(1) $\lim_{x\to 0} \frac{1}{x^2} - \frac{1}{tg^2 x} = \lim_{x\to 0} \frac{tg^2 x - x^2}{x^2 \, tg^2 x} \overset{VOAL}{=}$

$\lim_{x\to 0} \frac{x^2}{tg^2 x} \cdot \lim_{x\to 0} \frac{tg^2 x - x^2}{x^4} = \underline{\underline{1 \cdot \frac{2}{3}}}$

Taylor $\quad tg\, x = x + \frac{1}{3}x^3 + o(x^4)$

$\lim_{x\to 0} \frac{(x + \frac{1}{3}x^3 + o(x^4))^2 - x^2}{x^4} = \lim \frac{\cancel{x^2} + \frac{1}{3}x^4 + x\,o(x^4) + \frac{1}{3}x^4 + \frac{1}{9}x^6 + \frac{1}{3}x^3 o(x^4) + x\,o(x^4) + \frac{1}{3}x^3 o(x^4) + o(x^4)\,o(x^4) - \cancel{x^2}}{x^4}$

$= \lim_{x\to 0} \frac{\frac{2}{3}x^4 + \frac{1}{9}x^6 + o(x^4)}{x^4} \overset{VOAL}{=} \underline{\underline{\frac{2}{3}}}$

(2) $\lim_{x \to \frac{\pi}{2}} \left(tg\, x + \frac{1}{x - \frac{\pi}{2}} \right) = \lim_{x \to \frac{\pi}{2}} \frac{\sin x}{\cos x} + \frac{1}{x - \frac{\pi}{2}}$

$= \lim_{x \to \frac{\pi}{2}} \frac{(\sin x)(x - \frac{\pi}{2}) + \cos x}{(\cos x)(x - \frac{\pi}{2})} \overset{\text{VOAL}}{=} \lim_{x \to \frac{\pi}{2}} \frac{(x - \frac{\pi}{2})}{\cos x} \cdot \lim_{x \to \frac{\pi}{2}} \frac{(\sin x)(x - \frac{\pi}{2}) + \cos x}{(x - \frac{\pi}{2})^2}$

- $\lim_{x \to \frac{\pi}{2}} \frac{x - \frac{\pi}{2}}{\cos x} \underset{\frac{0}{0}}{\overset{L'H}{=}} \lim_{x \to \frac{\pi}{2}} \frac{1}{\sin x} = \underline{\underline{1}}$

- $\lim_{x \to \frac{\pi}{2}} \frac{(\sin x)(x - \frac{\pi}{2}) + \cos x}{(x - \frac{\pi}{2})^2} \underset{\frac{0}{0}}{\overset{L'H}{=}} \lim_{x \to \frac{\pi}{2}} \frac{(\cos x)(x - \frac{\pi}{2}) + \sin x - \sin x}{2(x - \frac{\pi}{2})}$

$= \lim_{x \to \frac{\pi}{2}} \frac{\cos x}{2} = \underline{\underline{0}}$

celkem $= \underline{\underline{0}}$

(3) $\lim\limits_{x\to 0}\left(\frac{2}{\pi}\arccos x\right)^{\cotg x} = \lim\limits_{x\to 0} e^{\cotg x \ln \frac{2}{\pi}\arccos x}$

$\lim\limits_{x\to 0}\frac{\cos x}{\sin x}\ln\frac{2}{\pi}\arccos x \overset{\text{VOAL}}{=} \lim\limits_{x\to 0}\overbrace{\frac{x}{\sin x}\cdot\cos x}^{\to 1}\cdot\lim\frac{\ln\left(\frac{2}{\pi}\arccos x\right)}{x}$

$\lim\limits_{x\to 0}\frac{\ln\left(\frac{2}{\pi}\overset{\nearrow\frac{\pi}{2}}{\arccos x}\right)}{x} \underset{\frac{0}{0}}{\overset{l'H}{=}} \lim\limits_{x\to 0}\frac{\frac{1}{\frac{2}{\pi}\arccos x}\cdot\frac{-1}{\sqrt{1-x^2}}}{1}$

$= \frac{-1}{\underline{\underline{1}}}$

(S) e^y spoj v -1

$\lim\limits_{y\to -1} e^y = \frac{1}{e}$

$\lim\limits_{x\to 0} g(x) = -1$

větou: $f(y) = e^y$

$g(x) = \cotg x \ln\left(\frac{2}{\pi}\arccos x\right)$

tedy " $= \underline{\underline{e^{-1}}}$

(4) $$\lim_{x\to 0} \frac{\cot g\, x - \frac{1}{x} + \frac{x}{3}}{x^3} = \lim \frac{\frac{\cos x}{\sin x} - \frac{1}{x} + \frac{x}{3}}{x^3} =$$

$$= \lim_{x\to 0} \frac{x\cos x - \sin x + \frac{1}{3} x^2 \sin x}{\sin x \cdot x^4} \overset{\text{VOAL}}{=} \lim_{x\to 0} \frac{x}{\sin x} \cdot \lim \frac{x\cos x - \sin x + \frac{1}{3}x^2 \sin x}{x^5}$$

$$\lim_{x\to 0} \frac{x\cos x - \sin x + \frac{1}{3}x^2 \sin x}{x^5} =$$

Taylor $$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)$$
$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5)$$

$$= \lim \frac{\cancel{x} - \cancel{\frac{x^3}{2}} + \frac{x^5}{24} + o(x^5) - \cancel{x} + \cancel{\frac{x^3}{6}} - \frac{x^5}{120} + o(x^5) + \cancel{\frac{1}{3}x^3} - \frac{1}{3\cdot 6}x^5 + o(x^5)}{x^5}$$

$$= \lim \frac{x^5\left(\frac{1}{24} - \frac{1}{120} - \frac{1}{18}\right) + o(x^5)}{x^5} = -\frac{1}{45}$$

(5) $\lim_{x \to 0} \frac{\operatorname{tg} x - x - \frac{x^3}{3}}{x^5} = \lim_{x \to 0} \frac{x + \frac{x^3}{3} + \frac{2}{15}x^5 - x - \frac{x^3}{3}}{x^5} = \underline{\underline{\frac{2}{15}}}$

$\operatorname{tg} x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + o(x^5)$

(6) $\lim\limits_{x \to 0} \dfrac{\arcsin x - \sin x}{\arctan x - \tan x} = \lim\limits_{x \to 0} \dfrac{\cancel{x} + \frac{x^3}{6} + o(x^3) - (\cancel{x} - \frac{x^3}{6} + o(x^3))}{\cancel{x} - \frac{x^3}{3} + o(x^3) - (\cancel{x} + \frac{x^3}{3} + o(x^3))}$

$= \lim\limits_{x \to 0} \dfrac{\frac{1}{3}x^3 + o(x^3)}{-\frac{2}{3}x^3 + o(x^3)} = \frac{1}{3} \cdot \frac{-3}{2} = -\frac{1}{2}$

(7) $T_{3,\frac{\pi}{2}}^{\cot g\, x}$

$f(x) = \cot g\, x$

$f'(x) = \dfrac{-1}{\sin^2 x}$

$f''(x) = 2 \cdot \dfrac{\cos x}{\sin x} \cdot \dfrac{1}{\sin^2 x} = \dfrac{2\cos x}{\sin^3 x}$

$f'''(x) = 2 \cdot \dfrac{-\sin x \sin^3 x - \cos x\, 3\sin^2 x \cos x}{\sin^6 x} = 2 \dfrac{-\sin^2 x - 3\cos^2 x}{\sin^4 x}$

$f\left(\frac{\pi}{2}\right) = 0$

$f'\left(\frac{\pi}{2}\right) = -1$

$f''\left(\frac{\pi}{2}\right) = 0$

$f'''\left(\frac{\pi}{2}\right) = 2 \dfrac{-1}{1} = -2$

Tedy Taylor: $-1\left(x-\frac{\pi}{2}\right) - \frac{2}{3!}\left(x-\frac{\pi}{2}\right)^3$

(8) vizte cvičení

(a)

$$\sum_{n=1}^{\infty} \left(\operatorname{arctg} \frac{1}{n}\right)\left(\operatorname{arctg}\left(2-\frac{1}{n}\right)\right) i^n (z-i)^n =: \sum a \qquad z \in \mathbb{C}$$

$$\text{"}\frac{1}{R}\text{"} = \limsup_{n\to\infty} \underbrace{\sqrt[n]{\operatorname{arctg}\frac{1}{n}}}_{\to 1 \text{ (viz další stranu)}} \underbrace{\sqrt[n]{\operatorname{arctg}\left(2-\frac{1}{n}\right)}}_{\to \operatorname{arctg} 2} \underbrace{\sqrt[n]{|i^n|}}_{=1} = 1$$

(užiti 2 policajtů volcha)

tedy $\sum$ AK pro $z \in \mathbb{C}$: $|z-i| < 1$

$\sum$ D $\qquad |z-i| > 1$

- co když $|z-i| = 1$?

$z:\quad i + e^{\alpha i}, \quad \alpha \in [0, 2\pi)$

tedy řešíme $$\sum_{n=1}^{\infty} \underbrace{\operatorname{arctg}\frac{1}{n}}_{\text{chová se jako } \frac{1}{n}} \underbrace{\operatorname{arctg}\left(2-\frac{1}{n}\right)}_{\text{jako konstanta}} \underbrace{i^n}_{=i,\,-1,\,-i,\,1} \underbrace{(i+e^{\alpha i}-i)^n}_{e^{\alpha i n} = \cos\alpha n + i\sin\alpha n}$$

(a) AK: $\sum \operatorname{arctg}\frac{1}{n} \operatorname{arctg}\left(2-\frac{1}{n}\right) \cdot \underbrace{|i^n| \cdot |e^{\alpha i n}|}_{1} =: \sum a_n$

LSK s $b_n = \frac{1}{n}$ $\qquad \lim_{n\to\infty} \frac{\operatorname{arctg}\frac{1}{n}}{\frac{1}{n}} \cdot \operatorname{arctg}\left(2-\frac{1}{n}\right) = \operatorname{arctg} 2$

$\sum \frac{1}{n}$ D

(Heine, VOLSF...)

tedy $\sum a_n$ K $\iff \sum b_n$ K $\Rightarrow$ $\boxed{\sum a_n \text{ D}}$

(b) NAK: $\alpha = 0$, pak $\sum \operatorname{arctg}\frac{1}{n} \operatorname{arctg}\left(2-\frac{1}{n}\right) i^n$

$$= i \sum_{n=1,3,5,7\ldots} \operatorname{arctg}\frac{1}{n} \operatorname{arctg}\left(2-\frac{1}{n}\right)(-1)^{\frac{n+1}{2}} + 1 \sum_{n=2,4,6\ldots} \operatorname{arctg}\frac{1}{n} \operatorname{arctg}\left(2-\frac{1}{n}\right)(-1)^{\frac{n}{2}}$$

(b) pokračování

$$\underbrace{\underbrace{\sum \Big(\underset{\downarrow \text{ monot.}\ 0}{\operatorname{arctg} \tfrac{1}{n}}\Big) (-1)^{\frac{n+1}{2}}}_{k:\ \text{Leibniz}} \operatorname{arctg}\left(2-\tfrac{1}{n}\right)}_{k:\ \text{Ábel}}$$

Druhá suma stejně. Tedy $k\sum + k\sum =$ konvergentní

(c) NAK $\alpha \neq 0$

Rozsypeme na 4 $\sum$, vynásobené konstantou $i, -1, -i$ a 1.

Řešíme $$\sum \underbrace{\operatorname{arctg} \tfrac{1}{n}}_{\to 0 \text{ monot.}} \operatorname{arctg}\left(2-\tfrac{1}{n}\right) \underbrace{e^{\alpha i n}}_{\text{om. č. součty}}$$

k Dirichlet

Ábel

celkem k

$\sqrt[n]{\operatorname{arctg}\frac{1}{n}} = \left(\operatorname{arctg}\frac{1}{n}\right)^{1/n}$ Heine $x_n = \frac{1}{n}$, $x_n \to 0$, $x_n \neq 0$ $\forall n \in \mathbb{N}$

$\lim\limits_{x\to 0} (\operatorname{arctg} x)^x = \lim\limits_{x\to 0} e^{x \ln(\operatorname{arctg} x)} \overset{"}{=} e^{0"} = 1$

$\lim\limits_{x\to 0} \cancel{=} \dfrac{\ln(\operatorname{arctg} x)}{\frac{1}{x}} \underset{\frac{něco}{\infty}}{\overset{l'H}{=}} \lim\limits_{x\to 0} \dfrac{\frac{1}{\operatorname{arctg} x} \cdot \frac{1}{1+x^2}}{-\frac{1}{x^2}}$

$= \lim\limits_{x\to 0} \dfrac{-x^2}{1+x^2} \cdot \dfrac{1}{\operatorname{arctg} x} = \lim\limits_{x\to 0} \dfrac{-x}{1+x^2} \cdot \dfrac{x}{\operatorname{arctg} x} \overset{VOAL}{=} 0 \cdot 1 = 0$

VOLSF $g(x) = x \ln(\operatorname{arctg} x)$ $\lim\limits_{x\to 0} g(x) = 0$

$f(y) = e^y$

$\lim\limits_{y\to 0} e^y = 1$ (S) e^y spoj v 0

(1a) $f(z) = \frac{1}{1-z}$; $z_0 = 0$

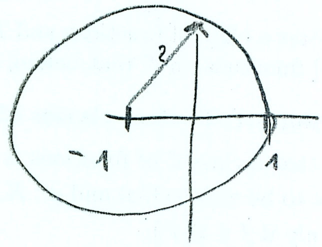
$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad |z| < 1$$

(1b) $z_0 = -1$

$$\frac{1}{1-z} = \frac{1}{1-(z+1)+1} = \frac{1}{2-(z+1)} = \frac{1}{2\left(1-\left(\frac{z+1}{2}\right)\right)}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z+1}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(z+1)^n}{2^{n+1}} \quad \text{pro} \quad \left|\frac{z+1}{2}\right| < 1$$

$$|z+1| < 2$$



(1c) $z_0 = i$

$$\frac{1}{1-z} = \frac{1}{1-(z-i)-i} = \frac{1}{(1-i)-(z-i)} = \frac{1}{1-i}\cdot\frac{1}{1-\left(\frac{z-i}{1-i}\right)}$$

$$= \frac{1}{1-i}\cdot\sum_{n=0}^{\infty}\frac{(z-i)^n}{(1-i)^n} = \sum_{n=0}^{\infty}\frac{(z-i)^n}{(1-i)^{n+1}}$$

$$\left|\frac{z-i}{1-i}\right| < 1$$

$$|z-i| < |1-i|$$

$$|z-i| < \sqrt{2}$$

(2a) $g(z) = \dfrac{1}{(1-z)^2}$; $z_0 = 0$

$$\frac{1}{(1-z)^2} = \left(\frac{1}{1-z}\right)' = \left(\sum_{n=0}^{\infty} z^n\right)' = \sum_{n=1}^{\infty} n z^{n-1}$$

$|z| < 1$

(2b) $z_0 = -1$

$$\frac{1}{(1-z)^2} = \left(\frac{1}{1-z}\right)' = \left(\sum_{n=0}^{\infty} \frac{(z+1)^n}{2^{n+1}}\right)' =$$

$$= \sum_{n=1}^{\infty} \frac{n(z+1)^{n-1}}{2^{n+1}} \qquad |z+1| < 2$$

(2c) $z_0 = i$

$$\frac{1}{(1-z)^2} = \left(\frac{1}{1-z}\right)' = \left(\sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}}\right)'$$

$$= \sum_{n=1}^{\infty} \frac{n(z-i)^{n-1}}{(1-i)^{n+1}} \qquad |z-i| < \sqrt{2}$$

(10)

(3a) $h(z) = \frac{z}{(1-z)^2}$; $z_0 = 0$

$$\frac{z}{(1-z)^2} = z \cdot \left(\frac{1}{1-z}\right)' = z \cdot \sum_{n=1}^{\infty} n z^{n-1} = \sum_{n=1}^{\infty} n z^n$$

$|z| < 1$

(3b) $z_0 = -1$

$$\frac{z}{(1-z)^2} = \frac{z-1 \quad +1}{(1-z)^2} = \frac{-1}{1-z} + \frac{1}{(1-z)^2}$$

$$= -\sum_{n=0}^{\infty} \frac{(z+1)^n}{2^{n+1}} + \sum_{n=1}^{\infty} \frac{n(z+1)^{n-1}}{2^{n+1}} = \sum_{n=0}^{\infty} \left(\frac{-1}{2^{n+1}} + \frac{n+1}{2^{n+2}}\right)(z+1)^n$$

$\left|\frac{z+1}{2}\right| < 1 \quad \rightarrow \quad |z+1| < 2$

(3c) $z_0 = i$

$$\frac{z}{(1-z)^2} = \frac{-1}{1-z} + \frac{1}{(1-z)^2} = -\sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}} + \sum_{n=1}^{\infty} \frac{n(z-i)^{n-1}}{(1-i)^{n+1}}$$

$|z-i| < \sqrt{2}$

$$= \sum_{n=0}^{\infty} \left(\frac{-1}{(1-i)^{n+1}} + \frac{n+1}{(1-i)^{n+2}}\right)(z-i)^n$$