(1)

$$\int_0^1 \frac{\ln(1+x)}{x}\,dx = \frac{\pi^2}{12}$$

(1) $\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$ $\quad$! kontrolovat Df

zvyšují !

$$f = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n+1}$$

(2) (d) $\quad h_n = \frac{1}{1}$ ok integrovatelná.

$$h_n \overset{?}{\geq} h_{n+1} \qquad \frac{x^n}{n+1} \overset{?}{\geq} \frac{x^{n+1}}{n+2}$$

$$\frac{n+2}{n+1} \geq x \in [0,1] \quad \text{ok } \checkmark$$

→ Lze prohodit $\sum$ a $\int$

(3) $\int_0^1 \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n+1}\,dx = \sum_{n=0}^{\infty} \int_0^1 (-1)^n \frac{x^n}{n+1}\,dx =$

$$= \sum_{n=0}^{\infty} (-1)^n \left[\frac{x^{n+1}}{(n+1)^2}\right]_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} = \underline{\underline{\frac{\pi^2}{12}}}$$

(2) $$\int_0^1 \frac{\ln(1-x)}{x} = -\frac{\pi^2}{6}$$

(1) $$\frac{\ln(1-x)}{x} = \frac{1}{x}\sum_{n=0}^{\infty} (-1)^n \frac{(-x)^{n+1}}{n+1} = (-1)\sum_{n=0}^{\infty} \frac{x^n}{n+1}$$

(2) $\boxed{\text{Levi}}$ pro fce - nezáporné, měřitelné

(3) $$\int_0^1 \frac{\ln(1-x)}{x} = \int_0^1 \sum_{n=0}^{\infty} (-1)\frac{x^n}{n+1} \overset{\text{Levi}}{=} (-1)\sum_{n=0}^{\infty} \int_0^1 \frac{x^n}{n+1} = -\sum_{n=0}^{\infty}\left[\frac{x^{n+1}}{(n+1)^2}\right]_0^1 =$$

$$= -\sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = -\frac{\pi^2}{6}$$

(3)

$$\int_0^\infty \frac{dx}{1+e^x}$$

(1) $$\frac{1}{1+e^x} = \frac{e^{-x}}{e^{-x}+1} = e^{-x}\sum_{n=0}^{\infty}\left(-e^{-x}\right)^n$$

(2) (a) geom. Σ, $\int_0^\infty \frac{1}{1+e^x}$ konvergujó z LSK

(3) $$\int_0^\infty \frac{1}{1+e^x} = \sum_{n=0}^{\infty}\int_0^\infty (-1)^n \underbrace{e^{-nx}\,e^{-x}}_{e^{-x(n+1)}} = \sum_{n=0}^{\infty}(-1)^n\left[\frac{e^{-x(n+1)}}{-(n+1)}\right]_0^\infty$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{1}{n+1} = \ln 2$$

(4)

$$\int_0^\infty \frac{x}{e^x+1}\,dx$$

(1) $\frac{x}{e^x(1+e^{-x})} = \frac{x}{e^x} \sum_{n=0}^{\infty} (-1)^n e^{-xn} = \sum_{n=0}^{\infty} \underbrace{(-1)^n x \, e^{-(n+1)x}}_{g_n(x)}$

(2) $\boxed{\text{Lebesgue}}$

$$\int_0^\infty \sum_n |g_n(x)|\, dx = \int_0^\infty \sum x e^{-(n+1)x} = \int_0^\infty \frac{x}{e^x-1}\,dx < \infty$$

(3) $\int f(x) = \sum_{n=0}^{\infty} \int_0^\infty (-1)^n x e^{-(n+1)x}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n+1)^2} = \underline{\underline{\frac{\pi^2}{12}}}$

(J)

$$\int_0^\infty \frac{x}{e^x-1}\,dx$$

(1) $$f(x) = \frac{x}{e^x}\cdot\frac{1}{1-e^{-x}} = xe^{-x}\sum_{n=0}^{\infty} e^{-nx} = \sum_{n=0}^{\infty} x e^{-(n+1)x}$$

(2) Levi

(3) $$\int_0^\infty f(x) = \int \sum x e^{-(n+1)x} = \sum \int_0^\infty x e^{-(n+1)x}$$

$$\int \underset{u}{x}\,\underset{v'}{e^{-(n+1)x}} = x\cdot\frac{e^{-(n+1)x}}{-(n+1)} - \int \frac{e^{-(n+1)x}}{-(n+1)}$$

$$u' = 1 \qquad v = \frac{e^{-(n+1)x}}{-(n+1)} \qquad\qquad \frac{e^{-(n+1)x}}{+(n+1)(n+1)}$$

$$= \sum \left[x\frac{e^{-(n+1)x}}{-(n+1)} - \frac{e^{-(n+1)x}}{(n+1)^2} \right]_0^\infty = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \frac{\pi^2}{6}$$

! Per partes je Newton !

(6)

$$\int_0^1 \frac{x^{p-1}}{1+x^q}\,dx$$

(1) $$\frac{x^{p-1}}{1+x^q} = \sum_{n=0}^{\infty} (-1)^n x^{p-1} x^{qn} = \sum_{n=0}^{\infty} (-1)^n x^{p-1+nq}$$

(2) (d) $h_1 = x^{p-1}$ $\quad p>0 \quad \int h_1$ konv.

$$h_n \overset{?}{\geq} h_{n+1} \qquad x^{p-1+nq} \geq x^{p-1+(n+1)q}$$

$$1 \geq x^q \qquad q>0 \ \checkmark$$

(3) $$\int_0^1 f(x) = \sum (-1)^n \int_0^1 x^{p-1+nq}\,dx = \sum (-1)^n \left[\frac{x^{p-1+nq+1}}{p+nq}\right]_0^1$$

$$= \underline{\underline{\sum_{n=0}^{\infty} (-1)^n \frac{1}{p+nq}}}$$

(7)

$$\int_0^1 \ln \frac{1}{1-x}$$

(1) $\ln \frac{1}{1-x} = -\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$

(2) Levi

(3) $\int f = \sum \int_0^1 \frac{x^n}{n} = \sum_{n=0}^{\infty} \left[\frac{x^{n+1}}{n(n+1)} \right]_0^1 = \sum_{n=0}^{\infty} \frac{1}{n(n+1)} = \underline{\underline{1}}$

(8)

$$\int_0^\infty \ln \frac{1+x}{1-x}\, dx$$

(1)

$$\left(\ln \frac{1+x}{1-x}\right)' = \frac{2}{1-x^2} \qquad \ln 1 = 0$$

$$\left(\ln \frac{1+x}{1-x}\right) = 2 \sum_{n=0}^{\infty} \frac{x^{2n}}{2n+1}$$

z mocninné Σ:

$$\frac{2}{1-x^2} = 2 \sum_{n=0}^{\infty} x^{2n} \qquad \text{|| obě strany } \int$$

$$\ln \frac{1+x}{1-x} = 2 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$$

(2) Levi

(3) $$\int f = 2 \sum_{n=0}^{\infty} \int \frac{x^{2n+1}}{2n+1} = 2 \sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)}$$

$$= 2 \sum_{n=0}^{\infty} \frac{1}{2n+1} - \frac{1}{2n+2} = 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \underline{\underline{2 \ln 2}}$$

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \dots$$

(9) $\displaystyle\int_0^1 \frac{1}{x} \ln \frac{1+x}{1-x}\, dx$

(1) $f(x) = 2 \cdot \frac{1}{x} \displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} = 2 \sum \frac{x^{2n}}{2n+1}$

(2) Lebesgue

(3) $\displaystyle\int f = \sum_{n=0}^{\infty} 2 \int_0^1 \frac{x^{2n}}{2n+1} = 2 \sum \left[\frac{x^{2n+1}}{(2n+1)^2} \right]_0^1 =$

$\displaystyle = 2 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{4}$

(10) $\int_0^\infty e^{-x} \cos \sqrt{x} \, dx = $ ~~...~~

(1) $\cos y = \sum_{n=0}^\infty (-1)^n \cdot \frac{y^{2n}}{(2n)!}$

$$f(x) = \sum_{n=0}^\infty (-1)^n e^{-x} \frac{x^n}{(2n)!} \qquad x \in (0, \infty)$$

(2) (d) $h_1 = e^{-x} \frac{1}{1} \quad \in L^1(0,\infty)$

$h_n \overset{?}{\geq} h_{n+1}$ NE ? $\quad \frac{e^{-x} x^n x}{(2n+2)!} \overset{?}{\leq} \frac{e^{-x} x^n}{2n!}$

NE $\quad x \leq (2n+2)(2n+1)$

(e) $\int_0^\infty \sum_{n=0}^\infty e^{-x} \frac{x^n}{(2n)!}$?

(b) $\sum \int_0^\infty e^{-x} \frac{x^n}{(2n)!} = \sum_{n=0}^\infty \frac{1}{(2n)!} \cdot n! < \infty$

d'Al $\quad \frac{\frac{(n+1)!}{(2n+2)!}}{\frac{n!}{2n!}} = \frac{2n(n+1)}{(2n+2)(2n+1)} \to \frac{1}{2}$ ✓

(3) $\int_0^\infty f = \sum_{n=0}^\infty (-1)^n \int_0^\infty e^{-x} \frac{x^n}{(2n)!} = \underline{\underline{\sum_{n=0}^\infty (-1)^n \frac{n!}{(2n)!}}}$

Víme: $\int_0^\infty e^{-x} x^n \, dx = n!$

(11)

$$\int_0^\infty e^{-ax} \sin bx \qquad |b| < a$$

(1) $$\sin y = \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n+1}}{(2n+1)!}$$

$$f(x) = e^{-ax} \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{(bx)^{2n+1}}{(2n+1)!}}_{g(x)}$$

(2) $$\sum_{n=0}^{\infty} e^{-ax} \frac{(bx)^{2n+1}}{(2n+1)!} = \sum | \quad |$$

$\sum$ $\boxed{\text{limes}}$ $$I_n = \int_0^\infty e^{-ax} x^n dx = \frac{n}{a} I_{n-1} \qquad I_0 = \frac{1}{a}$$

$$\int_0^\infty e^{-ax} \sum_{n=0}^{\infty} \frac{(bx)^{2n+1}}{(2n+1)!} \quad ?$$

(b) $$\sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \int_0^\infty e^{-ax} (|b|x)^{2n+1} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \frac{1}{a} \frac{1}{a} \cdot \frac{2}{a} \cdots \frac{(2n+1)}{a} |b|^{2n+1}$$

$$= \sum_{n=0}^{\infty} \left(\frac{|b|}{a}\right)^{2n+1} \cdot \frac{1}{a} < \infty$$

Geom. $\sum$ $\quad |b| < a$

(3) Výpočet

$$\int_0^\infty e^{-ax} \sin bx = \sum_{n=0}^{\infty} (-1)^n \left(\frac{b}{a}\right)^{2n} \cdot \frac{b}{a^2} =$$

$$= \frac{b}{a^2} \frac{1}{1 + \frac{b^2}{a^2}} = \underline{\underline{\frac{b}{a^2+b^2}}}$$