5

Mathyz

1)(a) $\cosh x + \sinh x = e^x$

$\hookrightarrow \frac{1}{2}(e^x + e^{-x} + e^x - e^{-x}) = \frac{1}{2} \cdot 2 \cdot e^x = e^x$

(b) $\cosh^2 x - \sinh^2 x = 1$

$\hookrightarrow (\cosh x - \sinh x)(\cosh + \sinh x) =$

$= \frac{1}{2}(e^x + e^{-x} - e^x + e^{-x})(e^x) =$

$= \frac{1}{2} \cdot 2 \cdot e^{-x} \cdot e^x = 1$

(c) $\cosh 2x = \cosh^2 x + \sinh^2 x$

$\cosh 2x = \frac{1}{2}(e^{2x} + e^{-2x})$

$\cosh^2 x + \sinh^2 x = \frac{1}{4}(e^x + e^{-x})^2 + \frac{1}{4}(e^x - e^{-x})^2$

$= \frac{1}{4}(e^{2x} + e^{-2x} + 2e^x e^{-x} + e^{2x} + e^{-2x} - 2e^x e^{-x})$

$= \frac{1}{4}(2e^{2x} + 2e^{-2x})$

(d) $\sinh 2x = 2 \sinh x \cosh x$

$\sinh 2x = \frac{1}{2}(e^{2x} - e^{-2x})$

$2 \sinh x \cosh x = 2 \cdot \frac{1}{2}(e^x - e^{-x}) \frac{1}{2}(e^x + e^{-x})$

$= \frac{1}{2}(e^{2x} - e^{-2x})$

(e) $\cosh(-x) = \frac{1}{2}(e^{-x} + e^{-(-x)}) = \frac{1}{2}(e^x + e^{-x}) = \cosh x$

(f) $\sinh(-x) = \frac{1}{2}(e^{-x} - e^{-(-x)}) = \frac{1}{2}(e^{-x} - e^x) =$

$= -\frac{1}{2}(e^x - e^{-x}) = -\sinh x$

Activity 5

(a) Superimpose the graphs of $y = \cosh x$ and $y = \sinh x$ on the screen of a graphics calculator. Do the curves ever intersect?

(b) Use a graphics calculator to sketch the function $f: x \mapsto \tanh x$ with domain $x \in \mathbb{R}$. What is the range of the function?

(c) Try to predict what the graphs of

$$y = \operatorname{sech} x, \; y = \operatorname{cosech} x \text{ and } y = \coth x$$

will look like. Check your ideas by plotting the graphs on a graphics calculator.

(2a)

2.5 Solving equations

Suppose $\sinh x = \frac{3}{4}$ and we wish to find the exact value of x.

Recall that $\cosh^2 x = 1 + \sinh^2 x$ and $\cosh x$ is always positive, so when $\sinh x = \frac{3}{4}$, $\cosh x = \frac{5}{4}$.

From Activity 1, we have $\sinh x + \cosh x = e^x$

so $$e^x = \frac{3}{4} + \frac{5}{4} = 2$$

and hence $x = \ln 2$.

Alternatively, we can write $\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$

so $\sinh x = \frac{3}{4}$ means

$$\frac{1}{2}\left(e^x - e^{-x}\right) = \frac{3}{4}$$

$$\Rightarrow \quad 2e^x - 3 - 2e^{-x} = 0$$

and multiplying by e^x

$$2e^{2x} - 3e^x - 2 = 0$$

$$\left(e^x - 2\right)\left(2e^x + 1\right) = 0$$

$$e^x = 2 \text{ or } e^x = -\frac{1}{2}$$

But e^x is always positive so $e^x = 2 \Rightarrow x = \ln 2$.

(2)(b) $\cosh x = \frac{13}{5}$

$$\frac{1}{2}(e^x + e^{-x}) = \frac{13}{5}$$

$$y + \frac{1}{y} = \frac{26}{5} \qquad e^x =: y$$

$$y^2 + 1 = \frac{26}{5}y$$

$$y^2 - \frac{26}{5}y + 1 = 0$$

$$y_{1,2} = \frac{\frac{26}{5} \pm \sqrt{\frac{26^2}{25} - 4}}{2}$$

$$= \frac{\frac{26}{5} \pm \sqrt{\frac{676}{25} - \frac{100}{25}}}{2}$$

$$= \frac{\frac{26}{5} \pm \frac{24}{5}}{2} = \begin{cases} 5 \\ \frac{1}{5} \end{cases}$$

$y_1 = 5$

$e^x = 5$

$\underline{\underline{x = \ln 5}}$

$y_2 = \frac{1}{5}$

$e^x = \frac{1}{5}$

$\underline{\underline{y = -\ln 5}}$

(2d) $4 \cosh x + \sinh x = 4$

$$4(e^x + e^{-x}) + e^x - e^{-x} = 4 \cdot 2 \qquad e^x = y$$

$$5e^x + 3e^{-x} = 8$$

$$5y + \frac{3}{y} = 8$$

$$5y^2 - 8y + 3 = 0$$

$$y_{1,2} = \frac{8 \pm \sqrt{64 - 60}}{10}$$

$$y_{1,2} = \frac{8 \pm 2}{10} = \begin{cases} 1 \\ \frac{3}{5} \end{cases}$$

$e^x = 1$

$x = \ln 1$

$\underline{\underline{x = 0}}$

$e^x = \frac{3}{5}$

$x = \ln \frac{3}{5}$

$\underline{\underline{x = \ln 3 - \ln 5}}$

Activity 6

Find the values of x for which

$$\cosh x = \frac{13}{5}$$

expressing your answers as natural logarithms.

Example

Solve the equation

(2c) $$2\cosh 2x + 10\sinh 2x = 5$$

giving your answer in terms of a natural logarithm.

Solution

$$\cosh 2x = \frac{1}{2}\left(e^{2x} + e^{-2x}\right); \quad \sinh 2x = \frac{1}{2}\left(e^{2x} - e^{-2x}\right)$$

So $$e^{2x} + e^{-2x} + 5e^{2x} - 5e^{-2x} = 5$$

$$6e^{2x} - 5 - 4e^{-2x} = 0$$

$$6e^{4x} - 5e^{2x} - 4 = 0$$

$$\left(3e^{2x} - 4\right)\left(2e^{2x} + 1\right) = 0$$

$$e^{2x} = \frac{4}{3} \text{ or } e^{2x} = -\frac{1}{2}$$

The only real solution occurs when $e^{2x} > 0$

So $$2x = \ln\frac{4}{3} \Rightarrow x = \frac{1}{2}\ln\frac{4}{3}$$

Exercise 2B

1. Given that $\sinh x = \frac{5}{12}$, find the values of

   (a) $\cosh x$ (b) $\tanh x$ (c) $\operatorname{sech} x$

   (d) $\coth x$ (e) $\sinh 2x$ (f) $\cosh 2x$

   Determine the value of x as a natural logarithm.

2. Given that $\cosh x = \frac{5}{4}$, determine the values of

   (a) $\sinh x$ (b) $\cosh 2x$ (c) $\sinh 2x$

   Use the formula for $\cosh(2x + x)$ to determine the value of $\cosh 3x$.

$\cosh^2 x - \sinh^2 x = 1$

$\cosh^2 x - \frac{25}{144}$ $\cosh \frac{13}{12}$

$\cosh^2 x \quad \frac{144+25}{144}$

$\cosh^2 x = \frac{169}{144}$

(3) $\sinh x = \frac{5}{12}$

(a) $\cosh x$ $\qquad \cosh x = \sqrt{1+\sinh^2 x}$

$\cosh x = \sqrt{1+\frac{25}{144}}$

$\cosh x = \sqrt{\frac{169}{144}}$

$\cosh x = \frac{13}{12}$

// $\cosh x \geq 0$ $\forall x$ lze v klidu odmocnit

(b) $\coth x = \frac{\cosh x}{\sinh x} = \frac{\frac{13}{12}}{\frac{5}{12}} = \frac{13}{5}$

(c) $\tanh x = \frac{5}{13}$

(d) $\sinh 2x = 2\sinh x \cosh x = 2 \cdot \frac{5}{12} \cdot \frac{13}{12} = \frac{65}{72}$

(e) $\cosh 2x = \cosh^2 x + \sinh^2 x = \frac{13^2}{12^2} + \frac{5^2}{12^2} = \frac{194}{144} = \frac{97}{72}$

(4) $\operatorname{arcsinh} x = \ln\left(x + \sqrt{x^2+1}\right)$

$\sinh x = \frac{1}{2}(e^x - e^{-x})$

položme $\quad y = \frac{1}{2}(e^x - e^{-x})$ $\qquad e^x =: z$

$2y = z - \frac{1}{z}$

$2yz = z^2 - 1$

$0 = z^2 - 2yz - 1$

$z_{1,2} = \frac{2y \pm \sqrt{4y^2+4}}{2}$

$z_{1,2} = y \pm \sqrt{y^2+1}$ $\qquad z_2 = y - \sqrt{y^2+1} < 0$

$\rightsquigarrow$ zahodíme

pak $\quad x = \ln\left(y + \sqrt{y^2+1}\right)$