9. cvičení

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Příklady částečně z minula

1. Ukažte, že funkce $F(\alpha) = \int_0^\infty \frac{e^{-\alpha x}}{1+x^2}\,\mathrm{d}x$ konverguje pro $\alpha \geq 0$ a pro $\alpha \in (0,\infty)$ splňuje diferenciální rovnici $F'' + F = \frac{1}{\alpha}$.

2. Vyšetřete průběh funkce
$$F(\alpha) = \int_0^\infty \frac{e^{-\alpha x}}{1+x^2}.$$

3. Spočtěte limity

   (a) Spočtěte $\lim_{\alpha\to 0+} F(\alpha)$.
   $$F(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\,\mathrm{d}x, \qquad \alpha \in (0,\infty)$$

   (b) Spočtěte $\lim_{\alpha\to 2+} F(\alpha)$.
   $$F(\alpha) = \int_0^\infty \frac{x}{2+x^\alpha}\,\mathrm{d}x, \qquad \alpha(2,\infty)$$

   (c) Spočtěte $\lim_{\alpha\to 1-} F(\alpha)$.
   $$F(\alpha) = \int_0^{\frac{\pi}{4}} \frac{1}{\ln(\alpha - \sin x)}\,\mathrm{d}x, \qquad \alpha \in (\sqrt{2}/2, 1)$$

4. Spočtěte

   (a)
   $$F(\alpha,\beta) = \int_0^\infty \frac{e^{-\alpha x^2} - e^{-\beta x^2}}{x^2}\,\mathrm{d}x \qquad (\alpha = \beta) \vee (\alpha,\beta \geq 0)$$
   Hint: $\int_0^\infty -e^{-\alpha x^2}\,\mathrm{d}x = -\frac{1}{2}\sqrt{\pi/\alpha}$

   (b)
   $$F(\alpha) = \int_0^\infty \frac{1 - e^{-\alpha x^2}}{x^2 e^{x^2}}\,\mathrm{d}x \qquad \alpha \in (-1,\infty)$$

   (c)
   $$F(\alpha,\beta) = \int_0^\infty \frac{\arctan \alpha x - \arctan \beta x}{x}\,\mathrm{d}x \qquad (\alpha = \beta) \vee (\alpha,\beta > 0) \vee (\alpha,\beta < 0)$$

   (d)
   $$F(\alpha,\beta) = \int_0^\infty e^{-\beta x} \frac{\sin \alpha x}{x}\,\mathrm{d}x \qquad \beta \in (0,\infty), \alpha \in \mathbb{R}$$
   Hint:dce dle α, $\int_0^\infty e^{-\beta x} \cos \alpha x\,\mathrm{d}x = \beta/(\alpha^2+\beta^2)$.

(e)

$$F(\alpha) = \int_0^\pi \frac{\ln(1+\alpha\cos x)}{\cos x}\,\mathrm{d}x \qquad \alpha \in (-1,1)$$

Hint: $\int_0^\pi \frac{\mathrm{d}x}{1+\alpha\cos x} = \frac{\pi}{\sqrt{1-\alpha^2}}$, $\arcsin y' = 1/\sqrt{1-y^2}$

(f)

$$F(\alpha) = \int_0^{\frac{\pi}{2}} \ln\frac{1+\alpha\cos x}{1-\alpha\cos x}\frac{\mathrm{d}x}{\cos x} \qquad \alpha \in (-1,1)$$

Hint: $\int_0^{\pi/2} \frac{2\,\mathrm{d}x}{1-\alpha^2\cos^2 x} = \frac{\pi}{\sqrt{1-\alpha^2}}$.

(g)

$$F(\alpha) = \int_0^{\frac{\pi}{2}} \frac{\arctan\,(\alpha \mathrm{tg}\ x)}{\mathrm{tg}\ x}\,\mathrm{d}x \qquad \alpha \in \mathbb{R}$$

Hint: $\int_0^{\pi/2} \frac{1}{1+\alpha^2 \mathrm{tg}\ ^2 x}\,\mathrm{d}x = \frac{\pi}{2}\frac{1}{1+\alpha}$.

(h)

$$F(\alpha,\beta) = \int_0^{\frac{\pi}{2}} \ln(\alpha^2\sin^2 x + \beta^2\cos^2 x)\,\mathrm{d}x \qquad (\alpha,\beta) \in \mathbb{R}^2 \setminus (0,0)$$

Hint: dce dle α, $\int_0^{\pi/2} \frac{2}{\alpha}\frac{\alpha^2\sin^2 x}{\alpha^2\sin^2 x+\beta^2\cos^2 x}\,\mathrm{d}x = \frac{\pi}{\alpha+\beta}$

Bonus

5. Vyšetřete průběh funkce

$$F(\alpha) = \int_0^1 \frac{\mathrm{d}x}{\sqrt{x^2+\alpha^2}}$$

6.

$$F(\alpha,\beta,\gamma) = \int_0^\infty \frac{e^{-\alpha x} - e^{-\beta x}}{x}\sin\gamma x\,\mathrm{d}x, \qquad \alpha,\beta > 0, \gamma \in \mathbb{R}$$

Hint: $\int_0^\infty - e^{-\alpha x}\sin\gamma x\,\mathrm{d}x = \frac{-\gamma}{\alpha^2+\gamma^2}$

7.

$$F(\alpha,\beta,\gamma) = \int_0^\infty e^{-\gamma x}\frac{\sin\alpha x - \sin\beta x}{x}\,\mathrm{d}x, \qquad \gamma > 0, \alpha,\beta \in \mathbb{R}$$

8.

$$F(\alpha,\beta,\gamma,\delta) = \int_0^\infty \frac{e^{-\alpha x}\cos\beta x - e^{-\gamma x}\cos\delta x}{x}\,\mathrm{d}x, \qquad \alpha,\gamma > 0, \beta,\delta \in \mathbb{R}$$