(1)

$$\int_0^\infty x \arctan e^{-x}\, dx \overset{*}{=} \sum_{k=0}^\infty \int_0^\infty (-1)^k \frac{x}{2k+1} e^{-(2k+1)x}\, dx =$$

$$\sum_{k=0}^\infty (-1)^k \int_0^\infty \frac{1}{(2k+1)^2} e^{-(2k+1)x}\, dx = \sum_{k=0}^\infty (-1)^k \frac{1}{(2k+1)^3}$$

$\Downarrow$

per partes

(*) Odůvodnění!

$$\sum \int |\ | = \sum_{k=0}^\infty \frac{1}{(2k+1)^3} < \infty$$

(2) $$I(a,b) = \int_0^\infty \frac{(\arctan bx)^2 - (\arctan ax)^2}{x}\, dx$$

- konverguje pro
  - $a = b$
  - $a \in \mathbb{R}$, $b \neq 0$
  - $a \neq 0$ $b \in \mathbb{R}$

pro $a, b > 0$, $a < b$

- $$\int_0^\infty \frac{\arctan^2(bx) - \arctan^2(ax)}{x}\, dx =$$

$$= \int_0^\infty \left[\frac{\arctan^2 yx}{x} \right]_a^b dx = \int_0^\infty \int_a^b \frac{2 \arctan yx}{1 + y^2 x^2}\, dy\, dx =$$

$$\overset{(*)}{=} \int_a^b \int_0^\infty 2 \frac{\arctan yx}{1 + y^2 x^2}\, dx\, dy = \int_a^b \left[\frac{\arctan^2 yx}{y} \right]_0^\infty dy = \frac{\pi^2}{4} \int_a^b \frac{1}{y}\, dy$$

$$= \frac{\pi^2}{4} \ln|b| - \ln|a|$$

- $a, b > 0$ $a < b$

$$\int_0^\infty - \left[\frac{\arctan^2 yx}{x} \right]_b^a = \dots = \frac{\pi^2}{4} \ln\left|\frac{b}{a}\right|$$

- Funkce $I(a,b)$ je sudá v obou proměnných, tedy $I(a,b) = \frac{\pi^2}{4} \ln\left|\frac{b}{a}\right|$ kdykoli $a \neq 0$, $b \neq 0$
- pro $a = 0 = b$ rovnou máme $I(0,0) = 0$.

(*) $f(x,y) \geq 0$; $M = (a,b) \times (0,\infty)$ je obdélník, $\to$ měřitelná

(3) $$\int_M \frac{1}{\sqrt{x^2+y^2}}\, d\lambda \qquad M: \quad x^2+y^2 < xz, \quad x^2+y^2 < y, \quad 0 < z < 1$$

válcové souřadnice

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = z$$

$$\left.\begin{array}{l} 0 < r^2 < z \cdot r\cos\varphi \\ 0 < r^2 < r\sin\varphi \end{array}\right\} \quad \alpha \in (0, \pi/2)$$

$$0 < z < 1$$

$$r < \min(z\cos\varphi, \sin\varphi)$$

(1) $\sin\alpha < z\cos\alpha \quad \rightarrow \quad z > \operatorname{tg}\alpha$ neb $1 > z > \operatorname{tg}\alpha$

$\rightarrow \alpha \in (0, \pi/4)$

$$\int_0^{\pi/4}\int_{\operatorname{tg}\alpha}^{1}\int_0^{\sin\alpha} \frac{r}{\sqrt{r^2}}\, dr\, dz\, d\alpha = \int_0^{\pi/4}\int_{\operatorname{tg}\alpha}^{1}\int_0^{\sin\alpha} 1\, dr\, dz\, d\alpha =$$

$$= \int_0^{\pi/4}\int_{\operatorname{tg}\alpha}^{1} \sin\alpha\, dz\, d\alpha = \int_0^{\pi/4} \sin\alpha\,(1-\operatorname{tg}\alpha)\, d\alpha =$$

$$= \int_0^{\pi/4} \sin\alpha - \frac{\sin^2\alpha}{\cos\alpha}\cdot\frac{\cos\alpha}{\cos\alpha}\, d\alpha = \left[-\cos\alpha\right]_0^{\pi/4} - \int_0^{\pi/4} \frac{\sin^2\alpha}{1-\sin^2\alpha}\cdot\cos\alpha\, d\alpha$$

$\sin\alpha = u$

$\cos\alpha = du$

$$= \left(-\frac{\sqrt{2}}{2}+1\right) - \int_0^{\sqrt{2}/2} \frac{u^2}{1-u^2}\, du = 1-\frac{\sqrt{2}}{2} + \int_0^{\sqrt{2}/2} \frac{1-u^2}{1-u^2} + \frac{-1}{1-u^2}\, du$$

$$= 1-\frac{\sqrt{2}}{2} + \left[u - \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right|\right]_0^{\sqrt{2}/2} = 1 - \frac{1}{2}\ln\left|\frac{1+\sqrt{2}/2}{1-\sqrt{2}/2}\right|$$

(2) $\sin\alpha > z\cos\alpha \to \operatorname{tg}\alpha > z$ neb $0 < z < 1$

$$\int_0^{\pi/4}\int_0^{\operatorname{tg}\alpha}\int_0^{z\cos\alpha} 1 \, dr\, dz\, d\alpha = \int_0^{\pi/4}\int_0^{\operatorname{tg}\alpha} z\cos\alpha \, dz\, d\alpha = \int_0^{\pi/4} \cos\alpha \frac{1}{2}\left[z^2\right]_0^{\operatorname{tg}\alpha} d\alpha$$

$$= \int_0^{\pi/4} \frac{1}{2}\cos\alpha \frac{\sin^2\alpha}{\cos^2\alpha} d\alpha = \int_0^{\sqrt{2}/2} \frac{1}{2}\frac{u^2}{1-u^2} du = \frac{1}{2}\cdot\left[-u + \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|\right]_0^{\sqrt{2}/2}$$

$\sin\alpha = u$

$\cos\alpha = du$

$$= \frac{1}{2}\left(-\sqrt{2}/2 + \frac{1}{2}\ln\left|\frac{1+\sqrt{2}/2}{1-\sqrt{2}/2}\right|\right)$$

(3) když $\operatorname{tg}\alpha > 1$, aplikujeme $z < 1$

$$\int_{\pi/4}^{\pi/2}\int_0^{1}\int_0^{z\cos\alpha} 1 \, dr\, dz\, d\alpha = \int_{\pi/4}^{\pi/2}\int_0^{1} z\cos\alpha \, dz\, d\alpha = \int_{\pi/4}^{\pi/2} \cos\alpha \frac{1}{2} \, d\alpha$$

$$= \frac{1}{2}\left[\sin\alpha\right]_{\pi/4}^{\pi/2} = \underline{\frac{1}{2}\left(1 - \frac{\sqrt{2}}{2}\right)}$$

(3) jiná varianta (vede na horší $\int$)

$z \in (0,1) \qquad \alpha \in (0, \pi/2) \qquad r < \min\{\cos\alpha \cdot z, \sin\alpha\}$

(1) $\cos\alpha \, z > \sin\alpha$

$\arctan z > \alpha \quad \to \quad \alpha \in (\quad , \quad)$

$$\int_0^1 \int_0^{\arctan z} \int_0^{\sin\alpha} 1 \, dr \, d\alpha \, dz$$

(2) $\sin\alpha > z\cos\alpha$

$\alpha > \arctan z$

$$\int_0^1 \int_{\arctan z}^{\pi/2} \int_0^{z\cos\alpha} 1 \, dr \, d\alpha \, dz$$

Výpočet: $$\int_0^1 \int_0^{\arctan z} \int_0^{\sin\alpha} 1 \, dr \, d\alpha \, dz = \int_0^1 \int_0^{\arctan z} \sin\alpha \, d\alpha \, dz =$$

$$= \int_0^1 \left[-\cos\alpha\right]_0^{\arctan z} dz = \int_0^1 -\cos(\arctan z) + 1 \, dz = 1 - \int_0^1 \cos \arctan z \, dz$$

$t = \arctan z$

$\tan t = z \qquad \frac{1}{\cos^2 t} dt = dz$

$$= 1 - \int_0^{\pi/4} \cos t \cdot \frac{1}{\cos^2 t} dt = 1 - \int_0^{\pi/4} \frac{\cos t}{1 - \sin^2 t} dt = 1 - \int_0^{\sqrt{2}/2} \frac{1}{1-u^2} du$$

$\sin t = u$

$\cos t \, dt = du$

$$= 1 - \left[\frac{1}{2} \ln\left|\frac{1+u}{1-u}\right|\right]_0^{\sqrt{2}/2} = 1 - \frac{1}{2} \ln\left|\frac{1+\sqrt{2}/2}{1-\sqrt{2}/2}\right|$$

$$\int_0^1 \int_{\arctan z}^{\pi/2} \int_0^{z\cos\alpha} 1 \, dr\, d\alpha\, dz = \int_0^1 \int_{\arctan z}^{\pi/2} z\cos\alpha \quad d\alpha\, dz =$$

$$= \int_0^1 z \Big[\sin\alpha \Big]_{\arctan z}^{\pi/2} dz = \int_0^1 z\Big(1 - \sin(\arctan z)\Big) dz =$$

$$= \int_0^1 z - z \sin\arctan z \, dz = \frac{1}{2} - \int_0^1 z \sin(\arctan z)\, dz$$

$$t = \arctan z$$

$$\operatorname{tg} t = z$$

$$\frac{1}{\cos^2 t} dt = dz$$

$$= \frac{1}{2} - \int_0^{\pi/4} \frac{\sin t}{\cos t} \cdot \frac{\sin t}{\cos^2 t} dt = \frac{1}{2} - \int_0^{\pi/4} \frac{\sin^2 t}{(1-\sin^2 t)^2} \cdot \cos t \, dt$$

$$u = \sin t$$

$$du = \cos t \, dt$$

$$= \frac{1}{2} - \int_0^{\sqrt{2}/2} \frac{u^2}{(1-u^2)^2} du = \frac{1}{2} - \int_0^{\sqrt{2}/2} -\frac{1}{4(u+1)} + \frac{1}{4(u+1)^2} + \frac{1}{4(u-1)} + \frac{1}{4(u-1)^2}$$

$$= \frac{1}{2} - \Big[-\frac{1}{4}\ln(u+1) + \frac{1}{4}\ln|u-1| + \frac{1}{4}\frac{-1}{(u+1)} + \frac{1}{4}\frac{-1}{(u-1)} \Big]_0^{\sqrt{2}/2}$$

$$= \frac{1}{2} - \Bigg(- \underbrace{\left(\frac{1}{4}\frac{-1}{1} + \frac{1}{4}\frac{-1}{-1} \right)}_{0} + -\frac{1}{4}\underbrace{\left(\frac{1}{\frac{\sqrt{2}}{2}+1} + \frac{1}{\frac{\sqrt{2}}{2}-1} \right)}_{-2\sqrt{2}} - \frac{1}{4}\ln\left|\frac{1+\sqrt{2}/2}{\frac{\sqrt{2}}{2}-1}\right| \Bigg)$$

$$= \frac{1}{2} - \frac{\sqrt{2}}{2} + \frac{1}{4}\ln\left|\frac{1+\sqrt{2}/2}{1-\sqrt{2}/2}\right|$$