Opravy (1)

$$F(a) = \int_0^1 \operatorname{sgn}(x-a)\, dx$$

je $F(a)$ spojitá pro $a \in \mathbb{R}$?

Zvolme pevné $a_0 \in \mathbb{R}$, pak ověření věty

(Sp-1) Pro s.v. $x \in (0,1)$ je $f(\cdot, x)$ spojitá v a_0 Ano

s výjimkou $x = a_0$, vyjde spojitost v a_0

(Sp-2) $\forall a \in \mathbb{R}$ je fce $f(a, \cdot)$ měřitelná (Ano, z definice)

(Sp-3) Majoranta

$$|\operatorname{sgn}(x-a)| \le 1 \in \mathcal{L}^1(0,1)$$

Tedy $F(a)$ je spojitá

Př 25.1.

- $$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x}\, dx$$

$s \in (0,\infty)$

substituce $x = r^2$ $dx = 2r\,dr$

$$\Gamma(s) = 2\int_0^\infty r^{2s-1} e^{-r^2} dr$$

- $$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\, dx$$

$p,q>0$ subst. $x = \cos^2\alpha$

$$B(p,q) = 2\int_0^{\pi/2} \cos^{2p-1}\alpha \ \sin^{2q-1}\alpha \, d\alpha$$

- $$\Gamma(p)\Gamma(q) = 4\int_{x>0,\, y>0} x^{2p-1} y^{2q-1} e^{-(x^2+y^2)}\, dx\,dy$$

pol. souř. $$= 4\int_{r>0,\ 0<\alpha<\frac{\pi}{2}} r^{2p+2q-1} e^{-r^2} \cos^{2p-1}\alpha \ \sin^{2q-1}\alpha \ d\alpha\, dr$$

$$= \Gamma(p+q)\, B(p,q).$$

$$F(a) = \int_0^\infty e^{-x^2} \cos 2ax \, dx$$

$\int$ konv $\forall a \in \mathbb{R}$ $\left(< \int_0^\infty e^{-x^2} dx < \infty \right)$

$$F'(a) = -2 \int_0^\infty \underbrace{e^{-x^2} \cdot x}_{u'} \cdot \underbrace{\sin 2ax}_{v} \, dx$$

$u = -\frac{1}{2} e^{-x^2}$ $\quad v' = 2a \cos 2ax$

podmínky za DÚ, majoranta $g(x) = xe^{-x^2}$

Per partes

$$F'(a) = -2 \underbrace{\left[-\frac{1}{2} e^{-x^2} \sin 2ax \right]_0^\infty}_{0} + 2 \int_0^\infty -\frac{1}{2} e^{-x^2} \cdot 2a \cos 2ax \, dx$$

$$= -2a \int_0^\infty e^{-x^2} \cos 2ax \, dx = -2a \, F(a)$$

tedy $F'(a) = -2aF(a)$

$$y' = -2xy$$

$$\frac{y'}{y} = -2x$$

$$\int \frac{1}{y} dy = \int -2x \, dx$$

$$\ln |y| = -x^2 + c$$

$$|y| = e^c e^{-x^2}$$

$$y = k e^{-x^2}$$

$k \in \mathbb{R}$

$\rightsquigarrow$ $F(a) = k \cdot e^{-a^2}$

z podm $a = 0$ $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

Celkem

$$F(a) = \frac{\sqrt{\pi}}{2} e^{-a^2}$$

$$\int_0^1 \frac{\ln x}{1-x}\,dx = \int_0^1 \ln x \sum_{n=0}^{\infty} x^n \,dx \overset{*}{=} \sum_{n=0}^{\infty} \int_0^1 x^n \ln x \,dx$$

$$\int_0^1 \underset{v'}{x^n} \underset{\substack{\downarrow \\ u}}{\ln x} = \underbrace{\left[\frac{x^{n+1}}{n+1} \ln x \right]_0^1}_{0-0} - \int_0^1 \frac{x^n}{n+1}\,dx$$

$v = \frac{x^{n+1}}{n+1} \quad u' = \frac{1}{x}$

$$= - \left[\frac{x^{n+1}}{(n+1)^2} \right]_0^1$$

$$= - \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = - \frac{\pi^2}{6}$$

TODO

$$F(a,b) = \int_0^\infty \frac{e^{-ax^2} - e^{-bx^2}}{x^2}\,dx$$

- konverguje pro $a = b$ a pro $a, b \geq 0$
- fixujme $b > 0$ pevné, pro $a \in (0,\infty)$ pak

$$\frac{\partial F}{\partial a}(a,b) = -\int_0^\infty e^{-ax^2}\,dx = -\frac{1}{2}\sqrt{\frac{\pi}{a}}$$

majoranta $G(x) = e^{-px^2}$ pro $a \in [p,\infty)$, $p > 0$, $p < b$

(ostatní podm. DÚ)

- pak $F(a,b) = -\sqrt{\pi}\sqrt{a} + c(b)$

dosadíme

$$\left.\begin{aligned} F(b,b) &= 0 \\ F(b,b) &= -\sqrt{\pi b} + c(b) \end{aligned}\right\} \quad F(a,b) = -\sqrt{\pi a} + \sqrt{\pi b}$$

toto máme pro $a, b > 0$

- Co když $a > 0$, $b \geq 0$:

Ukážeme, že $F(a,b)$ je spojitá v $[a, 0+]$:

$$f_a(b,x) = \frac{e^{-ax^2} - e^{-bx^2}}{x^2}, \quad x \in (0,\infty)$$

(Sp-1) $\forall x \in (0,\infty)$ je $f_a(\cdot, x)$ spoj v $0+$ (zjevně spoj.)

(Sp-2) $\forall b \in [0,\infty)$ je $f_a(b,\cdot)$ měřitelná $\Leftarrow$ je spoj v x

(Sp-3) majoranta

$$\frac{e^{-ax^2} - e^{-bx^2}}{x^2} \leq \frac{1}{x^2} \quad \text{pro } x \geq 1$$

$$\frac{e^{-ax^2} - 1}{x^2} + \frac{1 - e^{-bx^2}}{x^2} \leq |a| + |b| \quad \text{pro } x \in [0,1]$$

tedy $F(a,b)$ je spoj v $[a, 0+]$ a tedy $F(a,0] = -\sqrt{\pi}\sqrt{a} + 0 = -\sqrt{\pi a}$

- Situace $a \geq 0$ $b > 0$ je symetrická

$$\int_0^\infty e^{-x}\cos\sqrt{x}\,dx \qquad \cos y = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{y^{2n}}{(2n)!} \qquad y \in \mathbb{R}$$

$$\int_0^\infty \sum_{n=0}^{\infty} (-1)^n e^{-x} \frac{x^n}{(2n)!}\,dx \overset{(*)}{=} \sum_{n=0}^{\infty} (-1)^n \int_0^\infty e^{-x} \frac{x^n}{(2n)!}\,dx =$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!}\, n! \qquad \text{neb} \quad \int_0^\infty e^{-x} x^n = n!$$

(*) z Leb. věty

$$\left| \sum_{n=0}^{N} (-1)^n e^{-x} \frac{x^n}{(2n)!} \right| \le \underbrace{\sum_{n=0}^{\infty} e^{-x} \frac{x^n}{(2n)!}}_{g(x):=} \qquad g \in \mathcal{L}^1(0,\infty)$$

Jiné zdůvodnění příkladu, co už jsme kdysi dělali