(a) $\int_1^4 y'^2 \, dx$ $\int_1^4 y \, dx = 36$

$y(1) = 3 \qquad y(4) = 24$

$f = y'^2 \qquad g = y$

$H = y'^2 + \lambda y$

(EL): $y' 2y' - y'^2 - \lambda y = C$

$y'^2 - \lambda y = c$

$y' = \pm\sqrt{c + \lambda y}$

kr. for g: $g = y$

$\frac{\partial g}{\partial y} - \frac{d}{dx}\left(\frac{\partial g}{\partial y'}\right) = 0$

$1 = 0$

Jinak

$\lambda - \frac{d}{dx}(2y') = 0$

$\frac{d}{dx}(2y') = \lambda$

$2y' = \lambda x + k$

$y' = \frac{\lambda}{2}x + k$

$y = \frac{\lambda}{4}x^2 + kx + c$

pak

$3 = \frac{\lambda}{4} + k + c$

$24 = 4\lambda + 4k + c$

$\int_1^4 \frac{\lambda}{4}x^2 + kx + c \, dx = 36 \rightarrow \left[\frac{\lambda}{12}x^3 + \frac{k}{2}x^2 + cx\right]_1^4 = 36$

$\rightarrow 42\lambda + 60k + 24c = 288 \rightarrow \lambda = -4 \quad c = -16 \quad k = 20$

max

(b) $\int_0^1 y \, dx \qquad \int_0^1 \sqrt{1+y'^2} \, dx = \frac{\pi}{2}$

$$y(0) = y(1) = 0$$

$$H = y + \lambda \sqrt{1+y'^2}$$

(EL): $1 - \frac{d}{dx}\left(\lambda \frac{1}{\cancel{2}} \frac{1}{\sqrt{1+y'^2}} \cdot \cancel{2} y'\right) = 0$

$$\frac{\lambda y'}{\sqrt{1+y'^2}} = x + c$$

$$\to \quad y' = \pm \frac{x+c}{\sqrt{\lambda^2 - (x+c)^2}}$$

$$y = \pm \sqrt{\lambda^2 - (x+c)^2} + d$$

neboli $(y-d)^2 + (x+c)^2 = \lambda^2 \to$ kružnice

podmínky
$$0 = \pm \sqrt{\lambda^2 - c^2} + d$$
$$1 = \pm \sqrt{\lambda^2 - (1+c)^2} + d$$

$$\int_0^1 \sqrt{1 + \frac{(x+c)^2}{\lambda^2 - (x+c)^2}} = \frac{\pi}{2}$$

(c) $\int_0^{\pi} (y'^2 - y^2)\, dx \qquad y(0)=0 \qquad \int_0^{\pi} y\, dx = 1$

$y(\pi) = 1$

$H = y'^2 - y^2 + \lambda y$

$-2y + \lambda - \frac{d}{dx}(2y') = 0$

$\lambda = 2y'' + 2y$

$\lambda = y'' + y$

pak $\quad y = c\cos x + d\sin x + \lambda$

a $\quad 0 = c + \lambda$

$1 = -c + \lambda \quad \Rightarrow \quad \lambda = \frac{1}{2} \quad c = -\frac{1}{2}$

$\int_0^{\pi} -\frac{1}{2}\cos x + d\sin x + \frac{1}{2}\, dx = 1$

$\left[-\frac{1}{2}\sin x + (-d)\cos x + \frac{1}{2}x\right]_0^{\pi} = +d + d + \frac{\pi}{2}$

$2d + \frac{\pi}{2} = 1 \qquad d = \frac{1}{2} - \frac{\pi}{4}$

dohromady

$$y = -\frac{1}{2}\cos x + \left(\frac{1}{2} - \frac{\pi}{4}\right)\sin x + \frac{1}{2}$$

(d)

Solution Attempt 3. Let us severely penalize the use of unnecessary work by means of the cost

$$J = \int_0^{10} (uv)^2\, dt = \int_0^{10} \dot{v}^2 v^2\, dt, \tag{7.8}$$

subject to the end conditions $x(0) = v(0) = 0$ and the integral constraint (7.4).

Apply the Euler-Lagrange equation to the augmented Lagrangian

$$N = \dot{v}^2 v^2 + \lambda v \tag{7.9}$$

to obtain that $\lambda = 0$ [because $v(0) = 0$] and that when $v\dot{v} \neq 0$ (Exercise 7.6):

$$-\dot{v}/v = \ddot{v}/\dot{v}, \tag{7.10}$$

$$2v\dot{v} = k^2, \tag{7.11}$$

$$v = k\sqrt{t}, \tag{7.12}$$

$$x = 2kt^{3/2}/3, \tag{7.13}$$

giving from $x(10) = 100$ that $k = 15/\sqrt{10}$. Hence total work done under this control strategy is

$$W = v(10)^2/2 = (k\sqrt{10})^2/2 = 225/2 = 112.5, \tag{7.14}$$

no improvement over our previous attempt. Moreover, our control force

$$u = \dot{v} = k/2\sqrt{t}, \tag{7.15}$$

although positive, is unbounded at $t = 0$. Note that transversality requires (Exercise 7.14) that $\dot{v}(10)v(10) = 0$ and so (7.15) cannot be optimal for the cost (7.8).

(e) $\int_0^{10} y'^2 + y^2$

$\int_0^{10} y = 100$

$y(0) = 0$

$y(10) = 200$

$H = y'^2 + y^2 + \lambda y$

$$2y + \lambda - \frac{d}{dx}(2y') = 0$$

$$\lambda = y'' - y$$

<u>FSR</u> $e^x + \lambda = y$

$e^{-x} - \lambda = y$

tedy $y = k e^x + \ell e^{-x} + \lambda$

$0 = k + \ell + \lambda$

$$\int_0^{10} k e^x + \ell e^{-x} + \lambda = \left[k e^x - \ell e^{-x} + \lambda x \right]_0^{10} =$$

$$= -k + \ell + k e^{10} - \ell e^{-10} + 10\lambda = 100$$

$$k e^{10} + \ell e^{-10} + \lambda = 200$$

(f) $\int_0^1 y^2 \, dx = 2 \qquad y(0) = y(1) = 0$

$\int_0^1 y'^2 \, dx$

$H = y'^2 + \lambda y^2$

$2\lambda y - \frac{d}{dx}(2y') = 0$

$2y'' = 2\lambda y$

$y'' = \lambda y$

$\lambda > 0 \qquad$ FSŘ $\qquad y = e^{\pm\sqrt{\lambda}x}$

$\lambda > 0 \qquad y = c e^{\sqrt{\lambda}x} + d e^{-\sqrt{\lambda}x}$

$\lambda = 0 \qquad y = cx + d$

$\lambda < 0 \qquad y = c \cos\sqrt{|\lambda|}\, x + d \sin\sqrt{|\lambda|}\, x$

(a) $\begin{array}{l} 0 = c + d \\ 0 = c e^{\sqrt{\lambda}} + d e^{-\sqrt{\lambda}} \end{array} \to c = d = 0 \to$ neřeší $\int$

(b) $c = d = 0$ totéž

(c) $0 = c$ a $d = 0$ nebo $|\lambda| = k^2\pi^2$ a $c = 0$

Dak $\int_0^1 c \cos k\pi x + d \sin k\pi x = 2 \qquad k \in \mathbb{Z}$

$\left[\frac{c}{k\pi} \sin k\pi x - \frac{d}{k\pi} \cos k\pi x \right]_0^1 = 2$

(f) $+\dfrac{d}{k\alpha} + \dfrac{d}{2\alpha} = 2$

$$d = 2\alpha$$

(jeu urc. 2)

(g) $\int_0^T e^{-rx} y' \, dx \qquad \int_0^T \sqrt{y} \, dx = A$

$$H = e^{-rx} y + \lambda \sqrt{y}$$

$$e^{-rx} + \frac{1}{2}\lambda \frac{1}{\sqrt{y}} = 0$$

$$\frac{1}{\sqrt{y}} = \frac{-2}{\lambda} e^{-rx}$$

$$\cancel{2}\sqrt{y} = \frac{2}{\lambda}\frac{1}{r} e^{-rx} + c$$

$$\underline{\underline{y = \left(\frac{1}{\lambda r} e^{-rx} + c\right)^2}}$$

min $\int_0^{10} y'y \, dx$ $\quad\quad$ $\int_0^{10} y \, dx = 100$

(h)

$$H = y'y + \lambda y$$

$$y' + \lambda - \frac{d}{dx} y = 0$$

$$\lambda = 0$$

$$(i) \quad \int_0^{10} c y'^2 + y'y \, dx \qquad \int_0^{10} y = 100$$

$$y(0) = 0$$

$$y(10) = 20$$

$$H = cy'^2 + y'y + \lambda y$$

$$y' + \lambda - \frac{d}{dx}(2cy' + y) = 0$$

$$\cancel{y'} + \lambda - 2cy'' - \cancel{y'} = 0$$

$$y'' = \frac{\lambda}{2c}$$

$$y(x) = \frac{\lambda}{4c} x^2 + dx + b$$

$$0 = y(0) = b$$

$$\int_0^{10} y = \left[\frac{\lambda}{12c} x^3 + \frac{d}{2} x^2\right]_0^{10} = \frac{\lambda}{12c} 10^3 + \frac{d}{2} 100$$

$$= 100 \quad \rightarrow \quad \frac{\lambda}{12c} 10 + \frac{d}{2} = 1$$

$$10\lambda + 6cd = 12c$$

$$\frac{\lambda}{4c} 100 + 10d = 20$$

$$\rightarrow \lambda = 0, \; d = 2$$

$$\underline{\underline{y = 2x}}$$

EXERCISE

Variational problems

1. Test for extremum of the functional

$$I[y(x)] = \int_0^{\frac{\pi}{2}} (y'^2 - y^2)dx,\ y(0) = 0,\ y\left(\frac{\pi}{2}\right) = 1.$$

Hint: Euler's Equation (EE): $y'' + y = 0$, $y = c_1 \cos x + c_2 \sin x$ using B.C. $c_1 = 0, c_2 = 1$

Ans. $y = \sin x$

Find the extremal of the following functionals

2. $\int_{x_1}^{x_2} (y^2 + y'^2 - 2y \sin x)dx$

Hint: EE: $2y - 2\sin x - 2y'' = 0$

Ans. $y = c_1 e^x + c_2 e^{-x} + \frac{\sin x}{2}$

3. $\int_0^1 (y'^2 + 12xy)dx$, $y(0) = 0$, $y(1) = 1$.

Hint: EE: $y'' = 6x$, $y = x^3 + c_1 x + C_2$, $C = 0, c_2 = 0$

Ans. $y = x^3$

4. $\int_0^{\frac{\pi}{2}} (y'^2 - y^2 + 2xy)dy$, $y(0) = 0$, $y\left(\frac{\pi}{2}\right) = 0$

Hint: EE: $y'' + y = x$, $y = c_1 \cos x + c_2 \sin x + x$

Ans. $y = x - \frac{\pi}{2}\sin x$

5. $\int_{x_1}^{x_2} (y^2 + 2xyy')dx$, $y(x_1) = y_1$, $y(x_2) = y_2$

Hint: EE: $2y + 2xy' - 2(xy' + y) = 0$ i.e., $0 = 0$

Ans. Invalid problem

6. $\int_1^2 \frac{x^3}{y'^2}dx$, $y(1) = 1$, $y(2) = 4$

Ans. $y = x^2$

7. $\int_2^3 \frac{y'^2}{x^3}dx$, $y(2) = 1$, $y(3) = 16$

Hint: EE: $\frac{y''}{y'} = \frac{3}{x}$, $y' = cx^3$, $y = c_1 x^4 + c_2$

Ans. $y = \frac{3}{13}x^4 - \frac{35}{13}$

8. $\int_{x_0}^{x_1} (y^2 + y'^2 + 2ye^x)dx$

Ans. $y = Ae^x + Be^{-x} + \frac{1}{2}xe^x$

9. $\int_0^{\pi} (4y\cos x - y^2 + y'^2)dx$, $y(0) = 0$, $y(\pi) = 0$

Hint: EE: $y'' + y = 2\cos x$, $y = c_1 \cos x + c_2 \sin x + x\sin x$, $c_1 = 0$, $c_2 =$ arbitrary

Ans. $y = (C + x)\sin x$.

4.5 ISOPERIMETRIC PROBLEMS

In calculus, in problems of extrema with constraints it is required to find the maximum or minimum of a function of several variably $g(x_1, x_2, \ldots, x_n)$ where the variables $x_1, x_2, \ldots, x_n$ are connected by some given relation or condition known as a constraint.

The variational problems considered so far find the extremum of a functional in which the argument functions could be chosen arbitrarily except for possible end (boundary) conditions. However, the class of variational problems with subsidiary conditions or constraints imposed on the argument functions, apart from the end conditions, are branded as isoperimetric problems. In the original isoperimetric ("iso" for same, "perimetric" for perimeter) problem it is required to find a closed curve of given length which enclose maximum area. It is known even in ancient Greece that the solution to this problem is circle. This is an example of the extrema of integrals under constraint consists of maximumizing the area subject to the constraint (condition) that the length of the curve is fixed.

The simplest isoperimetric problem consists of finding a function $f(x)$ which extremizes the functional

$$I[y(x)] = \int_{x_1}^{x_2} f(x, y, y')dx \quad (1)$$

subject to the constraint (condition) that the second integral

$$J[y(x)] = \int_{x_1}^{x_2} g(x, y, y')dx \quad (2)$$

assumes a given prescribed value and satisfying the prescribed end conditions $y(x_1) = y_1$ and $y(x_2) = y_2$. To solve this problem, use the method of Lagrange's multipliers and form a new function

$$H(x, y, y') = f(x, y, y') + \lambda g(x, y, y') \quad (3)$$

where λ is an arbitrary constant known as the Lagrange multiplier. Now the problem is to find the extremal of the new functional,

$I^*[y(x)] = \int_{x_1}^{x_2} H(x, y, y')dx$, subject to no constraints (except the boundary conditions). Then the modified Euler's equation is given by

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0 \quad (4)$$

The complete solution of the second order Equation (4) contains, in general, two constants of integration say c_1, c_2 and the unknown Lagrange multiplier λ. These 3 constants c_1, c_2, λ will be determined using the two end conditions $y(x_1) = y_1$, $y(x_2) = y_2$ and given constraint (2).

Corollary: Parametric form: To find the extremal of the functional

$$I = \int_{t_1}^{t_2} f(x, y, \dot{x}, \dot{y}, t)dt$$

subject to the constraint

$$J = \int_{t_1}^{t_2} g(x, y, \dot{x}, \dot{y}, t)dt = \text{constant}$$

solve the system of two Euler equations given by

$$\frac{\partial H}{\partial x} - \frac{d}{dt}\left(\frac{\partial H}{\partial \dot{x}}\right) = 0 \quad \text{and} \quad \frac{\partial H}{\partial y} - \frac{d}{dt}\left(\frac{\partial H}{\partial \dot{y}}\right) = 0$$

resulting in the solution $x = x(t)$, $y = y(t)$, which is the parametric representation of the required function $y = f(x)$ which is obtained by elimination of t. Here $\dot{x} = \frac{dx}{dt}$ and $\dot{y} = \frac{dy}{dt}$ and

$$H(x, y, \dot{x}, \dot{y}, t) = f(x, y, \dot{x}, \dot{y}, t) + \lambda\, g(x, y, \dot{x}, \dot{y}, t)$$

The two arbitrary constants c_1, c_2 and λ are determined using the end conditions and the constraint.

4.6 STANDARD ISOPERIMETRIC PROBLEMS

Circle

Example 1: Isoperimetric problem is to determine a closed curve C of given (fixed) length (perimeter) which encloses maximum area.

Solution: Let the parametric equation of the curve C be

$$x = x(t), \quad y = y(t) \quad (1)$$

where t is the parameter. The area enclosed by curve C is given by the integral

$$I = \frac{1}{2}\int_{t_1}^{t_2} (x\dot{y} - \dot{x}y)dt \quad (2)$$

where $\dot{x} = \frac{dx}{dt}$, $\dot{y} = \frac{dy}{dt}$. We have $x(t_1) = x(t_2) = x_0$ and $y(t_1) = y(t_2) = y_0$, since the curve is closed. Now the total length of the curce C is given by

$$J = \int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2}dt \quad (3)$$

Form $\quad H = \frac{1}{2}(x\dot{y} - \dot{x}y) + \lambda\sqrt{\dot{x}^2 + \dot{y}^2} \quad (4)$

Here λ is the unknown Lagrangian multiplier. Problem is to find a curve with given perimeter for which area (2) is maximum. Euler equations are

$$\frac{\partial H}{\partial x} - \frac{d}{dt}\left(\frac{\partial H}{\partial \dot{x}}\right) = 0 \quad (5)$$

and

$$\frac{\partial H}{\partial y} - \frac{d}{dt}\left(\frac{\partial H}{\partial \dot{y}}\right) = 0 \quad (6)$$

Differentiating H in (4) w.r.t. $x, \dot{x}, y, \dot{y}$ and substituting them in (5) and (6), we get

$$\frac{1}{2}\dot{y} - \frac{d}{dt}\left(-\frac{1}{2}y + \frac{\lambda\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right) = 0 \quad (7)$$

$$-\frac{1}{2}\dot{x} - \frac{d}{dt}\left(\frac{1}{2}x + \frac{\lambda\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right) = 0 \quad (8)$$

Integrating (7) and (8) w.r.t. 't', we get

$$y - \frac{\lambda\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = c_1 \quad (9)$$

and

$$x + \frac{\lambda\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = c_2 \quad (10)$$

where c_1 and c_2 are arbitrary constants. From (9) and (10) squaring $(y - c_1)$ and $(x - c_2)$ and adding, we get

$$(x - c_2)^2 + (y - c_1)^2 = \left(\frac{-\lambda\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right)^2 + \left(\frac{\lambda\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right)^2$$

$$= \lambda^2\frac{(\dot{x}^2 + \dot{y}^2)}{(\dot{x}^2 + \dot{y}^2)} = \lambda^2$$

i.e., $(x - c_2)^2 + (y - c_1)^2 = \lambda^2$

which is the equation of circle. Thus we have obtained the well-known result that the closed curve of given perimeter for which the enclosed area is a maximum is a circle.

Catenary

Example 2: Determine the shape an absolutely flexible, inextensible homogeneous and heavy rope of given length L suspended at the points A and B

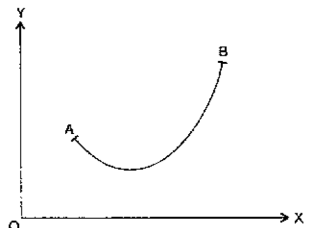

Fig. 4.6

Solution: The rope in equilibrium take a shape such that its centre of gravity occupies the lowest position. Thus to find minimum of y-coordinate of the centre of gravity of the string given by

$$I[y(x)] = \frac{\int_{x_1}^{x_2} y\sqrt{1+y'^2}dx}{\int_{x_1}^{x_2}\sqrt{1+y'^2}dx} \quad (1)$$

subject to the constraint

$$J[y(x)] = \int_{x_1}^{x_2}\sqrt{1+y'^2}dx = L = \text{constant} \quad (2)$$

Thus to minimize the numerator in R.H.S. of (1) subject to (2). Form

$$H = y\sqrt{(1+y'^2)} + \lambda\sqrt{1+y'^2} = (y+\lambda)\sqrt{1+y'^2} \quad (3)$$

where λ is Lagrangian multiplier. Here H is independent of x. So the Euler equation is

$$H - y'\frac{\partial H}{\partial y'} = \text{constant} = k_1$$

i.e., $(y+\lambda)(\sqrt{1+y'^2}) - y'(y+\lambda)\cdot\frac{1}{2}\frac{2y'}{\sqrt{1+y'^2}} = k_1$

$$(y+\lambda)\left\{(1+y'^2) - y'^2\right\} = k_1(\sqrt{1+y'^2})$$

or $$y + \lambda = k_1\sqrt{1+y'^2} \quad (4)$$

Put $y' = \sinh t$, where t is a parameter, in (4)

Then $\quad y + \lambda = k_1\sqrt{1+\sinh^2 ht} = k_1\cosh t \quad (5)$

Now $\quad dx = \frac{dy}{y'} = \frac{k_1\sinh t\, dt}{\sinh t} = k_1 dt$

Integrating $\quad x = k_1 t + k_2 \quad (6)$

Eliminating 't' between (5) and (6), we have

$$y + \lambda = k_1\cosh t = k_1\cosh\left(\frac{x-k_2}{k_1}\right) \quad (7)$$

Equation (7) is the desired curve which is a catenary.

Note: The three unknowns λ, k_1, k_2 will be determined from the two boundary conditions (curve passing through A and B) and the constraint (2).

WORKED OUT EXAMPLES

Example 1: Find the extremal of the function $I[y(x)] = \int_0^{\pi}(y'^2 - y^2)dx$ with boundary conditions $y(0) = 0, y(\pi) = 1$ and subject to the constraint $\int_0^{\pi} y\, dx = 1$.

Solution: Here $f = y'^2 - y^2$ and $g = y$. So choose $H = f + \lambda g = (y'^2 - y^2) + \lambda y$ where λ is the unknown Lagrange's multiplier. The Euler's equation for H is

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0$$

Using derivatives of H w.r.t. y and y', we get

$$(-2y + \lambda) - \frac{d}{dx}(2y') = 0$$

$$\text{or} \quad y'' + y = \lambda$$

whose general solution is

$$y(x) = CF + PI = (c_1\cos x + c_2\sin x) + (\lambda) \quad (1)$$

The three unknowns c_1, c_2, λ in (1) will be determined using the two boundary conditions and the given constraint. From (1)

$$0 = y(0) = c_1 + c_2\cdot 0 + \lambda \quad \text{or} \quad c_1 + \lambda = 0$$

$$1 = y(\pi) = -c_1 + c_2\cdot 0 + \lambda \quad \text{or} \quad -c_1 + \lambda = 1$$

Solving $\lambda = \frac{1}{2}, c_1 = -\lambda = -\frac{1}{2}$

Now from the given constraint

$$\int_0^{\pi} y\, dx = 1, \quad \text{we have}$$

$$\int_0^{\pi}(c_1\cos x + c_2\sin x + \lambda)dx = 1$$

$$c_1\sin x - c_2\cos x + \lambda x\Big|_0^{\pi} = 1$$

$$(0 + c_2 + \lambda\pi) - (0 - c_2 + 0) = 1$$

or $$2c_2 = 1 - \pi\lambda = \left(1 - \frac{\pi}{2}\right)$$

Thus the required extremal function $y(x)$ is

$$y(x) = -\frac{1}{2}\cos x + \left(\frac{1}{2} - \frac{\pi}{4}\right)\sin x + \frac{1}{2}.$$

Example 2: Show that the extremal of the isoperimetric problem $I[y(x)] = \int_{x_1}^{x_2} y'^2\, dx$ subject to the condition $J[y(x)] = \int_{x_1}^{x_2} y\, dx = \text{constant} = k$ is a parabola. Determine the equation of the parabola passing through the points $P_1(1, 3)$ and $P_2(4, 24)$ and $k = 36$.

Solution: Here $f = y'^2$ and $g = y$. So form

$$H = f + \lambda g = y'^2 + \lambda y.$$

The Euler equation for H is

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0$$

$$\lambda - \frac{d}{dx}(2y') = 0$$

$$\text{or} \quad y'' - \frac{\lambda}{2} = 0$$

Integrating twice,

$$y(x) = \frac{\lambda}{2}\frac{x^2}{2} + c_1 x + c_2 \quad (1)$$

which is a parabola. Here c_1 and c_2 are constants of integration. To determine the particular parabola, use B.C's $y(1) = 3$ and $y(4) = 24$ (i.e., passing through points P_1 and P_2) and the given constraint. From (1)

$$3 = y(1) = \frac{\lambda}{4} + c_1 + c_2 \quad (2)$$

Again from (1)

$$24 = y(4) = 4\lambda + 4c_1 + c_2 \quad (3)$$

Now from the constraint

$$\int_{x_1=1}^{x_2=4} y(x)dx = 36$$

or $$\int_1^4\left(\frac{\lambda}{4}x^2 + c_1 x + c_2\right)dx = 36$$

i.e., $$\frac{\lambda}{4}\cdot\frac{x^3}{3} + c_1\frac{x^2}{2} + c_2 x\Big|_1^4 = 36$$

or $$42\lambda + 60c_1 + 24c_2 = 288 \quad (4)$$

From (2) & (3):

$$\lambda - c_2 = 12$$

and from (3) & (4)

$$2\lambda - c_2 = 8$$

Solving $\lambda = -4, c_2 = -16, c_1 = 20$. Thus the specific parabola satisfying the given B.C.'s (passing through P_1 and P_2) is

$$y = -\frac{4}{4}x^2 + 20x - 16$$

i.e., $\quad y = -x^2 + 20x - 16.$

EXERCISE

1. Find the curve of given length L which joins the points $(x_1, 0)$ and $(x_2, 0)$ and cuts off from the first quadrant the maximum area.

Ans. $(x-c)^2 + (y-d)^2 = \lambda^2, c = \frac{x_1+x_2}{2}, a = \frac{(x_2-x_1)}{2}, \lambda^2 = d^2 + a^2, \sqrt{d^2+a^2} \cot^{-1}\left(\frac{d}{a}\right) = \frac{L}{2}$.

2. Determine the curve of given length L which joins the points $(-a, b)$ and (a, b) and generates the minimum surface area when it is revolved about the x-axis.

Ans. $y = c\cosh\frac{x}{c} - \lambda$, where $c = \frac{a}{\sinh^{-1}\left(\frac{L}{2}\right)}, \lambda = \frac{c}{2}\sqrt{4+L^2} - b$

Ma 530

Calculus of Variations II

Calculus of Variations with Constraints

We begin with some examples.

Example 1

What curve through the points (x_1, y_1) and (x_2, y_2) of given length L has the maximum area between the curve and the x –axis?

If $y(x)$ is a **single-valued function** of x, then

$$A[y(x)] = \int_{x_1}^{x_2} y(x)dx \tag{1}$$

whereas

$$L = \int_{x_1}^{x_2} \sqrt{1 + (y')^2}\, dx \tag{2}$$

Therefore the problem is to maximize (1) subject to the constraint (2) and the conditions $y(x_1) = y_1, y(x_2) = y_2$.

Example 2

If in the previous example we do not assume that $y(x)$ is a single-valued function of x, then it is convenient to suppose that x and y are given parametrically, i.e.,

$$x = x(t), \quad y = y(t) \quad t_1 \le t \le t_2$$

where $x(t_1) = x_1, x(t_2) = x_2, y(t_1) = y_1$, and $y(t_2) = y_2$. Then we have the constraint

$$\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = L \tag{3}$$

Also

$$dA = \frac{1}{2}(xdy - ydx)$$

since by Green's Theorem

$$\oint_C Pdx + Qdy = \iint_R (Q_x - P_y)dA$$

so that

$$\oint_C \frac{1}{2}(xdy - ydx) = \frac{1}{2}\iint_R [1 - (-1)]dA = \iint_R dA = \text{area of } R$$

Thus

$$A[x(t), y(t)] = \frac{1}{2}\int_{t_1}^{t_2}\left[x\frac{dy}{dt} - y\frac{dx}{dt}\right]dt \tag{4}$$

The problem is to minimize (4) subject to the constraint (3) and the conditions $x(t_1) = x_1, x(t_2) = x_2$, $y(t_1) = y_1$, and $y(t_2) = y_2$.

The above two examples illustrate problems in which one desires to minimize (or maximize) a given integral subject to a constraint. Several examples of such problems are

$$\delta\int_{x_1}^{x_2} F(x, y, y')dx = 0 \;\text{ subject to }\; \int_{x_1}^{x_2} G(x, y, y')dx = \text{constant} \tag{5}$$

$$\delta\int_{x_1}^{x_2} F\big(t, x, y, \dot{x}, \dot{y}\big)dx = 0 \;\text{ subject to }\; \int_{x_1}^{x_2} G\big(t, x, y, \dot{x}, \dot{y}\big)dx = \text{constant} \tag{6}$$

$$\delta\int_{x_1}^{x_2} F(x, u, v, u', v')dx = 0 \;\text{ subject to }\; \phi(u, v) = 0 \tag{7}$$

where $u = u(x), v = v(x)$.

We deal with (5) first. Thus our problem is to make

$$I[y(x)] = \int_{x_1}^{x_2} F(x, y, y')dx = \text{minimum or maximum}$$

where y is prescribed as $y(x_1) = y_1$ and $y(x_2) = y_2$ subject to the condition

$$J[y(x)] = \int_{x_1}^{x_2} G(x, y, y')dx = k$$

where k is a given constant.

This problem cannot be attacked by the earlier method of forming $y + \epsilon\eta$ where η vanishes on the boundary only, for in general such functions do not satisfy the subsidiary condition in a neighborhood of $\epsilon = 0$ except at $\epsilon = 0$. Since we have **two** requirements, we therefore consider the function

$$y(x) + \epsilon_1\eta_1(x) + \epsilon_2\eta_2(x)$$

where η_1 and η_2 have continuous derivatives and

$$\eta_1(x_1) = \eta_1(x_2) = \eta_2(x_1) = \eta_2(x_2) = 0.$$

Then we have

$$\Phi(\epsilon_1, \epsilon_2) = I[y(x) + \epsilon_1\eta_1(x) + \epsilon_2\eta_2(x)] = \int_{x_1}^{x_2} F(x, y + \epsilon_1\eta_1 + \epsilon_2\eta_2, y' + \epsilon_1\eta_1' + \epsilon_2\eta_2')dx$$

subject to

$$\Psi(\epsilon_1, \epsilon_2) = J[y(x) + \epsilon_1\eta_1(x) + \epsilon_2\eta_2(x)] - k = \int_{x_1}^{x_2} G(x, y + \epsilon_1\eta_1 + \epsilon_2\eta_2, y' + \epsilon_1\eta_1' + \epsilon_2\eta_2')dx - k = 0$$

Thus we desire that the function $\Phi(\epsilon_1, \epsilon_2)$ take on a relative minimum or maximum at all $\epsilon_1 = \epsilon_2 = 0$ with respect to all sufficiently small values of ϵ_1 and ϵ_2 for which $\Psi(\epsilon_1, \epsilon_2) = 0$.

This problem is of the form we treated earlier by means of Lagrange multipliers. In particular, if λ is the Lagrange multiplier, then we require

$$\left.\frac{\partial(\Phi+\lambda\Psi)}{\partial\epsilon_1}\right|_{\epsilon_1=\epsilon_2=0} = 0 \qquad (1)$$

$$\left.\frac{\partial(\Phi+\lambda\Psi)}{\partial\epsilon_2}\right|_{\epsilon_1=\epsilon_2=0} = 0 \qquad (2)$$

and

$$\Psi(\epsilon_1,\epsilon_2) = 0$$

Now (1) $\Rightarrow$

$$\int_{x_1}^{x_2}[F_y\eta_1 + F_{y'}\eta_1']dx + \lambda\int_{x_1}^{x_2}[G_y\eta_1 + G_{y'}\eta_1']dx = 0$$

and (2) $\Rightarrow$

$$\int_{x_1}^{x_2}[F_y\eta_2 + F_{y'}\eta_2']dx + \lambda\int_{x_1}^{x_2}[G_y\eta_2 + G_{y'}\eta_2']dx = 0$$

Let

$$[H]_y = H_y - \frac{d}{dx}H_{y'}$$

Then as before an integration by parts yields

$$\int_{x_1}^{x_2}\{[F]_y + \lambda[G]_y\}\eta_1 dx = 0 \qquad (3)$$

and

$$\int_{x_1}^{x_2}\{[F]_y + \lambda[G]_y\}\eta_2 dx = 0 \qquad (4)$$

If $[G]_y \neq 0$, then we can, say, choose η_2 such that $\int_{x_1}^{x_2}[G]_y\eta_2 dx \neq 0$, and thus λ may be chosen so that (4) holds. However, since η_1 is arbitrary, λ will not be such that (3) holds. Therefore it follows from (3) that

$$[F]_y + \lambda[G]_y = 0$$

or

$$\frac{\partial(F+\lambda G)}{\partial y} - \frac{d}{dx}\left(\frac{\partial(F+\lambda G)}{\partial y'}\right) = 0$$

is the necessary condition. The general solution of this equation will involve two constants of integration and the constant parameter λ. Thus we have 3 constants to satisfy the 3 conditions $y(x_1) = y_1, y(x_2) = y_2$, and $\int_{x_1}^{x_2} G(x,y,y')dx = k$.

The above results may be summarized as follows:

Theorem. In order to minimize (or maximize) $\int_{x_1}^{x_2} F dx$ subject to a constraint $\int_{x_1}^{x_2} G dx = k$ we first write $H = F + \lambda G$, where λ is a constant, and minimize (or maximize) $\int_{x_1}^{x_2} H dx$ subject to no constraints. Carry the Lagrange multiplier λ through the calculations, and determine it, together with the constants of integration arising in the Euler equation, so that the constraint $\int_{x_1}^{x_2} G dx = k$ holds, and the end conditions are satisfied.

Example Maximize

$$A[y(x)] = \int_{x_1}^{x_2} y(x)dx \qquad (1)$$

subject to

$$L = \int_{x_1}^{x_2}\sqrt{1+(y')^2}\,dx \qquad (2)$$

and $y(x_1) = y_1, y(x_2) = y_2$.

Solution: Without loss of generality we may move the axis and change the scale so that the curve is to pass through $(0,0)$ and $(1,0)$. Thus we must maximize

$$\int_0^1 y(x)dx$$

subject to $y(0) = y(1) = 0$ and the constraint

$$\int_0^1\sqrt{1+(y')^2}\,dx = L \quad \text{where } L > 1$$

We form

$$H = y + \lambda\left(1+(y')^2\right)^{\frac{1}{2}}$$

The Euler equation

$$-H_y + \frac{d}{dx}H_{y'} = 0$$

implies

$$-1 + \lambda\frac{d}{dx}\left\{\frac{y'}{\left(1+(y')^2\right)^{\frac{1}{2}}}\right\} = 0$$

Then integrating we have

$$\lambda\frac{y'}{\left(1+(y')^2\right)^{\frac{1}{2}}} = x - c_1$$

or

$$\{\lambda^2 - (x-c_1)^2\}(y')^2 = (x-c_1)^2$$

Therefore

$$y' = \pm\frac{(x-c_1)}{\sqrt{\lambda^2-(x-c_1)^2}}$$

so

$$y = \pm\left(\lambda^2-(x-c_1)^2\right)^{\frac{1}{2}} + c_2$$

or finally

$$(y-c_2)^2 + (x-c_1)^2 = \lambda^2$$

The required curves are arcs of circles. We have three constants to determine. They are determined so that the arc passes through $(0,0),(0,1)$ and has length L.

Remark: When $L = \frac{\pi}{2}$, we have a semicircle since the circumference of the circle is $2\pi\lambda$. For $L > \pi/2$ y is no longer a single-valued function of x. For such a case it is convenient to employ a parametric representation expressing x and y as functions of t, i.e., $x = x(t)$ and $y = y(t)$. We are led to the problem

Maximize

$$I = \frac{1}{2}\int_{t_1}^{t_2}\left(x\dot{y} - y\dot{x}\right)dt$$

where $x(t_1) = 0, x(t_2) = 1, y(t_1) = y(t_2) = 0$ and

$$J = \int_{t_1}^{t_2}\left[\left(\dot{x}\right)^2 + \left(\dot{y}\right)^2\right]^{\frac{1}{2}} dt = L$$

Remark: It is easily seen that the problem

$$\delta\int_{x_1}^{x_2} F\left(t,x,y,\dot{x},\dot{y}\right)dx = 0 \text{ subject to } \int_{x_1}^{x_2} G\left(t,x,y,\dot{x},\dot{y}\right)dx = \text{constant}$$

leads to the Euler equations

$$\frac{d}{dt}H_{\dot{x}} - H_x = 0$$
$$\frac{d}{dt}H_{\dot{y}} - H_y = 0$$

where $H = F + \lambda G$.
For our problem

$$H = \frac{1}{2}\left(x\dot{y} - y\dot{x}\right) + \lambda\left(\dot{x}^2 + \dot{y}^2\right)^{\frac{1}{2}}$$

These again leads to arcs of circles.

Remark: For the problem

$$\delta\int_{x_1}^{x_2} F(x,u,v,u',v')dx = 0 \text{ subject to } \phi(u,v) = 0$$

the Euler equations are

$$\frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) - \frac{\partial F}{\partial u} - \lambda\phi = 0$$
$$\frac{d}{dx}\left(\frac{\partial F}{\partial v'}\right) - \frac{\partial F}{\partial v} - \lambda\phi = 0$$

prevents division by zero in the proof. An analogous proviso pertains here for a similar reason. Thus, a necessary condition for solution to (1) and (2) may be stated as follows: *If the function x^* is an optimum solution to* (1) *and* (2) *and if x^* is not an extremal for the constraining integral* (2), *then there is a number λ such that $x^*(t)$, λ satisfy* (1) *and* (2).

Example 1.

$$\min \int_0^1 [x'(t)]^2\, dt$$

$$\text{subject to} \quad \int_0^1 x(t)\, dt = B, \quad x(0) = 0, \quad x(1) = 2.$$

The augmented integrand is $(x')^2 - \lambda x$. Its Euler equation $\lambda + 2x'' = 0$ has the solution

$$x(t) = -\lambda t^2/4 + c_1 t + c_2.$$

Three constants are to be determined—λ, c_1, c_2—using the integral constraint and boundary conditions:

$$\int_0^1 x\, dt = \int_0^1 (-\lambda t^2/4 + c_1 t + c_2)\, dt = B,$$

$$x(0) = c_2 = 0, \qquad x(1) = -\lambda/4 + c_1 + c_2 = 2.$$

Hence

$$c_1 = 6B - 4, \qquad c_2 = 0, \qquad \lambda = 24(B - 1).$$

Example 2. For

$$\max \int_0^T x\, dt$$

$$\text{subject to} \quad \int_0^T \left[1 + (x')^2\right]^{1/2} dt = B, \quad x(0) = 0, \quad x(T) = 0.$$

The augmented integrand $x - \lambda[1 + (x')^2]^{1/2}$ has Euler equation

$$1 = -d\left(\lambda x'/\left[1 + (x')^2\right]^{1/2}\right)/dt.$$

Separate the variables and integrate:

$$t = -\lambda x'/\left[1 + (x')^2\right]^{1/2} + k.$$

Solve for x' algebraically:

$$x' = (t - k)/\left[\lambda^2 - (t - k)^2\right]^{1/2}.$$

Let $u = \lambda^2 - (t - k)^2$, so $du = -2(t - k)\, dt$. Then

$$x(t) = \int x'(t)\, dt = -\int du/2u^{1/2} = -u^{1/2} + c,$$

so

$$(x - c)^2 + (t - k)^2 = \lambda^2.$$

The solution traces out part of a circle. The constraints k, c, λ are found to satisfy the two endpoint conditions and the integral constraint.

The Lagrange multiplier associated with (1) and (2) has a useful interpretation as the marginal value of the parameter B; that is, the rate at which the optimum changes with an increase in B. For instance, in the resource extraction problem (3) and (4), λ represents the profit contributed by a marginal unit of the resource. In Example 2, λ represents the rate at which area increases with string length.

To verify the claim, note that the optimal path $x^* = x^*(t; B)$ depends on the parameter B. Assume x^* is continuously differentiable in B. Define $V(B)$ as the optimal value in (1) and (2). Then

$$V(B) = \int_{t_0}^{t_1} F(t, x^*, x^{*\prime})\, dt$$

$$= \int_{t_0}^{t_1} \left[F(t, x^*, x^{*\prime}) - \lambda G(t, x^*, x^{*\prime})\right] dt + \lambda B \tag{12}$$

since (2) is satisfied, where

$$x^* = x^*(t; B), \qquad x^{*\prime} = \partial x^*/\partial t. \tag{13}$$

Differentiating (12) totally with respect to B and taking (13) into account gives

$$V'(B) = \int_{t_0}^{t_1} \left[(F_x^* - \lambda G_x^*)h + (F_{x'}^* - \lambda G_{x'}^*)h'\right] dt + \lambda. \tag{14}$$

where

$$h = \partial x^*/\partial B, \qquad h' = \partial x^{*\prime}/\partial B = \partial^2 x^*/\partial t\, \partial B. \tag{15}$$

But since the augmented integrand in (12) satisfies the Euler equation (11), the integral in (14), after integrating the last term by parts, is zero for any continuously differentiable function h satisfying the endpoint conditions. It follows that

$$V'(B) = \lambda \tag{16}$$

as claimed. (To see that the function h defined in (15) is admissible, one need only observe that the optimal path corresponding to any modified B must be feasible.)

M2A2 Problem Sheet 1 - Calculus of Variations

Solutions

1. *Conservation of 'energy'.*

The Euler-Lagrange equation corresponding to a functional $F(y, y', x)$ is

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} = 0.$$

Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}(F - y'\frac{\partial F}{\partial y'}) = \frac{\partial F}{\partial x}.$$

Hence, in the case that F is independent of x, show that

$$F - y'\frac{\partial F}{\partial y'} = \text{constant}.$$

Solution *Evaluate the expression $\frac{\mathrm{d}}{\mathrm{d}x}(F - y'\frac{\partial F}{\partial y'})$ explicitly. We get:*

$$(\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'})F - y''\frac{\partial F}{\partial y'} - y'\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} = \frac{\partial F}{\partial x},$$

as required; the other terms on the left cancel in pairs, provided y satisfies the Euler-Lagrange equation. If $\frac{\partial F}{\partial x}$ is identically zero, then $E = (F - y'\frac{\partial F}{\partial y'})$ is constant. If F is the Lagrangian of a mechanical system, the quantity $-E$ is called the energy.

2. *The hanging rope.*

A rope hangs between the two points $(x, y) = (\pm a, 0)$ in a curve $y = y(x)$, so as to minimise its potential energy

$$\int_{-a}^{a} mgy\sqrt{1 + y'^2}\mathrm{d}x$$

while keeping its length constant:

$$\int_{-a}^{a} \sqrt{1 + y'^2}\mathrm{d}x = L$$

Of course $L > 2a$. Find and solve the Euler-Lagrange equation.

Solution *We seek to extremise the integral*

$$\int_{-a}^{a} F\mathrm{d}x = \int_{-a}^{a} (y - \lambda)\sqrt{1 + y'^2}\mathrm{d}x,$$

where λ is a Lagrange multiplier. The Euler-Lagrange equation is

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} = 0,$$

which is

$$\sqrt{1 + y'^2} = \frac{\mathrm{d}}{\mathrm{d}x}(y - \lambda)\frac{y'}{\sqrt{1 + y'^2}},$$

a second order ode for y. But F is independent of x, so by the result of question 1, $E = (F - y'\frac{\partial F}{\partial y'})$ is constant. That is,

$$(y - \lambda)\sqrt{1 + y'^2} - (y - \lambda)\frac{y'^2}{\sqrt{1 + y'^2}} = \frac{y - \lambda}{\sqrt{1 + y'^2}} = K^{-1}, \quad \text{a constant.}$$

That is

$$y'^2 = K^2(y - \lambda)^2 - 1.$$

Thus $y = \lambda + K^{-1}\cosh(K(x - x_0))$. Here the constants x_0, and λ are found from the two boundary conditions $y(\pm a) = 0$, giving $x_0 = 0$, and $\lambda = -K^{-1}\cosh(Ka)$, while K is found from the length of the rope:

$$\int_{-a}^{a} \cosh^2(Kx)dx = \frac{1}{4}\sinh(2Ka) + a = L.$$

3. *The relativistic particle* A particle moving with speed near c, the speed of light, has Lagrangian

$$L = -m_0c^2\sqrt{1 - \frac{|\dot{\mathbf{x}}|^2}{c^2}} - U(\mathbf{x}).$$

Show that the equation of motion can be interpreted as Newton's 2nd law, but with a mass depending on the speed of the particle-

$$m = \frac{m_0}{\sqrt{1 - \frac{\dot{x}^2}{c^2}}}.$$

The constant m_0 is called the 'rest mass' of the particle. Use the result of question 1 to find a conserved quantity - the relativistic energy of the particle. Find the leading approximation to this Lagrangian in the case

$$\frac{\dot{\mathbf{x}}^2}{c^2} << 1.$$

Solution *The 'momentum' is*

$$\frac{\partial L}{\partial \dot{\mathbf{x}}} = \frac{m_0\dot{\mathbf{x}}}{\sqrt{1 - \frac{|\dot{\mathbf{x}}|^2}{c^2}}},$$

(In components,

$$\frac{\partial L}{\partial \dot{\mathbf{x}}_i} = \frac{m_0\dot{\mathbf{x}}_i}{\sqrt{1 - \frac{|\dot{\mathbf{x}}|^2}{c^2}}}),$$