$$f(x,y) = \begin{cases} \frac{x^3-2y^3}{4x^2+y^2} & [x,y] \neq [0,0] \\ 0 & [x,y] = [0,0] \end{cases}$$

$D_f = \mathbb{R}^2$

- $$\frac{\partial f}{\partial x} = \frac{3x^2(4x^2+y^2) - (x^3-2y^3)\cdot 8x}{(4x^2+y^2)^2} = \frac{12x^4 + 3x^2y^2 - 8x^4 + 16xy^3}{(4x^2+y^2)^2}$$

$$= \frac{4x^4 + 3x^2y^2 + 16xy^3}{(4x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{-6y^2(4x^2+y^2) - (x^3-2y^3)\cdot 2y}{(4x^2+y^2)^2} = \frac{2y(-x^3 - 1y^3 - 12yx^2)}{(4x^2+y^2)^2}$$

obaji pro $\mathbb{R}^2 \setminus \{[0,0]\}$, $\frac{\partial f}{\partial x}$ i $\frac{\partial f}{\partial y}$ spoj. na $\mathbb{R}^2 \setminus \{[0,0]\}$ → máme tot. dif

- $$\frac{\partial f}{\partial x}[0,0] = \lim_{t\to 0} \frac{\frac{(0+t)^3 - 2\cdot 0^3}{4(0+t^2)+0^2} - 0}{t} = \lim_{t\to 0} \frac{t^3}{4t^3} = \underline{\underline{\frac{1}{4}}}$$

$$\frac{\partial f}{\partial y}[0,0] = \lim_{t\to 0} \frac{\frac{0-2(0+t)^3}{4\cdot 0 + (0+t)^2} - 0}{t} = \lim_{t\to 0} \frac{-2t^3}{t^3} = \underline{\underline{-2}}$$

- tot. dif v $[0,0]$

$$\lim_{h\to[0,0]} \frac{\frac{h_1^3 - 2h_2^3}{4h_1^2+h_2^2} - 0 - \frac{1}{4}h_1 + 2h_2}{\sqrt{h_1^2+h_2^2}} =$$

$$= \lim_{h\to[0,0]} \frac{h_1^3 - 2h_2^3 - \frac{1}{4}h_1 4h_1^2 - \frac{1}{4}h_1h_2^2 + 8h_2h_1^2 + 2h_2^3}{\sqrt{(h_1^2+h_2^2)}(4h_1^2+h_2^2)}$$

$$= \lim_{h\to[0,0]} \frac{h_1h_2}{\sqrt{h_1^2+h_2^2}} \cdot \frac{-\frac{1}{4}h_2 + 8h_1}{4h_1^2+h_2^2}$$

zkusme $h_1 = h_2$ $\quad \lim \frac{h_1^2}{\sqrt{2}|h_1|} \cdot \frac{\frac{31}{4}h_1}{5h_1^2} = \pm\frac{\frac{31}{4}}{5}$ (lim zleva, zprava)

což není 0 → tot dif v $[0,0]$ ∄

- mimo $[0,0]$ máme

$$d_f[x,y](h_1,h_2) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\cdot(h_1,h_2)$$

$$\begin{array}{ccc} & F_1 & F_2 \\ F(x,y,z) = (& x^2y + y + xz^2, & (z+1)e^{xy}) \end{array}$$

$G: \mathbb{R}^2 \to \mathbb{R}^3$ v $(0,1)$ dá:

$$\begin{pmatrix} 2 & 1 \\ 3 & -1 \\ 1 & 1 \end{pmatrix}$$

(a) dei F v $(0,0,0)$ $\qquad$ • $D_F = \mathbb{R}^3$

$\frac{\partial F_1}{\partial x} = 2xy + z^2 \qquad \frac{\partial F_1}{\partial y} = x^2 + 1 \qquad \frac{\partial F_1}{\partial z} = 2xz$

pro $\mathbb{R}^3$

$\frac{\partial F_2}{\partial x} = (z+1)e^{xy} \cdot y \qquad \frac{\partial F_2}{\partial y} = (z+1)e^{xy} \cdot x \qquad \frac{\partial F_2}{\partial z} = e^{xy}$

parc. derivace jsou spoj. na $\mathbb{R}^3 \to \exists$ tot. dif

pro $(0,0,0)$:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

pro $G \circ F$ $\quad$ ($G(F(0,0,0))$) $\qquad$ // $G \circ F: \mathbb{R}^3 \to \mathbb{R}^3$

součin matic:

$$\begin{pmatrix} 2 & 1 \\ 3 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 0 & 2 & 1 \\ 0 & 3 & -1 \\ 0 & 1 & 1 \end{pmatrix}}}$$

(b) der F_1 v $(0,0,0)$ podle vektoru $(1,-1,1)$

• der $F_1 \longmapsto (0,1,0) \cdot (1,-1,1) = \underline{\underline{-1}}$

$\uparrow$ vektor

// platí, neb F_1 má spoj. p. der a tedy tot. dif

$F: \mathbb{R}^3 \to \mathbb{R}^2$

$$F(x,y,z) = \begin{cases} \left(\overbrace{x \sin y \cdot \sin z}^{F_1},\ \overbrace{x^2 \sin \frac{1}{x^2+y^2+z^2}}^{F_2} \right) & [x,y,z] \neq [0,0,0] \\ (0,0) & [x,y,z] = [0,0,0] \end{cases}$$

(a) dif v $[\pi, 1, 0]$ + matici

(b) $\frac{\partial F_2}{\partial x}$ $[0,0,0]$

• $\mathbb{R}^3 \setminus \{[0,0,0]\}$

$$\frac{\partial F_1}{\partial x} = \sin y \sin z$$

$$\frac{\partial F_1}{\partial y} = x \cos y \sin z$$

$$\frac{\partial F_1}{\partial z} = x \sin y \cos z$$

$$\frac{\partial F_2}{\partial x} = 2x \sin \frac{1}{x^2+y^2+z^2} + x^2 \cos \frac{1}{x^2+y^2+z^2} \cdot 2x \cdot (-1) \frac{1}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial F_2}{\partial y} = x^2 \cos \frac{1}{x^2+y^2+z^2} \cdot \frac{-2y}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial F_2}{\partial z} = x^2 \cos \frac{1}{x^2+y^2+z^2} \cdot \frac{-2z}{(x^2+y^2+z^2)^2}$$

Reprezentující matice

$$\begin{pmatrix} \sin y \sin z & x \cos y \sin z & x \sin y \cos z \\ 2x \sin\left(\frac{1}{\ }\right) - \frac{2x^3}{(\ )^2} \cos\left(\frac{1}{\ }\right) & \frac{-2x^2 y}{(\ )^2} \cos \frac{1}{(\ )} & \frac{-2zx^2}{(\ )^2} \cos \frac{1}{(\ )} \end{pmatrix}$$

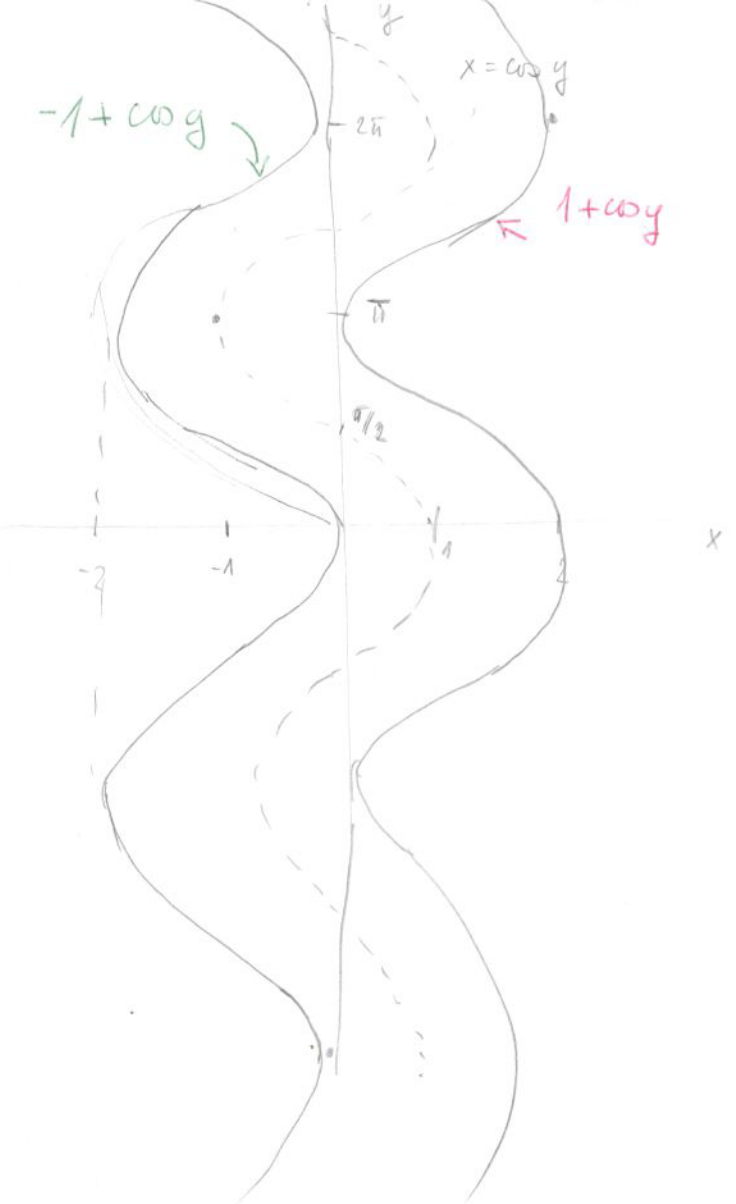
spec v bodě $[\pi, 1, 0]$:

$$\begin{pmatrix} 0 & 0 & \pi \cdot \sin 1 \\ 2\pi \sin \frac{1}{1+\pi^2} + \pi^2 \cos \frac{1}{1+\pi^2} \cdot \frac{-2}{(1+\pi^2)^2} & \pi^2 \cos \frac{1}{1+\pi^2} \cdot \frac{-2}{(1+\pi^2)^2} & 0 \end{pmatrix}$$

• $\frac{\partial F_2}{\partial x}(0,0,0)$:

$$\lim_{t \to 0} \frac{(0+t)^2 \sin \frac{1}{(0+t)^2+0^2+0^2} - 0}{t} = \lim_{t \to 0} \underbrace{t}_{0} \cdot \underbrace{\sin \frac{1}{t^2}}_{\text{om.}} = 0$$

$$f(x,y) = \arccos(x - \cos y)$$



(1)

$D_f: \quad -1 \le x - \cos y \le 1$

$$-1 + \cos y \le x \le 1 + \cos y$$

$$D_f = \{[x,y] \in \mathbb{R}^2 : 1 - \cos y \le x \le 1 + \cos y\}$$

(2) Derivace (pro body mimo hranici)

pro

$$-1 + \cos y < x < 1 + \cos y$$

$$\frac{\partial f}{\partial x} = \frac{-1}{\sqrt{1-(x-\cos y)^2}} \cdot 1$$

$$\frac{\partial f}{\partial y} = \frac{-1}{\sqrt{1-(x-\cos y)^2}} \cdot \sin y$$

(3) $\frac{\partial f}{\partial y}$ v bodech $[0, k\pi]$, $k \in \mathbb{Z}$

z definice

$$\frac{\partial f}{\partial y}([0,k\pi]) = \lim_{t\to 0} \frac{\arccos(0 - \cos(k\pi + t)) - \arccos(0 - \cos(k\pi))}{t}$$

$$= \lim_{t\to 0} \frac{\arccos(-\cos(k\pi+t)) - \arccos(-\cos k\pi)}{t}$$

$$\overset{L'H}{\underset{\frac{0}{0}}{=}} \lim_{t\to 0} \frac{\frac{-1}{\sqrt{1-(\cos(k\pi+t))^2}} \cdot (+\sin(k\pi+t)) \cdot 1}{1} =$$

$$= \lim_{t\to 0} \frac{-\sin(k\pi+t)}{\sqrt{\sin^2(k\pi+t)}} = \lim_{t\to 0} \frac{-\sin k\pi + t}{|\sin(k\pi+t)|}$$

$$\lim_{t \to 0+} \frac{-\sin(k\pi + t)}{|\sin(k\pi + t)|} = \begin{cases} -1 & k \text{ sudé} \\ 1 & k \text{ liché} \end{cases}$$

$$\lim_{t \to 0-} = \begin{cases} 1 & k \text{ sudé} \\ -1 & k \text{ liché} \end{cases}$$

Závěr: $\frac{\partial f}{\partial y}$ v $[0, 2\pi]$ neexistuje

(4) Tečná rovina

$$\frac{\partial f}{\partial x}[1,0] = \frac{-1}{\sqrt{1-(1-\cos 0)^2}} = -1$$

$$\frac{\partial f}{\partial y}[1,0] = \frac{-\sin 0}{\sqrt{1-(1-\cos 0)^2}} = 0$$

$$f([1,0]) = \arccos(1-\cos 0) = \pi/2$$

$$\underline{\underline{z = \frac{\pi}{2} + (-1)(x-1)}}$$

$F: \mathbb{R}^2 \to \mathbb{R}^2$

$$F(x,y) = \begin{cases} \left((x^2+y^2)(e^x-1) \,,\, \dfrac{x^2}{x^2+y^2} \right) & [x,y] \neq [0,0] \\ (0,0) & [x,y] = [0,0] \end{cases}$$

$G: \mathbb{R}^2 \to \mathbb{R}$

$G(s,t) = st$

pro $\mathbb{R}^2 \setminus \{[0,0]\}$

- $\dfrac{\partial F_1}{\partial x} = 2x(e^x-1) + e^x(x^2+y^2)$ $\qquad \dfrac{\partial F_1}{\partial y} = 2y(e^x-1)$

$\dfrac{\partial F_2}{\partial x} = \dfrac{2x(x^2+y^2) - x^2 \cdot 2x}{(x^2+y^2)^2} = \dfrac{2xy^2}{(x^2+y^2)^2}$ $\qquad \dfrac{\partial F_2}{\partial y} = \dfrac{-x^2 \cdot 2y}{(x^2+y^2)^2}$

$\forall$ parc. der. spoj. na $\mathbb{R}^2 \setminus \{[0,0]\}$ $\to$ $\exists$ tot. dif.

v $[1,1]$:

$$J_{F_1} = \begin{pmatrix} 2(e-1)+e\cdot 2 = 4e-2 & 2(e-1) \\ \frac{2}{4} = \frac{1}{2} & \frac{-2}{4} = -\frac{1}{2} \end{pmatrix}$$

$$|J_{F_1}| = -\tfrac{1}{2}(4e-2) - \tfrac{1}{2}\cdot 2(e-1) = -2e+1 - e+1 = \underline{\underline{-3e+2}}$$

- $\dfrac{\partial(G(F))}{\partial x}(0,-3)$, $\dfrac{\partial(G(F))}{\partial y}(0,-3)$

der F v $(0,-3)$: $\begin{pmatrix} 9 & 0 \\ 0 & 0 \end{pmatrix}$

$(F(0,-3)) = (0,0)$

der G: $\dfrac{\partial G}{\partial s} = t$ $\qquad \dfrac{\partial G}{\partial t} = s$ $\qquad$ v $(0,0)$: $(0,0)$

$\to$ spojité, máme tot. dif.

tedy G∘F má matici

$$\begin{pmatrix} 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

tedy $\qquad \dfrac{\partial (G \circ F)}{\partial x} = \dfrac{\partial (G \circ F)}{\partial y} = \underline{\underline{0}}$

$$f(x,y) = \log \frac{1 - \sqrt[3]{x}}{|y| + \sqrt[3]{x}}$$

(1) D_f: $|y| + \sqrt[3]{x} \neq 0$

$|y| \neq -\sqrt[3]{x}$

problém jen pro $x \leq 0$

pak $y \neq \pm \sqrt[3]{x}$

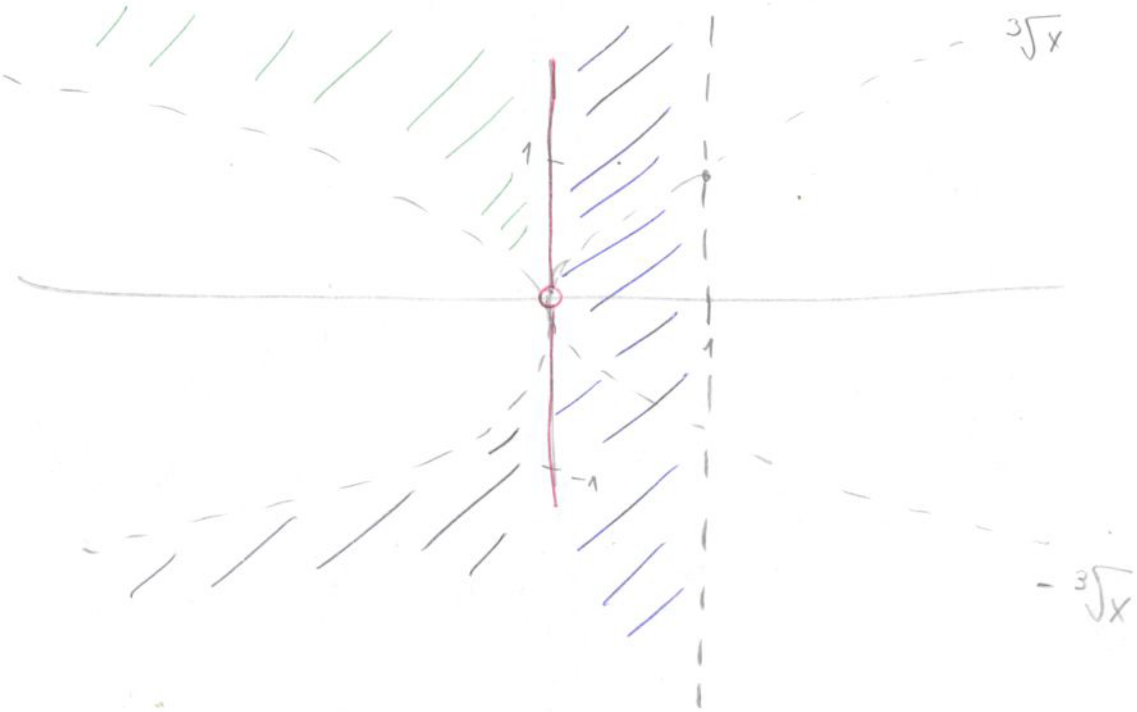
$$\frac{1-\sqrt[3]{x}}{|y|+\sqrt[3]{x}} > 0$$

$1-\sqrt[3]{x} > 0$ & $|y| + \sqrt[3]{x} > 0$

nebo $1-\sqrt[3]{x} < 0$ & $|y| + \sqrt[3]{x} < 0$

• $1 > x$ & $|y| > -\sqrt[3]{x}$

nebo • $1 < x$ $|y| < -\sqrt[3]{x}$ → nelze



dohromady

$D_f = \{[x,y] \in \mathbb{R}^2 : x \in (0,1)$
nebo $(x \leq 0$ &
$y \in (-\infty, \sqrt[3]{x}) \cup (-\sqrt[3]{x}, \infty))\}$

• $x \in (0,1)$ $|y| > -\sqrt[3]{x}$ vždy
nezáp záporné

• $x = 0$ $|y| > 0$ platí pro $y \neq 0$

$x \in (-\infty, 0)$ $|y| > -\sqrt[3]{x}$

• $y > 0$ $y > -\sqrt[3]{x}$

• $y < 0$ $-y > -\sqrt[3]{x}$

$\sqrt[3]{x} > y$

(2)

$$\frac{\partial f}{\partial x} = \frac{|y| + \sqrt[3]{x}}{1 - \sqrt[3]{x}} \cdot \frac{-\frac{1}{3}\frac{1}{\sqrt[3]{x^2}}(|y| + \sqrt[3]{x}) - (1 - \sqrt[3]{x}) \cdot \frac{+1}{3}\frac{1}{\sqrt[3]{x^2}}}{(|y| + \sqrt[3]{x})^2}$$

$$= \frac{-|y| - 1}{(|y| + \sqrt[3]{x})(1 - \sqrt[3]{x})\, 3\sqrt[3]{x^2}} \qquad \text{pro } D_f : x \neq 0,$$

$$\frac{\partial f}{\partial y} = \frac{|y| + \sqrt[3]{x}}{1 - \sqrt[3]{x}} \cdot \frac{-(1 - \sqrt[3]{x}) \cdot \operatorname{sgn}(y)}{(|y| + \sqrt[3]{x})^2}$$

$$= \frac{-\operatorname{sgn} y}{|y| + \sqrt[3]{x}} \qquad \text{pro } D_f : y \neq 0$$

(3) $\frac{\partial f}{\partial x}$ pro $x = 0$, $y \in \mathbb{R} \setminus \{0\}$ z definice

$$\frac{\partial f}{\partial x}(0, y) = \lim_{t \to 0} \frac{\ln \frac{1 - \sqrt[3]{0+t}}{|y| + \sqrt[3]{0+t}} - \ln \frac{1 - 0}{|y| + 0}}{t}$$

$$= \lim_{t \to 0} \frac{1}{t} \cdot \ln \frac{\frac{1 - \sqrt[3]{t}}{|y| + \sqrt[3]{t}}}{\frac{1}{|y|}} =$$

$$= \lim_{t \to 0} \frac{1}{t} \ln \frac{(1 - \sqrt[3]{t})|y|}{|y| + \sqrt[3]{t}} =$$

$$\overset{L'H}{=} \lim_{t \to 0} \frac{|y| + \sqrt[3]{t}}{1 - \sqrt[3]{t}} \cdot \frac{-\frac{1}{3}\frac{1}{\sqrt[3]{t^2}}(|y| + \sqrt[3]{t}) - (1 - \sqrt[3]{t})\frac{1}{3}\frac{1}{\sqrt[3]{t^2}}}{(|y| + \sqrt[3]{t})^2} =$$

$$= \lim_{t \to 0} \frac{1}{|y| + \sqrt[3]{t}} \cdot \frac{1}{1 - \sqrt[3]{t}} \underbrace{\frac{1}{\sqrt[3]{t^2}}}_{\frac{1}{0^+}} \cdot (-|y| - 1) = -(|y| + 1) \cdot \infty$$

$$= \underline{\underline{-\infty}}$$

$\frac{\partial f}{\partial y}$ pro $y=0$, $x \in (0,1)$

$$\frac{\partial f}{\partial y}(x,0) = \lim_{t\to 0} \frac{\ln \frac{1-\sqrt[3]{x}}{|t|+\sqrt[3]{x}} - \ln \frac{1-\sqrt[3]{x}}{\sqrt[3]{x}}}{t}$$

$$= \lim_{t\to 0} \frac{\ln \frac{\sqrt[3]{x}}{|t|+\sqrt[3]{x}}}{t} = \lim_{t\to 0} \underbrace{\frac{\ln \frac{\sqrt[3]{x}}{|t|+\sqrt[3]{x}}}{\frac{\sqrt[3]{x}}{|t|+\sqrt[3]{x}} - 1}}_{\to 1} \cdot \underbrace{\frac{\frac{\sqrt[3]{x}-|t|-\sqrt[3]{x}}{|t|+\sqrt[3]{x}}}{t}}_{\frac{-|t|}{t\,(|t|+\sqrt[3]{x})}}$$

$$\Rightarrow \begin{cases} \lim_{t\to 0-} = +1 \cdot \frac{1}{\sqrt[3]{x}} \\ \lim_{t\to 0+} = -1 \cdot \frac{1}{\sqrt[3]{x}} \end{cases}$$

(4) tečna

$$f((-1,3)) = \ln \frac{2}{3-1} = 0$$

$$\frac{\partial f}{\partial x}((-1,3)) = \frac{-4}{+3 \cdot 2 \cdot 2} = -1/3$$

$$\frac{\partial f}{\partial y}((-1,3)) = \frac{-1}{3-1} = -1/2$$

$$z = 0 + {}^{-}1/3\,(x+1) - 1/2\,(y-3)$$