

$$(1) \sum \underbrace{\sin \frac{1}{n} - \arcsin \frac{1}{n}}_{a_n}$$

"pro $x = \frac{1}{n}$ zürtime brucha"

$$\sin x = x - \frac{x^3}{6} + O(x^4)$$

$$\arcsin x = x + \frac{1}{6}x^3 + O(x^4)$$

$x \rightarrow 0$

$$f(x) = \sin x - \arcsin x$$

$$= x - \frac{x^3}{6} + O(x^4)$$

$$- x - \frac{1}{6}x^3 + O(x^4)$$

$$= -\frac{2}{6}x^3 + O(x^4)$$

LS&S

$$b_n = \frac{1}{n^3}$$

$$\lim_{n \rightarrow \infty} \frac{|\sin \frac{1}{n} - \arcsin \frac{1}{n}|}{\frac{1}{n^3}} = \frac{1}{3} < 1 \Rightarrow$$

Heine $x_n = \frac{1}{n}$ $x_n \rightarrow 0$ $\frac{1}{n} \neq 0$ $\forall n \in \mathbb{N}$, $0 < \frac{1}{n} \leq 1$, ledy $\frac{1}{n} \in D_f$

$$\lim_{x \rightarrow 0} \frac{|-\frac{2}{6}x^3 + O(x^4)|}{x^3} = +\frac{1}{3} \in (0, \infty)$$

$$\sum a_n \text{ A\ddot{e}} \Leftrightarrow \sum b_n \text{ \& } \sum \frac{1}{n^3} \text{ \&}$$

Zalver $\sum a_n \text{ A\ddot{e}}$

Prop. $\sin x \leq \arcsin x$ me $x \in (0, 1)$

ledy $a_n \leq 0$

$$\sum 2 \tan\left(\frac{1}{n^{1/5}}\right) - \sin\left(\frac{1}{n^{1/5}}\right) - \frac{1}{n^{3/5}}$$

$$f(x) = 2 \tan x - \sin x - x^3 \quad (x = \frac{1}{n^{1/5}})$$

$$\begin{aligned} f(x) &= 2 \left[\left(x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + o(x^6) \right) - \left(x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^6) \right) \right] - x^3 \\ &= 2x^5 \left(\frac{2}{15} - \frac{1}{120} \right) + o(x^6) = \frac{1}{4}x^5 + o(x^6) \end{aligned}$$

$$b_n = \frac{1}{n} \quad \sum \frac{1}{n} \text{ D}$$

$$\lim_{n \rightarrow \infty} \frac{|a_n|}{b_n} = \frac{1}{4} \in (0, \infty) \quad \text{Heine } x_n = \frac{1}{n^{1/5}}$$

$$= \lim_{x \rightarrow 0} \frac{\left| \frac{1}{4}x^5 + o(x^5) \right|}{x^5} = \frac{1}{4}$$

$$\sum |a_n| \text{ k} \Leftrightarrow \sum b_n \text{ k}$$

$$\text{Zelver } \sum |a_n| \text{ D}$$

$$\sum \underbrace{\sin\left(\frac{1}{n} - \arcsin \frac{1}{n}\right)}_{a_n}$$

$$(x = 1/n)$$

$$f(x) = \sin(x - \arcsin x)$$

$$x - \arcsin x = x - x - \frac{x^3}{6} + o(x^4)$$

$$f(x) = -\frac{x^3}{6} + o(x^4) + o(x^3)$$

$$b_n = \frac{1}{n^3}$$

$$\sum b_n \text{ k}$$

$$\lim_{n \rightarrow \infty} \frac{|a_n|}{b_n} = 1/6 \in (0, \infty)$$

$$\text{Heine } x_n = 1/n$$

$$\lim_{x \rightarrow 0} \frac{\left| -\frac{x^3}{6} + o(x^3) \right|}{x^3} = 1/6$$

$$\text{Zurück } \sum |a_n| \text{ k}$$

$$(4) \sum \underbrace{\ln \frac{1}{n^\beta} - \ln \left(\sin \frac{1}{n^\beta} \right)}_{a_n} \quad \beta > 0$$

$$a_n = \ln \frac{\frac{1}{n^\beta}}{\sin \frac{1}{n^\beta}} = - \ln n^\beta \sin \frac{1}{n^\beta}$$

zavešćue funkcije $(x = \frac{1}{n^\beta})$

$$f(x) = - \ln \left(\frac{1}{x} \sin x \right)$$

$$\begin{aligned} \frac{1}{x} \sin x &= \left[\frac{1}{x} \left(x - \frac{x^3}{6} + o(x^4) \right) \right] & x \rightarrow 0 \\ &= \left[1 - \frac{x^2}{6} + o(x^3) \right] \end{aligned}$$

$$\ln \left(\frac{1}{x} \sin x \right) = - \frac{x^2}{6} + \underbrace{o(x^3) + o \left(-\frac{x^2}{6} + o(x^3) \right)}_{o(x^2)}$$

$$b_n = \frac{1}{n^{2\beta}} \quad \sum n^{-2\beta} \text{ k } \Leftrightarrow -2\beta < -1; \beta > \frac{1}{2}$$

$\lim_{n \rightarrow \infty} \frac{|a_n|}{b_n} = \frac{1}{6} \in \mathbb{R} \setminus \{0\}$ Heine $x_n = \frac{1}{n^\beta}, x_n \rightarrow 0, \frac{1}{n^\beta} \neq 0 \forall n$
 $0 < \frac{1}{n^\beta} \leq 1 \rightarrow \frac{1}{n^\beta} \in D_f$

$$\lim_{x \rightarrow 0} \frac{\left| -\frac{x^2}{6} + o(x^2) \right|}{x^2} = \frac{1}{6}$$

$$\sum |a_n| \text{ k } \Leftrightarrow \sum b_n \text{ k } \Leftrightarrow \beta > \frac{1}{2}$$

Záver $\sum a_n$ Až $\Leftrightarrow \beta > \frac{1}{2}$

$$\sum \underbrace{\left(\sin \frac{1}{n} - \frac{1}{n} \right) \frac{1}{n^k}}_{a_n}$$

potrebna: funkcija ($x = \frac{1}{n}$)

$$f(x) = \sin x - x$$

$$\text{Razvoj: } f(x) = -x + x - \frac{x^3}{3!} + o(x^3)$$

$$= -\frac{x^3}{3!} + o(x^3)$$

$$b_n = \frac{1}{n^\alpha} \cdot \frac{1}{n^3} = n^{-\alpha-3} \quad \sum n^{-\alpha-3} \text{ k } \Leftrightarrow \begin{matrix} -\alpha-3 < -1 \\ \underline{-2 < \alpha} \end{matrix}$$

$$\lim \frac{|a_n|}{b_n} = \lim \frac{\left| \left(\sin \frac{1}{n} - \frac{1}{n} \right) \frac{1}{n^k} \right|}{\frac{1}{n^3} \cdot \frac{1}{n^k}} = \frac{1}{6} \in (0, \infty)$$

Heine $x_n = \frac{1}{n} \quad x_n \rightarrow 0 \quad \frac{1}{n} \neq 0 \quad \frac{1}{n} \in \mathbb{D}_f (= \mathbb{R})$

$$\lim_{x \rightarrow 0} \frac{\left| -\frac{x^3}{6} + o(x^3) \right|}{x^3} = +\frac{1}{6}$$

$$\sum |a_n| \text{ k } \Leftrightarrow \sum b_n \text{ k}$$

$$\text{Záver: } \sum a_n \text{ A k } \Leftrightarrow \alpha > -2$$

$$\sum \underbrace{\sqrt{n+2} - 2\sqrt{n+1} + \sqrt{n}}_{a_n}$$

$$\sqrt{n} \left(\sqrt{1 + \frac{2}{n}} - 2\sqrt{1 + \frac{1}{n}} + 1 \right)$$

$$(x = \frac{1}{n})$$

$$f(x) = \sqrt{1+2x} - 2\sqrt{1+x} + 1$$

$$x \rightarrow 0$$

$$= 1 + \frac{1}{2}(2x) - \frac{1}{8}(2x)^2 + o(x^2) - 2\left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + o(x^2)\right) + 1$$

$$\binom{1/2}{2} = \frac{\frac{1}{2}(-\frac{1}{2})}{2!} = -\frac{1}{8}$$

$$= x^2 \left(-\frac{1}{8} + \frac{2}{8}\right) + o(x^2)$$

$$= -\frac{1}{4}x^2 + o(x^2)$$

$$b_n = \sqrt{n} \cdot \frac{1}{n^2} = \frac{1}{n^{3/2}}$$

$$\sum \frac{1}{n^{3/2}} \quad k$$

$$\lim_{n \rightarrow \infty} \frac{|a_n|}{b_n} = \lim_{n \rightarrow \infty} \frac{|\sqrt{n} (\sqrt{1+2/n} - 2\sqrt{1+1/n} + 1)|}{\sqrt{n} \cdot \frac{1}{n^2}} = \frac{1}{4} \in (0, \infty)$$

$$\text{Heine } x_n = \frac{1}{n} \quad x_n \rightarrow 0$$

$$\lim_{x \rightarrow 0} \frac{|-\frac{1}{4}x^2 + o(x^2)|}{x^2} = \frac{1}{4}$$

$$\sum |a_n| \text{ k} \Leftrightarrow \sum b_n \text{ k}$$

$$\text{Zusatz: } \sum |a_n| \text{ k}$$

$$\sum \underbrace{\left[e - \left(1 + \frac{1}{n}\right)^n \right]^p}_{a_n}$$

$$e - e^{n \ln\left(1 + \frac{1}{n}\right)} = +e \left(1 - e^{n \ln\left(1 + \frac{1}{n}\right) - 1} \right)$$

$$(x = 1/n)$$

$$f(x) = +e \left[1 - e^{\frac{1}{x} \ln(1+x) - 1} \right]$$

$$\ln(1+x) = x - \frac{x^2}{2} + o(x^2)$$

$$\frac{1}{x} \ln(1+x) - 1 = 1 - \frac{x}{2} + \frac{1}{x} o(x^2) - 1 = -\frac{x}{2} + o(x)$$

$$e^{\frac{1}{x} \ln(1+x) - 1} = 1 + \left(-\frac{x}{2}\right) + o(x) + o\left(-\frac{x}{2} + o(x)\right) - 1 \quad x \rightarrow 0$$

$$= -\frac{x}{2} + o(x)$$

$$f(x) = +e \left(-\frac{x}{2} + o(x)\right)$$

$$b_n = \left(\frac{1}{n}\right)^p$$

$$\lim_{n \rightarrow \infty} \frac{|a_n|}{b_n} = \lim_{n \rightarrow \infty} \frac{\left| \left[e - \left(1 + \frac{1}{n}\right)^n \right]^p \right|}{\frac{1}{n^p}}$$

Heine $x_n = 1/n, 1/n \rightarrow 0, 1/n \neq 0$
 $0 < \frac{1}{n} \leq 1 \quad \frac{1}{n} \in \mathbb{D}f$

$$\lim_{x \rightarrow 0} \frac{\left| \left[+e \left(-\frac{x}{2} + o(x)\right) \right]^p \right|}{x^p} = \lim_{x \rightarrow 0} \frac{e^p \left(\frac{x^p}{2^p} + (o(x))^p \right)}{x^p} = \left(\frac{e}{2}\right)^p \in (0, \infty)$$

$$\rightarrow \sum |a_n| \text{ k} \Leftrightarrow \sum \frac{1}{n^p} \text{ k} \Leftrightarrow p > 1$$

$$\text{Zusatz: } \sum a_n \text{ A.k.} \Leftrightarrow p > 1$$

$$\sum (e^{\frac{1}{\sqrt{n}}} - 1 - \frac{1}{\sqrt{n}}) \left(\arctan \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n}} \right)$$

$$(x = \frac{1}{\sqrt{n}})$$

$$f(x) = (e^{x^2} - 1 - x^2) (\arctan x^2 - x)$$

$$= \left(1 + x^2 + \frac{x^4}{2!} + o(x^4) - 1 - x^2 \right) \left(x^2 + \frac{1}{6} x^6 + o(x^6) - x \right)$$

$$= -\frac{x^5}{2} + o(x^5)$$

$$b_n = \left(\frac{1}{\sqrt{n}} \right)^5 = n^{-5/2} \quad \sum b_n <$$

$$\lim_{n \rightarrow \infty} \frac{|a_n|}{b_n} = \frac{1}{2} \in (0, \infty)$$

$$\text{Weil } x_n = \frac{1}{\sqrt{n}}$$

$$\lim_{x \rightarrow 0} \frac{\left| -\frac{x^5}{2} + o(x^5) \right|}{x^5} = \frac{1}{2}$$

$$\text{Zaher: } \sum |a_n| <$$

$$\sum \sin\left(\frac{1}{\sqrt{n}}\right) - \ln\left(1 + \frac{1}{\sqrt[3]{n}}\right)$$

$$(x = \frac{1}{\sqrt{n}})$$

$$f(x) = \sin x^3 - \ln(1 + x^2)$$

$$= x^3 + o(x^3) - (x^2 + o(x^2))$$

$$= -x^2 + o(x^2)$$

$$\text{LSE } b_n = \left(\frac{1}{\sqrt{n}}\right)^2 = \frac{1}{n^{2/3}} \quad \sum b_n \rightarrow \infty$$

$$\lim_{n \rightarrow \infty} \frac{|a_n|}{b_n}$$

$$\text{Heine } x_n = \frac{1}{\sqrt{n}}$$

$$\lim_{x \rightarrow 0} \frac{|-x^2 + o(x^2)|}{x^2} = 1 \in (0, \infty)$$

$$\text{Zurück } \sum |a_n| \rightarrow \infty$$

$$\sum \frac{1}{\sqrt{u}} + \ln \left(\sqrt{1+\frac{1}{u}} - \frac{1}{\sqrt{u}} \right)$$

$$\left(\frac{1}{\sqrt{u}} = x \right)$$

$$f(x) = x + \ln \left(\sqrt{1+x^2} - x \right)$$

$$-x + \sqrt{1+x^2} = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \mathcal{O}(x^6) - x$$

$$\begin{aligned} x + \ln \left(\sqrt{1+x^2} - x \right) &= x + \left(-x + \frac{x^2}{2} - \frac{x^4}{8} + \mathcal{O}(x^6) \right) - \frac{1}{2} \left(-x + \frac{x^2}{2} - \frac{x^4}{8} + \mathcal{O}(x^6) \right)^2 \\ &\quad + \frac{1}{3} \left(-x + \frac{x^2}{2} - \frac{x^4}{8} + \mathcal{O}(x^6) \right)^3 + \mathcal{O} \left(\left(-x + \frac{x^2}{2} - \frac{x^4}{8} + \mathcal{O}(x^6) \right)^3 \right) \end{aligned}$$

$$\begin{aligned} &= \frac{x^2}{2} - \frac{x^4}{8} + \mathcal{O}(x^6) - \frac{1}{2} \left(x^2 - x^3 + \frac{x^4}{4} + \mathcal{O}(x^6) \right) \\ &\quad + \frac{1}{3} \left(-x^3 + \frac{3}{2}x^4 + \mathcal{O}(x^6) \right) + \mathcal{O}(x^3) \end{aligned}$$

$$= x^3 \left(\frac{1}{2} - \frac{1}{3} \right) + \mathcal{O}(x^3) = \frac{1}{6}x^3 + \mathcal{O}(x^3)$$

$$b_n = \frac{1}{\sqrt{n^3}} = n^{-3/2} \quad \sum b_n <$$

$$\lim_{n \rightarrow \infty} \frac{|a_n|}{b_n} = \frac{1}{6} \in (0, \infty)$$

$$\text{Heine } x_n = \frac{1}{\sqrt{n}}$$

$$\lim_{x \rightarrow 0} \frac{\left| \frac{1}{6}x^3 + \mathcal{O}(x^3) \right|}{x^3} = \frac{1}{6}$$

Zieler: $\sum |a_n| <$