mathbb{R}$

- Zafixujme $[-r, r]$, $r > 0$

pro fix n: $\quad \sigma_n = \sup\limits_{x\in[-r,r]} |n \sin\frac{x}{n} - x| \qquad (g_n' \leq 0)$

$= |n \sin\frac{r}{n} - r|$

max v krajích

f_n jsou liché

- $\lim\limits_{n\to\infty} \sigma_n = \lim |\underbrace{n \sin\frac{r}{n}}_{n \frac{\sin\frac{r}{n}}{\frac{r}{n}} \cdot \frac{r}{n}} - r| = 0$ ✓

$\rightarrow f_n \rightrightarrows$ na $[-r, r]$

$\rightarrow f_n \overset{\text{loc}}{\rightrightarrows}$ na $\mathbb{R}$

$nx\,e^{-nx^2}$ na $\mathbb{R}$

- fix x: $\lim\limits_{n\to\infty} \dfrac{nx}{e^{+nx^2}} = 0$

$x \neq 0$:

$$\lim_{y\to\infty} \frac{yx}{e^{+yx^2}} \overset{L'H}{\underset{\frac{\infty}{\infty}}{=}} \lim_{y\to\infty} \frac{x}{e^{yx^2}\cdot x^2} = 0$$

$x=0$ $\lim\limits_{n\to\infty} 0\cdot e^0 = 0$ $\qquad f_n \to 0$

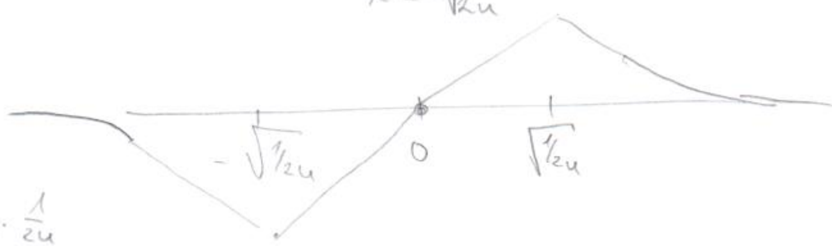
- $\sigma_n = \sup\limits_{x\in\mathbb{R}} |\underbrace{nxe^{-nx^2}}_{f_n(x)} - 0|$

fix n

$$f_n' = ne^{-nx^2} + nxe^{-nx^2}(-2nx) = e^{-nx^2}(n - 2n^2x^2)$$

$$n(1 - 2nx^2) = 0$$
$$2nx^2 = 1$$
$$x^2 = \frac{1}{2n}$$
$$x = \pm\sqrt{\frac{1}{2n}}$$

// f_n jsou liché



$$f_n\left(\pm\sqrt{\tfrac{1}{2n}}\right) = \pm n\cdot\sqrt{\frac{1}{2n}}\, e^{-n\cdot\frac{1}{2n}} =$$
$$= \pm\frac{\sqrt{n}}{\sqrt{2}}\, e^{-1/2} = \pm\frac{\sqrt{n}}{\sqrt{2e}}$$

$\lim\limits_{x\to\pm\infty} f_n(x) = 0$ $\qquad$ // $\lim\limits_{x\to\infty} \dfrac{nx}{e^{nx^2}} \overset{L'H}{\underset{\frac{\infty}{\infty}}{=}} \lim\limits_{x\to\infty} \dfrac{n}{e^{nx^2}\cdot 2xn} = 0$

- Závěr 1: $\lim\limits_{n\to\infty} \sigma_n = \lim \sqrt{\frac{n}{2e}} = \infty \neq 0$

$\to$ f_n na $\mathbb{R}$ __Ne__konverguje stejnom.

• lok:

v 0: na $(-r,r)$, $r>0$ $f_n \not\rightrightarrows 0$, protože

$\exists n_0 : \forall n \geq n_0 :$ $\sqrt{\frac{1}{2n}} \in (-r,r)$

na $[\delta,\infty)$:

$$\sigma_n = \sup_{x\in[\delta,\infty)} |f_n| = f_n(\delta) = n\delta e^{-n\delta^2}$$

$$\lim_{n\to\infty} n\delta e^{-n\delta^2} = 0 \qquad f_n \rightrightarrows 0 \text{ na } [\delta,\infty)$$

na $(-\infty,-\delta)$ analogicky (f_n jsou liché)

Závěr 2: $f_n \rightrightarrows 0$ na $[\delta,\infty)$, $(-\infty,-\delta]$

$f_n \not\rightrightarrows$ $\mathbb{R}$

$f_n \overset{loc}{\rightrightarrows}$ $(-\infty,0)$, $(0,\infty)$

$f_n \to 0$

$x \arctan(nx)$ na $\mathbb{R}$

- f_n jsou sudé, $f_n \geq 0$
- fix x: $\lim\limits_{n\to\infty} x \arctan nx = \begin{cases} x\cdot \pi/2 & x>0 \\ 0 & x=0 \\ x\cdot -\pi/2 & x<0 \end{cases} = \frac{\pi}{2}|x|$

- $\underline{\text{fix } n}$ $\quad \tau_n = \sup\limits_{x\in\mathbb{R}} \Big|\underbrace{x\arctan(nx) - \frac{\pi}{2}|x|}_{g_n(x)\leq 0}\Big|$

stačí vyšetřovat $x>0$ (f_n sudé):

$$= \sup_{x\in\mathbb{R}} -\left(x\arctan(nx) - \frac{\pi}{2}\cdot x\right) = \sup \underbrace{-x\left(\arctan(nx) - \pi/2\right)}_{h_n(x)}$$

$$h_n(x) = x\,\Big(\underbrace{\tfrac{\pi}{2} - \arctan nx}_{\operatorname{arccot}(nx)}\Big)$$

$\boxed{\text{Hint}}$ (a) $y>0$: $\quad \frac{\pi}{2} - \arctan y = \arctan\frac{1}{y}$ (lze ověřit derivací)

tedy: $\quad x\left(\frac{\pi}{2} - \arctan(nx)\right) = x\cdot \arctan\frac{1}{nx}$
$x>0$

(b) $y>0$: $\quad 0 < \frac{\arctan y}{y} < 1 \quad \Rightarrow \quad \arctan y < y$

$$x\left(\tfrac{\pi}{2} - \arctan(nx)\right) = x \arctan\frac{1}{nx} < x\cdot\frac{1}{nx} = \frac{1}{n}$$
pro $x>0$:

Dohromady: $\quad h_n(0) = 0, \quad h_n(-x) = h_n(x)$

$$\tau_n \leq 1/n$$

$\lim 1/n = 0 \quad \Rightarrow \quad \boxed{f_n \rightrightarrows \frac{\pi}{2}|x| \text{ na } \mathbb{R}}$

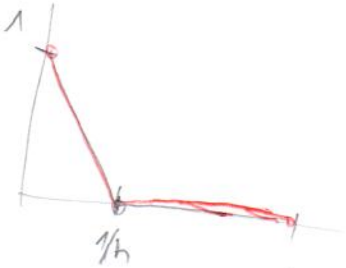
(5a) "⇒" zřejmé

"⇍" Např. funkce $f_n(x) = \begin{cases} 0 & x \neq 1/2 \\ 1-1/n & x = 1/2 \end{cases}$

pak na $[0,1/2)$ a $(1/2,1]$ jsou $f_n \equiv 0$, tedy $f_n \rightrightarrows 0$

ale na $[0,1]$ $f_n \to f = \begin{cases} 0 & x \neq 1/2 \\ 1 & x = 1/2 \end{cases}$, f nespojitá → spor.

(5b) Ukažme funkci
"⇍" f_n



$f \equiv 0$

Pak $\int_0^1 |f_n - f| = \int_0^1 f_n = \frac{1}{2} \frac{1}{n} \to 0$

ale $f_n \not\rightrightarrows 0$ (ani bodově $f_n \to f$ =)

"⇒": Máme větu, příp. v míře Lebesgueovu větu.

Nebo přímo: $\forall \varepsilon\ \exists n_0: \forall x, \forall n \geq n_0: |f_n(x) - f(x)| < \varepsilon$.

Pak $\left|\int_0^1 f_n - f\right| \leq \int_0^1 |f_n - f| < \varepsilon \cdot 1$ (f_n, f spoj., interval om.)

$\to \int_0^1 f_n - f = 0 \quad \to \int_0^1 f_n \to \int_0^1 f$

(5c) "⇒" zřejmé.

Zvolme $\varepsilon > 0$. Pak $\exists n_0: \forall n \geq n_0\ \forall x \in [0,1] \setminus E: |f_n(x) - f(x)| < \varepsilon$.

Zafixujme $n \geq n_0$ a $x \in [0,1]$. Pak ze spoj. f_n a f $\exists \tilde{x}_n \in [0,1] \setminus E$:

$|f_n(\tilde{x}_n) - f_n(x)| < \varepsilon$, $|f(\tilde{x}_n) - f(x)| < \varepsilon$. ⇓ $\exists \delta_1$ a δ_2 ze na okolí...

kdyby δ neexistovalo, tak $B(x,\delta) \subset E$ a E by nebylo míry 0.

Celkem:

$|f_n(x) - f(x)| \leq |f_n(x) - f_n(\tilde{x}_n)| + |f_n(\tilde{x}_n) - f(\tilde{x}_n)| + |f(\tilde{x}_n) - f(x)| < \varepsilon + \varepsilon + \varepsilon$

■