(405) Vypočtěte

$$\int_1^2 x^2\,dx.$$

Řešení:

$$\int_1^2 x^2\,dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}.$$

(407) Vypočtěte

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \operatorname{tg}^2 x \, dx.$$

Řešení:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \operatorname{tg}^2 x \, dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin^2 x}{\cos^2 x} \, dx \left| \begin{array}{c} t = \operatorname{tg} x \\ dt = \frac{1}{\cos^2 x} dx \\ \sin x = \frac{t}{\sqrt{1+t^2}} \\ \frac{\pi}{3} \rightsquigarrow \sqrt{3} \\ \frac{\pi}{6} \rightsquigarrow \frac{\sqrt{3}}{3} \end{array} \right| =$$

$$= \int_{\frac{\sqrt{3}}{3}}^{\sqrt{3}} \frac{t^2}{1+t^2} \, dt = \int_{\frac{\sqrt{3}}{3}}^{\sqrt{3}} 1 \, dt - \int_{\frac{\sqrt{3}}{3}}^{\sqrt{3}} \frac{1}{1+t^2} \, dt =$$

$$= [t - \operatorname{arctg} t]_{\frac{\sqrt{3}}{3}}^{\sqrt{3}} = \frac{2}{3}\sqrt{3} - \frac{\pi}{6}.$$

(408) Vypočtěte

$$\int_0^1 \left(\frac{e^x}{e^{2x}+3} + \frac{1}{\cos^2 x} \right) dx.$$

Řešení:

Využijeme aditivity integrálu a pro přehlednost zadaný integrál rozdělíme na dvě části.

$$I_1 = \int_0^1 \frac{e^x}{e^{2x}+3}\, dx \left| \begin{array}{c} t = e^x \\ dt = e^x\, dx \\ 1 \rightsquigarrow e \\ 0 \rightsquigarrow 1 \end{array} \right| =$$

$$= \int_1^e \frac{1}{t^2+3}\, dt = \frac{1}{3} \int_1^e \frac{1}{\left(\frac{t}{\sqrt{3}}\right)^2 + 1}\, dt \left| \begin{array}{c} s = \frac{t}{\sqrt{3}} \\ ds = \frac{1}{\sqrt{3}} dt \\ e \rightsquigarrow \frac{e}{\sqrt{3}} \\ 1 \rightsquigarrow \frac{1}{\sqrt{3}} \end{array} \right| =$$

$$= \frac{\sqrt{3}}{3} \int_{\frac{1}{\sqrt{3}}}^{\frac{e}{\sqrt{3}}} \frac{1}{s^2+1}\, ds = \frac{\sqrt{3}}{3} [\operatorname{arctg} s]_{\frac{1}{\sqrt{3}}}^{\frac{e}{\sqrt{3}}} =$$

$$= \frac{\sqrt{3}}{3} \left(\operatorname{arctg} \frac{e\sqrt{3}}{3} - \frac{\pi}{6} \right),$$

$$I_2 = \int_0^1 \frac{1}{\cos^2 x}\, dx = [\operatorname{tg} x]_0^1 = \operatorname{tg} 1.$$

Celkem tedy

$$\int_0^1 \left(\frac{e^x}{e^{2x}+3} + \frac{1}{\cos^2 x} \right) dx = I_1 + I_2 = \frac{\sqrt{3}}{3} \left(\operatorname{arctg} \frac{e\sqrt{3}}{3} - \frac{\pi}{6} \right) + \operatorname{tg} 1.$$

(412) Vypočtěte

$$\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx.$$

Řešení:

$$\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx \left| \begin{array}{c} t=\sqrt{x} \\ dt=\frac{1}{2\sqrt{x}}\,dx \\ \infty \rightsquigarrow \infty \\ 1 \rightsquigarrow 1 \end{array} \right| =$$

$$= 2\int_1^\infty e^{-t}\,dt = 2[-e^{-t}]_1^\infty =$$

$$= -2\left(\lim_{t\to\infty} e^{-t} - e^{-1}\right) = \frac{2}{e}.$$

(414) Vypočtěte

$$\int_1^\infty \frac{dx}{x}.$$

Řešení:

$$\int_1^\infty \frac{dx}{x} = [\ln x]_1^\infty = \lim_{x\to\infty} \ln x - \ln 1 = \infty \quad \Rightarrow \text{ integrál diverguje.}$$

(415) Vypočtěte

$$\int_0^\infty \sin x \, dx.$$

Řešení:

$$\int_0^\infty \sin x \, dx = [-\cos x]_0^\infty = -\lim_{x\to\infty} \cos x + \cos 0 \quad \text{limita neexistuje} \Rightarrow \text{integrál diverguje.}$$

(416) Vypočtěte

$$\int_{-\infty}^{0} e^x \, dx.$$

Řešení:

$$\int_{-\infty}^{0} e^x \, dx = [e^x]_{-\infty}^{0} = 1 - \lim_{x \to -\infty} e^x = 1.$$

(422) Vypočtěte

$$\int_1^\infty \frac{\operatorname{arctg} x}{x^2+1}\,dx.$$

Řešení:

$$\int_1^\infty \frac{\operatorname{arctg} x}{x^2+1}\,dx \left| \begin{array}{c} \operatorname{arctg} x = t \\ \frac{1}{1+x^2}\,dx = dt \\ \infty \rightsquigarrow \frac{\pi}{2} \\ 1 \rightsquigarrow \frac{\pi}{4} \end{array} \right| = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} t\,dt = \left[\frac{t^2}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{\pi^2}{8} - \frac{\pi^2}{32} = \frac{3\pi^2}{32}.$$

(423) Vypočtěte

$$\int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}}.$$

Řešení:

$$\int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}} = \int_{-\infty}^{\infty} \frac{e^x}{e^{2x}+1}\,dx \left| \begin{array}{c} t = e^x \\ dt = e^x\,dx \\ \infty \rightsquigarrow \infty \\ -\infty \rightsquigarrow 0 \end{array} \right| = \int_0^{\infty} \frac{dt}{t^2+1} = [\operatorname{arctg} t]_0^{\infty} = \frac{\pi}{2}.$$

(432) Vypočtěte

$$\int_1^2 \frac{dx}{x \ln x}.$$

Řešení:

$$\int_1^2 \frac{dx}{x \ln x} \left| \begin{array}{c} t = \ln x \\ dt = \frac{1}{x}\, dx \\ 1 \rightsquigarrow 0 \\ 2 \rightsquigarrow \ln 2 \end{array} \right| = \int_0^{\ln 2} \frac{dt}{t} = [\ln t]_0^{\ln 2} =$$

$$= \ln(\ln 2) - \lim_{t \to 0} \ln t = \ln(\ln 2) + \infty \quad \Rightarrow \text{ integrál diverguje.}$$

(434) Vypočtěte

$$\int_a^b \frac{dx}{\sqrt{x^2-a^2}}.$$

Řešení:

$$\int_a^b \frac{dx}{\sqrt{x^2-a^2}} \left| \begin{array}{c} t = x + \sqrt{x^2-a^2} \\ dt = \left(1 + \frac{\sqrt{x^2-a^2}+x}{\sqrt{x^2-a^2}}\right) dx \\ a \rightsquigarrow a \\ b \rightsquigarrow b + \sqrt{b^2-a^2} \end{array} \right| = \int_a^{b+\sqrt{b^2-a^2}} \frac{1}{t}\, dt =$$

$$= \left[\ln|t|\right]_a^{b+\sqrt{b^2-a^2}} = \ln\left(b + \sqrt{b^2-a^2}\right) - \ln a = \ln \frac{b+\sqrt{b^2-a^2}}{a}.$$