

$$(1a) \quad y' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} y + \begin{pmatrix} -x^2 \\ 2x \end{pmatrix}$$

$$\begin{pmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{pmatrix} \sim \begin{pmatrix} 1-\lambda & -1 \\ 0 & -1+(3-\lambda)(\lambda-1) \end{pmatrix} \sim \begin{pmatrix} 1-\lambda & -1 \\ 0 & \underbrace{-\lambda^2+4\lambda-4}_{x^2+2x-2} \end{pmatrix}$$

$$y_2: \quad \begin{aligned} \lambda^2 - 4\lambda + 4 &= 0 \\ (\lambda - 2)^2 &= 0 \\ \lambda &= 2 \end{aligned}$$

$$y_{2,H} = c_1 e^{2x} + c_2 x e^{2x}$$

$0+0i$ reiner L鰏en

$$\rightarrow y_{2p} = Ax^2 + Bx + C$$

$$-y_{2p}'' + 4y_{2p}' - 4y_{2p} = x^2 + 2x - 2$$

$$-4Ax^2 + x(8A - 4B) + 4B - 4C - 2A = x^2 + 2x - 2$$

$$\underline{A = -\frac{1}{4}}$$

$$-2 - 4B = 2$$

$$\underline{B = -1}$$

$$-4 - 4C + \frac{1}{2} = -2$$

$$-4C = \frac{3}{2}$$

$$\underline{C = -\frac{3}{8}}$$

$$y_2 = c_1 e^{2x} + c_2 x e^{2x} - \frac{x^2}{4} - x - \frac{3}{8}$$

$$y_1 = -2x - (3-\lambda)y_2$$

$$y_1 = -2x - 3y_2 + y_2'$$

$$y_1 = -c_1 e^{2x} - c_2 e^{2x} \cdot x + c_2 e^{2x} + \frac{3}{4}x^2 + \frac{x}{2} + \frac{1}{8}$$

$$x_i, c_{1,2} \in \mathbb{R}$$

Od druhého řádku odečteme třetí řádek a poté od třetího řádku odečteme $(\lambda - 4)$ -násobek druhého řádku

$$\sim \begin{pmatrix} \lambda - 2 & -1 & 1 \\ \lambda^2 + 2 & 1 & 0 \\ -\lambda^2 - \lambda - 7 & \lambda - 4 & 0 \end{pmatrix} \sim \begin{pmatrix} \lambda - 2 & -1 & 1 \\ \lambda^2 + 2 & 1 & 0 \\ -\lambda^3 + 3\lambda^2 - 3\lambda + 1 & 0 & 0 \end{pmatrix}$$

V posledním řádku máme nyní rovnici $-x''' + 3x'' - 3x' + x = 0$, v matici je tedy rovnou charakteristický polynom této rovnice. Dostáváme tedy řešení

$$x(t) = c_1 e^t + c_2 t e^t + c_3 t^2 e^t, \quad t \in \mathbb{R}, c_1, c_2, c_3 \in \mathbb{R}.$$

Z druhého řádku matice dopočítáme y :

$$y(t) = -x'' - 2x = -e^t(3c_1 + 2c_2 + 2c_3 + t(3c_2 + 4c_3) + t^2 \cdot 3c_3), \quad t \in \mathbb{R}.$$

Z první rovnice dopočítáme z :

$$z(t) = e^t(-2c_1 - 3c_2 - 2c_3 + t(-2c_2 - 6c_3) + t^2 \cdot (-2c_3)), \quad t \in \mathbb{R}.$$

Příklad 3. Řešte soustavu

$$\begin{aligned} 2y'' + 3z'' - 7y - 6z &= t + 1 \\ 4y'' + 3z'' - 4y - 3z &= 2t. \end{aligned}$$

Řešení. Napíšeme si λ -matici s pravou stranou:

$$\begin{aligned} &\begin{pmatrix} 2\lambda^2 - 7 & 3\lambda^2 - 6 & | & t + 1 \\ 4\lambda^2 - 4 & 3\lambda^2 - 3 & | & 2t \end{pmatrix} \sim \begin{pmatrix} 2\lambda^2 - 7 & 3\lambda^2 - 6 & | & t + 1 \\ 2\lambda^2 + 3 & 3 & | & t - 1 \end{pmatrix} \\ &\sim \begin{pmatrix} 2\lambda^2 - 7 - (\lambda^2 - 2)(2\lambda^2 + 3) & 0 & | & t + 1 - (t - 1)'' + 2(t - 1) \\ 2\lambda^2 + 3 & 3 & | & t - 1 \end{pmatrix} \\ &= \begin{pmatrix} -2\lambda^4 + 3\lambda^2 - 1 & 0 & | & 3t - 1 \\ 2\lambda^2 + 3 & 3 & | & t - 1 \end{pmatrix}. \end{aligned}$$

Řešení charakteristického polynomu v prvním řádku jsou $\lambda_{1,2} = \pm 1$, $\lambda_{3,4} = \pm \sqrt{2}/2$. Tedy řešení homogenní rovnice

$$-2y^{(4)} + 3y'' - y = 0$$

jsou

$$ae^t + be^{-t} + ce^{\sqrt{2}/2t} + de^{-\sqrt{2}/2t}.$$

Partikulární řešení nehomogenní rovnice bude ve tvaru $y_p(t) = rt + s$, což po dosazení do rovnice dává $y_p(t) = -3t + 1$. Tedy

$$y(t) = ae^t + be^{-t} + ce^{\sqrt{2}/2t} + de^{-\sqrt{2}/2t} - 3t + 1.$$

Z druhé rovnice pak máme

$$\begin{aligned} 3z(t) &= t - 1 - 2y'' - 3y = t - 1 - 2\left(ae^t + be^{-t} + \frac{1}{2}ce^{\sqrt{2}/2t} + \frac{1}{2}de^{-\sqrt{2}/2t}\right) \\ &\quad - 3\left(ae^t + be^{-t} + ce^{\sqrt{2}/2t} + de^{-\sqrt{2}/2t} - 3t + 1\right) \\ &= 10t - 4 - 5ae^t - 5be^{-t} - 4ce^{\sqrt{2}/2t} - 4de^{-\sqrt{2}/2t}. \end{aligned}$$

$$(2b) \quad y_1' = y_2 + \sin x$$

$$y_2' = -y_1 + \cos x$$

$$\cdot (\rightarrow) \begin{pmatrix} 1 & -1 & | & \sin x \\ 1 & 1 & | & \cos x \end{pmatrix} \sim \begin{pmatrix} 0 & -1-\lambda^2 & | & \sin x - \lambda \cos x \\ 1 & 1 & | & \cos x \end{pmatrix}$$

$$-y_2 - y_2'' = \frac{\sin x - (\cos x)'}{2 \sin x} \quad y_{2H} = C_1 \cos x + C_2 \sin x$$

$$-1 - \lambda^2 = 0 \quad \lambda = \pm i$$

$$-1 = \lambda^2$$

$$\text{PS: } 2 \sin x = e^{0x} (0 \cos 1x + 2 \sin 1x)$$

i' je korjen

$$y_{2p} = x (A \cos x + B \sin x)$$

Imaginir

$$2A \sin x - 2B \cos x = 2 \sin x$$

$$B = 0 \quad A = 1$$

$$y_{2p} = C_1 \cos x + C_2 \sin x + \underline{x \cos x}$$

$$y_1 = \cos x - y_2'$$

$$y_1 = x \sin x + \underline{C_1 \sin x - C_2 \cos x}$$

$$C_1, C_2 \in \mathbb{R}$$

(2c)

$$y_1' = y_1 + 2y_2 + \sin x$$

$$y_2' = -y_1 - y_2$$

$$y(0) = (0, 0)$$

$$\rightarrow \left(\begin{array}{cc|c} 1-\lambda & 2 & -\sin x \\ -1 & -1-\lambda & 0 \end{array} \right) \sim \left(\begin{array}{cc|c} 0 & 2+(-1)(1+\lambda)(1-\lambda) & -\sin x \\ 1 & 1+\lambda & 0 \end{array} \right)$$

$$\left(\begin{array}{cc|c} 0 & \lambda^2 + 1 & -\sin x \\ 1 & 1+\lambda & 0 \end{array} \right)$$

$$y_2: \quad \lambda^2 + 1 = 0 \\ \lambda = \pm i$$

$$y_{2,h} = c_1 \cos x + c_2 \sin x$$

Spec-PS: $-\sin x = e^{0x} (0 \cos 1x - 1 \sin 1x)$
 $0 + 1i$ je \leftarrow unabh. L_{inh}

$$y_{sp} = x^1 (A \cos x + B \sin x)$$

$$y_{sp}'' + y_{sp}' = -\sin x$$

$$y_2 = c_1 \cos x + c_2 \sin x + \frac{1}{2} x \cos x$$

$$-2A \sin x + 2B \cos x = -\sin x \\ \rightarrow B = 0 \quad A = \frac{1}{2}$$

part $y_1 = -y_2 - y_2'$

$$y_1 = (-c_2 + c_1) \sin x + (-c_1 - c_2) \cos x + \frac{1}{2} (x \cos x + \cos x - x \sin x)$$

Partic. $0 = c_1 \quad c_2 = -\frac{1}{2}$

$$0 = -c_2 - \frac{1}{2}$$

allgemein $y_1 = \frac{1}{2} \sin x + \frac{1}{2} \cos x - \frac{1}{2} (x \cos x + \cos x - x \sin x)$

$$y_2 = -\frac{1}{2} \sin x + \frac{1}{2} x \cos x$$

$x \in \mathbb{R}$

(2d)

$$u' = 4u + 3v - 3w$$

$$v' = -3u - 2v + 3w$$

$$w' = 3u + 3v - 2w + 2e^{-x}$$

$$\begin{array}{l} \text{3.} \\ + \\ (4-2) \end{array} \begin{array}{c} u \quad v \quad w \\ \left(\begin{array}{ccc|c} 4-\lambda & 3 & -3 & 0 \\ -3 & -2-\lambda & 3 & 0 \\ 3 & 3 & -2-\lambda & -2e^{-x} \end{array} \right) \sim$$

$$\sim \begin{array}{c} \left(\begin{array}{ccc|c} 4-\lambda & 3 & -3 & 0 \\ 0 & 4-(2+\lambda)(4-\lambda) & -9+12-3\lambda & 0 \\ 0 & 1-\lambda & 1-\lambda & -2e^{-x} \end{array} \right) \begin{array}{l} \sim \\ \cdot (-3) \end{array}$$

$$\sim \begin{array}{c} \left(\begin{array}{ccc|c} 4-\lambda & 3 & -3 & 0 \\ 0 & (\lambda-1)^2 & 3(1-\lambda) & 0 \\ 0 & (\lambda-1)^2 - 3(1-\lambda) & 0 & 6e^{-x} \end{array} \right) \sim \begin{array}{c} \left(\begin{array}{ccc|c} 4-\lambda & 3 & -3 & 0 \\ 0 & (\lambda-1)^2 & 3(1-\lambda) & 0 \\ 0 & \underbrace{(\lambda-1)(\lambda+2)}_{\lambda^2 + \lambda - 2} & 0 & 6e^{-x} \end{array} \right)$$

$$v: (\lambda-1)(\lambda+2) = 0 \\ \lambda_1 = 1 \quad \lambda_2 = -2$$

$$w_4 = c_1 e^x + c_2 e^{-2x}$$

$$p.s.: 6e^{-x} = e^{-x}(6 \cos 0x + 0 \sin 0x)$$

-1+0i neue L\u00f6sungen

$$v_p = A e^{-x}$$

$$v'' + v' - 2v = 6e^{-x}$$

$$-2A e^{-x} = 6e^{-x} \quad A = -3$$

$$v = \underline{c_1 e^x + c_2 e^{-2x}} - 3e^{-x}$$

$$w: 3w - 3w' = -v'' + 2v' - v$$

$$-w' + w = \frac{1}{3}(12e^{-x} - 9c_2 e^{-2x})$$

variabel konstant $w_4 = k e^x$

$$-k' e^x - k e^x + k e^x = 4e^{-x} - 3c_2 e^{-2x}$$

$$k' = -e^{-x}(4e^{-x} - 3c_2 e^{-2x})$$

$$k = e^{-3x}(2e^x - c_2) + c_3$$

$$w = \underline{2e^{-x} - c_2 e^{-2x} + c_3 e^x}$$

$$u: \quad -3u = (2 + \lambda) v - 3w$$

$$u = -\frac{1}{3}(2v + v' - 3w)$$

$$u = \underline{\underline{-e^x(c_1 - c_3) + e^{2x}(-c_2) + 3e^{-x}}}$$

$$x_1, c_3 \in \mathbb{R}$$

(2e)

$$u' = u + v + z + w$$

$$v' = u + v + z + w + 1$$

$$z' = u + v + z + w + 2$$

$$w' = u + v + z + w + 3$$

$$\begin{pmatrix} 1-\lambda & 1 & 1 & 1 & 0 \\ 1 & 1-\lambda & 1 & 1 & -1 \\ 1 & 1 & 1-\lambda & 1 & -2 \\ 1 & 1 & 1 & 1-\lambda & -3 \end{pmatrix} \sim \begin{pmatrix} 1-\lambda & 1 & 1 & 1 & 0 \\ 0 & (1-\lambda)(\lambda-1)+1 & (1-\lambda)(\lambda-1) & (1-\lambda) & -(1-\lambda) \\ 0 & \lambda & -\lambda & 0 & -1 \\ 0 & 0 & \lambda & -\lambda & -1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1-\lambda & 1 & 1 & 1 & 0 \\ 0 & -\lambda^2+2\lambda & \lambda & \lambda & 1-\lambda \\ 0 & 0 & -\lambda^2+2\lambda+1 & -\lambda & -\lambda+1 & -\lambda+2 \\ 0 & 0 & \lambda & -\lambda & -1 \end{pmatrix}$$

$$\sim \begin{pmatrix} u & v & z & w & \\ 1-\lambda & 1 & 1 & 1 & 0 \\ 0 & -\lambda^2+2\lambda & \lambda & \lambda & 1-\lambda \\ 0 & 0 & -\lambda^2+2\lambda+1 & -\lambda & -2\lambda+3 \\ 0 & 0 & -\lambda^2+4\lambda-3 & 0 & -2\lambda+2 \end{pmatrix}$$

das charakteristische

$$-\lambda^4 + 4\lambda^3 = +12$$

$$+\lambda(-\lambda+4) = 0$$

$$\lambda_1 = 0 \quad \lambda_2 = 4$$

$$z_4 = c_1 + c_2 e^{4x}$$

$$-1 = e^{0x} (-1 \cos 0x + 0 \sin 0x) + 0 + 0i \text{ für } \lambda = 0$$

$$z_p = Ax$$

$$z_p' = A$$

$$z_p'' = 0$$

$$4A = +12 \quad A = \frac{1}{2}$$

$$z = c_1 + c_2 e^{4x} + \frac{x}{2}$$

$$\Delta z - \Delta w = -1$$

$$4c_2 e^{4x} + \frac{1}{2} + 1 = w'$$

$$w' = 4c_2 e^{4x} + \frac{3}{2}$$

$$w = \underline{c_2 e^{4x} + \frac{3}{2}x + c_3}$$

$$\Delta v - \Delta z = -1$$

$$v' = -1 + z'$$
$$v' = 4c_2 e^{4x} - \frac{1}{2}$$

$$v = \underline{c_2 e^{4x} - \frac{1}{2}x + c_4}$$

$$u - u' = -v - z - w$$

$$-u' + u = -c_2 e^{4x} + \frac{1}{2}x - c_4 - c_1 - c_2 e^{4x} - \frac{x}{2} - c_2 e^{4x} - \frac{3}{2}x - c_3$$

$$-u' + u = -3c_2 e^{4x} - \frac{3}{2}x - (c_1 + c_3 + c_4)$$

integr. faktor

$$u' - u = c_2 e^{4x} + \frac{3x}{2} + (c_1 + c_3 + c_4) \cdot e^{-x}$$

$$(u e^{-x})' = 3c_2 e^{3x} + \frac{3}{2}x e^{-x} + (c_1 + c_3) e^{-x}$$

$$u e^{-x} = \underline{c_2 e^{3x}} - (c_1 + c_3 + c_4) e^{-x} - \frac{3}{2} e^{-x} x - \frac{3}{2} e^{-x}$$

$$u = \underline{c_2 e^{4x} - \frac{3}{2}x + \frac{3}{2} - c_1 - c_3 - c_4}$$