(1)(i) $$\sum_{n=1}^{\infty} \frac{x^{n+2}}{n+1} = x \sum_{n=1}^{\infty} \frac{x^{n+1}}{n+1} = x\,\underline{\left(-x - \ln(1-x)\right)}$$

$$\left(\sum_{n=1}^{\infty} \frac{x^{n+1}}{n+1} \right)' = \sum_{n=1}^{\infty} x^{n} = \frac{x}{1-x}$$

$$\underline{|x| < 1}$$

$$\Rightarrow \sum \frac{x^{n+1}}{n+1} = \int \frac{x-1+1}{1-x} = \int -1 + \frac{1}{1-x}\,dx$$

$$= -x - \ln|1-x| + c$$

c: $$0 = -0 - \ln|1-0| + c \Rightarrow c = 0$$

(ii) $$\sum_{n=1}^{\infty} (-1)^{n-1} n^2 x^n = x \sum_{n=1}^{\infty} (-1)^{n-1} n \cdot n\, x^{n-1}$$

$$= x \left(\sum_{n=1}^{\infty} (-1)^{n-1} n\, x^{n} \right)' = x\, \underline{\frac{(1+x)^2 - x\,2(x+1)}{(1+x)^4}}$$

$$x \sum_{n=1}^{\infty} (-1)^{n-1} n\, x^{n-1} = x \left(\sum_{n=1}^{\infty} (-1)^{n-1} x^{n} \right)' = x \frac{1}{(1+x)^2}$$

$$\sum_{n=1}^{\infty} (-1)(-1)^{n} x^{n} = -\frac{1}{1+x}$$

celkem $$x\,\frac{1+x-2x}{(1+x)^3} = \underline{\frac{x(1-x)}{(1+x)^3}}, \quad |x| < 1$$

(iii) $$\sum_{n=1}^{\infty} \frac{x^{3n+1}}{3n+1}$$

$$\left(\sum_{n=1}^{\infty} \frac{x^{3n+1}}{3n+1}\right)' = \sum_{n=1}^{\infty} x^{3n} = \sum_{n=1}^{\infty} (x^3)^n = \frac{x^3}{1-x^3}$$

$|x| < 1$

$$\int \frac{x^3}{1-x^3} = -\int \frac{1-x^3-1}{1-x^3}dx = -\int 1 dx + \int \frac{1}{1-x^3} dx$$

$$= \int -1 dx + \int \frac{1}{(1-x)(1+x+x^2)} dx = \int -1 dx + \int \frac{1}{3}\frac{1}{1-x}dx + \frac{1}{3}\int \frac{x+2}{x^2+x+1}dx$$

$$\frac{A}{1-x} + \frac{Bx+C}{x^2+x+1}$$

$$Ax^2 + Ax + A + Bx - Bx^2 + C - Cx = 1$$

$$A + C = 1$$

$$A + B - C = 0$$

$$A - B = 0$$

$$C = 1 - A$$

$$A + B - 1 + A = 0$$

$$A - B = 0$$

$$2A + B = 1 \qquad C = 2/3$$

$$A - B = 0 \qquad B = 1/3$$

$$3A = 1 \qquad A = 1/3$$

$$= -x + \frac{1}{3}\ln|1-x| + \frac{1}{3}\cdot\frac{1}{2}\int \frac{2x+1}{x^2+x+1}dx + \frac{1}{3}\int \frac{1}{2}\frac{3}{x^2+x+1}dx$$

$$= -x - \frac{1}{3}\ln|1-x| + \frac{1}{6}\ln(x^2+x+1) + \underbrace{\frac{1}{2}\int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}dx}$$

$$= -x - \frac{1}{3}\ln|1-x| + \frac{1}{6}\ln(x^2+x+1) + \frac{\arctan\frac{2x+1}{\sqrt{3}}}{\sqrt{3}} + C \qquad \frac{1}{2}\cdot\frac{4}{3}\int \frac{1}{\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)^2+1}dx = \frac{2}{3}\cdot\frac{\sqrt{3}}{2}\arctan\frac{2x+1}{\sqrt{3}}$$

(1iii) Výpočet c: dosadíme $x = 0$

$$0 = \frac{\operatorname{arctan} \frac{1}{\sqrt{3}}}{\sqrt{3}} + c \qquad c = -\frac{\pi}{6\sqrt{3}}$$

2. vzorová písemka

(j) $f(x) = \dfrac{1}{\cos x}$

Řešení: Použijeme substituci $y = \sin x$. Potom $dy = \cos x \, dx$ a platí

$$\int \frac{1}{\cos x} \, dx = \int \frac{\cos x}{\cos^2 x} \, dx = \int \frac{\cos x}{1 - \sin^2 x} \, dx = \int \frac{dy}{1 - y^2} \overset{C}{=} \frac{1}{2} \ln \left| \frac{1+y}{1-y} \right| =$$

$$= \frac{1}{2} \ln \left| \frac{1 + \sin x}{1 - \sin x} \right| = \ln \left| \operatorname{tg} \frac{x}{2} + \frac{\pi}{4} \right|$$

dělení mnohočlenem
(wolfram

(2a) $$\int \frac{x^5-x^4-2x^3+5x^2-3x+1}{x^4-3x^3+4x^2-3x+1}\,dx =$$

$$= \int x+2 + \underbrace{\frac{2x-1}{(x-1)^2(x^2-x+1)}}_{I_3}\,dx =$$

$$\frac{x^2}{2} \qquad 2x$$

$$I_3 = \int \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{Cx+D}{x^2-x+1}\,dx$$

wolfram

$$\rightarrow = \int \frac{1}{(x-1)^2} + \frac{1}{x-1} + \frac{-1-x}{x^2-x+1}\,dx$$

$$= \frac{-1}{x-1} + \ln|x-1| + (-1)\,\frac{1}{2}\underbrace{\int \frac{2x-1+3}{x^2-x+1}\,dx}_{\ln(x^2-x+1) + \underbrace{\int \frac{3}{(x-\frac{1}{2})^2+\frac{3}{4}}\,dx}_{I_4}}$$

$$I_4 = 3\cdot\frac{1}{\frac{3}{4}}\int \frac{1}{\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)^2+1}\,dx$$

$$= 4\cdot\frac{\sqrt{3}}{2}\,\operatorname{arctan}\frac{2x-1}{\sqrt{3}} + c$$

celkem

$$\underline{\frac{x^2}{2} + 2x - \frac{1}{x-1} + \ln|x-1| - \frac{1}{2}\ln(x^2-x+1) - \sqrt{3}\,\operatorname{arctan}\frac{2x-1}{\sqrt{3}}}$$

$$\underline{\underline{+c}}$$

(2b) $\int \frac{e^{5x}}{e^{2x}+1}\,dx = \int \frac{y^4}{1+y^2}\,dy = \int \frac{y^4+y^2-y^2}{1+y^2}\,dy$

$y = e^x$

$dy = e^x\,dx$

$$= \int y^2 - \frac{y^2+1}{1+y^2} + \frac{1}{1+y^2}\,dy$$

$$= \frac{y^3}{3} - y + \arctan y + c$$

$$= \frac{e^{3x}}{3} - e^x + \arctan e^x + c$$

(r) $\displaystyle\int \frac{1-\ln^2 x}{x(2+\ln^2 x)^2}\,dx = \int \frac{1-y^2}{(2+y^2)^2}\,dy = \int \frac{-2-y^2+3}{(2+y^2)^2}\,dy$

$y = \ln x$

$dy = \frac{1}{x}\,dx$

$$= \int \frac{-1}{2+y^2} + \frac{3}{(2+y^2)^2}\,dy = \int \frac{-1}{2\left(1+\left(\frac{y}{\sqrt{2}}\right)^2\right)}\,dy + \int \frac{3}{4\left(1+\left(\frac{y}{\sqrt{2}}\right)^2\right)^2}\,dy$$

$z = \frac{y}{\sqrt{2}} \quad dz = \frac{1}{\sqrt{2}}\,dy$

$$= -\frac{1}{2}\cdot\sqrt{2}\arctan\frac{y}{\sqrt{2}} + \frac{3}{4}\cdot\sqrt{2}\int\frac{1}{(1+z^2)^2}\,dy$$

$\downarrow$ pamatujeme nebo odvodíme vzorec

$$\frac{3\sqrt{2}}{4}\,\frac{1}{2}\left(\arctan z + \frac{z}{1+z^2}\right) + C$$

$$\text{celkem} = -\frac{\sqrt{2}}{2}\arctan\frac{\ln x}{\sqrt{2}} + \frac{3\sqrt{2}}{8}\left(\arctan\frac{\ln x}{\sqrt{2}} + \frac{\frac{\ln x}{\sqrt{2}}}{1+\frac{\ln^2 x}{2}}\right) + C$$

(2d) $\int x \sqrt[3]{2x+3}\, dx = \int \frac{t^3-3}{2} \cdot t \cdot \frac{3t^2}{2}\, dt =$

$t = \sqrt[3]{2x+3}$

$t^3 = 2x+3$

$\frac{t^3-3}{2} = x$

$\frac{3t^2}{2} = dx$

$= \frac{1}{4} \int 3t^6 - 9t^3\, dt =$

$= \frac{1}{4} \left(3\frac{t^7}{7} - 9\frac{t^4}{4} \right) + c$

$= \frac{3}{28} (2x+3)^{7/3} - \frac{9}{16} (2x+3)^{4/3} + c$

$x \in \mathbb{R}$

(366) Pomocí vhodné substituce vypočtěte

$$\int \frac{1}{x}\sqrt{\frac{x+1}{x-1}}\,dx.$$

Řešení:

$$\int \frac{1}{x}\sqrt{\frac{x+1}{x-1}}\,dx \left| \begin{array}{c} t^2 = \frac{x+1}{x-1} \\ x = \frac{1+t^2}{t^2-1} \\ dx = -\frac{4t}{(t^2-1)^2}\,dt \end{array} \right| = \int \frac{t^2-1}{t^2+1} t \frac{-4t}{(t^2-1)^2}\,dt =$$

$$= \int \frac{-4t^2}{(t^2+1)(t^2-1)}\,dt = \int \left(-\frac{1}{t-1} + \frac{1}{t+1} - \frac{2}{t^2+1}\right) dt =$$

$$= -\ln|t-1| + \ln|t+1| - 2\operatorname{arctg} t + C =$$

$$= -\ln\left|\sqrt{\frac{x+1}{x-1}} - 1\right| + \ln\left|\sqrt{\frac{x+1}{x-1}} + 1\right| - 2\operatorname{arctg}\sqrt{\frac{x+1}{x-1}} + C =$$

$$= 2\ln\left|\sqrt{|x+1|} - \sqrt{|x-1|}\right| - 2\operatorname{arctg}\sqrt{\frac{x+1}{x-1}} + C.$$

(2e) $\int \sqrt{x^2+4}\, dx = \int \sqrt{4\sinh^2 y + 4} \cdot 2\cosh y \, dy$

$x = 2\sinh y \qquad \operatorname{argsinh} \frac{x}{2} = y$

$dx = 2\cosh y \, dy \qquad = \int \sqrt{4} \sqrt{\cosh^2 y} \cdot 2\cosh y \, dy$

$= 4 \int \cosh^2 y \, dy = 4 \int \left(\frac{e^y + e^{-y}}{2}\right)^2 dy =$

$\int e^{2y} + e^{-2y} + 2e^y e^{-y} \, dy = \frac{1}{2} e^{2y} - \frac{1}{2} e^{-2y} + 2y + c$

$= \underline{\underline{\frac{1}{2} e^{2\operatorname{argsinh}\frac{x}{2}} - \frac{1}{2} e^{-2\operatorname{argsinh}\frac{x}{2}} + 2\operatorname{argsinh}\frac{x}{2} + c}}$

$x \in \mathbb{R}$

(2g) $\displaystyle\int \frac{x}{\sqrt{x^2-5x+6}}\,dx$

$x \in (-\infty, 2) \cup (3, \infty)$

$x^2-5x+6 = (x-3)(x-2)$

$$y = \sqrt{\frac{x-3}{x-2}} \qquad x = \frac{3-2y^2}{1-y^2} \qquad dx = \frac{2y}{(1-y^2)^2}\,dy$$

$$= \int \frac{3-2y^2}{(1-y^2)} \cdot \frac{1}{y} \cdot \frac{2\cancel{y}}{(1-y^2)^2}\,dy$$

$$= 2\int \frac{3-2y^2}{(1-y^2)^3} = 2\int \frac{2-2y^2}{(1-y^2)^3} + 2\int \frac{1}{(1-y^2)^3}\,dy$$

$$= 4\int \frac{1}{(1-y^2)^2} + 2\int \frac{1}{(1-y^2)^3}\,dy$$

$$= \frac{2}{16}\int \frac{11}{1+y} + \frac{11}{(1+y)^2} + \frac{2}{(1+y)^3} - \frac{11}{y-1} + \frac{11}{(y-1)^2} - \frac{2}{(y-1)^3}\,dy$$

$$= \frac{2}{16}\left[11\ln|1+y| + \frac{-11}{1+y} + \frac{-1}{(1+y)^2} - 11\ln|y-1| + \frac{-11}{y-1} + \frac{1}{(y-1)^2} \right] + C$$

dosadime za y ...

(2h) $\displaystyle\int \frac{\sqrt{x^2+x+1}}{x^2}\,dx \qquad x \neq 0$

$$t = \sqrt{x^2+x+1} + x$$

$$x = \frac{t^2-1}{1+2t} \qquad \sqrt{x^2+x+1} = t - \frac{t^2-1}{1+2t}$$

pak $\quad dx = \underbrace{\frac{2t(1+2t)-2(t^2-1)}{(1+2t)^2}}_{\displaystyle\frac{2t+4t^2-2t^2+2}{(1+2t)^2}} dt$

$$= \int \frac{\frac{t+2t^2-t^2+1}{1+2t}}{\frac{(t^2-1)^2}{\cancel{(1+2t)^2}}} \cdot \frac{2t^2+2t+2}{\cancel{(1+2t)^2}}\,dt =$$

$$= \int \frac{2(t^2+t+1)^2}{(t-1)^2(t+1)^2(1+2t)}\,dt =$$

wolfram

$$= 2\int \frac{-1/4}{t+1} + \frac{1}{2t+1} - \frac{1/4}{(t+1)^2} + \frac{1/4}{t-1} + \frac{3/4}{(t-1)^2}\,dt$$

$$= 2\left[-\frac{1}{4}\ln|t+1| + \frac{1}{2}\ln|2t+1| + \frac{1}{4}\frac{1}{t+1} + \frac{1}{4}\ln|t-1| + \frac{3}{4}\frac{-1}{(t-1)}\right] + C$$

dosadit za t

(2i)

$$\int \frac{1}{\sin^2 x + \sin x \cos x + \cos^2 x}\, dx$$

$t = \operatorname{tg} x \qquad x \in \left(-\frac{\pi}{2} + 2\pi, \frac{\pi}{2} + 2\pi\right)$

$dx = \frac{dt}{t^2+1}$

$$= \int \frac{1}{\frac{t^2}{1+t^2} + \frac{t}{1+t^2} + \frac{1}{1+t^2}} \cdot \frac{1}{t^2+1}\, dt =$$

$$= \int \frac{1}{t^2+t+1}\, dt = \int \frac{1}{\left(t+\frac{1}{2}\right)^2 + \frac{3}{4}}\, dt =$$

$$= \int \frac{1}{\frac{3}{4}} \cdot \frac{1}{\left(\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)^2 + 1}\, dt = \frac{4}{3} \cdot \frac{\sqrt{3}}{2} \arctan \frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$

$$= \frac{2}{\sqrt{3}} \arctan \frac{2\operatorname{tg} x + 1}{\sqrt{3}} + c \qquad =: F$$

Lepení

D_f: $\sin x \cos x \neq -1$

$\sin 2x \neq -2$ ✓

$\operatorname{tg} x$:

$c_{-1} \quad -\frac{\pi}{2} \quad c_0 \quad \frac{\pi}{2} \quad c_1$

$$\lim_{x \to \frac{\pi}{2}-} F(x) = \lim \frac{2}{\sqrt{3}} \arctan \frac{2\operatorname{tg} x + 1}{\sqrt{3}} + c_0 = \frac{2}{\sqrt{3}} \frac{\pi}{2} + c_0$$

$$\lim_{x \to \frac{\pi}{2}+} \frac{2}{\sqrt{3}} \arctan \frac{2\operatorname{tg} x + 1}{\sqrt{3}} + c_1 = \frac{-2}{\sqrt{3}} \frac{\pi}{2} + c_1$$

(2i)

$$\frac{2}{\sqrt{3}}\frac{\pi}{2}+C_k = \frac{-2}{\sqrt{3}}\frac{\pi}{2}+C_{k+1}$$

$$\boxed{\frac{2}{\sqrt{3}}\pi + C_k = C_{k+1}}$$

$C_0 = C_0$

$C_1 = \frac{2}{\sqrt{3}}\pi + C_0$

$C_2 = 2\cdot\frac{2}{\sqrt{3}}\pi + C_0$

$C_3 = 3\frac{2}{\sqrt{3}}\pi + C_0$

$C_n = n\frac{2}{\sqrt{3}}\pi + C_0$

tedy

$$F(x) = \begin{cases} \frac{2}{\sqrt{3}}\operatorname{arctan}\frac{2\operatorname{tg}x+1}{\sqrt{3}} + n\frac{2}{\sqrt{3}}\pi + C_0 & x\in\left(-\frac{\pi}{2}+n\pi; \frac{\pi}{2}+n\pi\right) \\ (n+1)\frac{2}{\sqrt{3}}\pi + C_0 & x = \frac{\pi}{2}+n\pi \end{cases}$$

$n\in\mathbb{Z}$

(2k) $\displaystyle\int \frac{1}{(1+\sin x)(1-\cos x)}\,dx$

$t = \operatorname{tg}\frac{x}{2} \qquad x\in(-\pi+2k\pi,\ \pi+2k\pi),\ k\in\mathbb{Z}$

Obecně:

$$= \int \frac{1}{\left(1+\frac{2t}{1+t^2}\right)\left(1-\frac{1-t^2}{1+t^2}\right)}\cdot\frac{2}{1+t^2}\,dt$$

$$= 2\int \frac{1}{(1+2t+t^2)(1+t^2-1+t^2)}\,dt = \cancel{2}\int \frac{1}{\cancel{2}(t+1)^2t^2}\,dt$$

udělám

$$= \int \frac{1}{t^2}+\frac{2}{t+1}+\frac{1}{(t+1)^2}-\frac{2}{t}\,dt =$$

$$= -\frac{1}{t} + 2\ln|t+1| - \frac{1}{t+1} - 2\ln|t| + C$$

$t = \operatorname{tg}\frac{x}{2}$

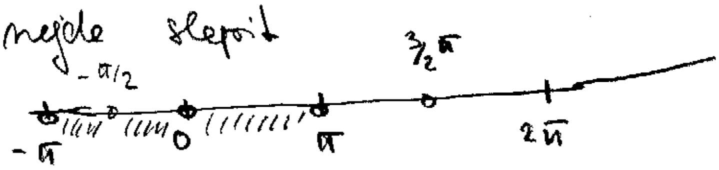
Podmínky $\quad \sin x \neq -1 \qquad x \neq \frac{3}{2}\pi + 2k\pi$

$\qquad\qquad \cos x \neq 1 \qquad x \neq 0 + 2k\pi \qquad k\in\mathbb{Z}$

Lepení v $\pi + 2k\pi$:

$$\lim_{x\to\pi-} -\frac{1}{\operatorname{tg}\frac{x}{2}} + 2\ln\Big|\underbrace{\operatorname{tg}\frac{x}{2}}_{\infty}+1\Big| - \frac{1}{\operatorname{tg}\frac{x}{2}+1} - 2\ln\left|\operatorname{tg}\frac{x}{2}\right| + C_0$$

nejde slepit

Substituce probíhala na 

$(\text{—})(\text{—})(\text{———}) + 2k\pi$