$\boxed{\text{Konvergence}}$

$\boxed{\text{Spojitost}}$

$$F(y) = \int_0^\infty \frac{x}{2+xy}\,dx$$

· Ukažte, že $F(y)$ je spojitá na svém definičním oboru

(1) Rozbor

$$f(x,y) = \frac{x}{2+xy} \quad \text{je spojitá na } \mathbb{R}\times\mathbb{R}$$

$[a,b) = [0,\infty)$

$J = ?$

(A) Hledání J: Pro jaká y konvergují $\int_0^\infty \frac{x}{2+xy}\,dx$?

Použijeme lim. srovnávací kritérium (LSK)

$[a,b) = [0,\infty)$, $f = \frac{x}{2+xy}$

Problem. body: $0,\ \infty$

0: f je v 0 spojitá $\to$ OK ✓

∞: LSK, $g(x) = \frac{x}{x^y} = x^{1-y}$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\frac{x}{2+xy}}{\frac{x}{x^y}} = \lim_{x\to\infty} \frac{x^y}{2+x^y} =$$

$$= \lim_{x\to\infty} \frac{x^y}{x^y} \frac{1}{\frac{2}{x^y}+1} = \begin{cases} 1 & y>0 \\ \frac{1}{3} & y=0 \\ 0 & y<0 \end{cases}$$

(a) $y \geq 0$ $\to$ $\int f(x)$ K $\Leftrightarrow$ $\int g(x)$ K

to je ale tehdy, když

$1-y < -1$

$\underline{\underline{2 < y}}$

(b) $y<0$ nevíme nic

odhadneme LSK: $g(x) = x$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim \frac{\frac{\cancel{x}}{2+xy}}{\cancel{x}} = \lim_{x\to\infty} \frac{1}{2+xy} = \frac{1}{2}$$

pak $\int f(x)$ k $\Leftrightarrow$ $\int g(x)$ k

ale $\int_0^\infty x$ D $\to$ tedy $\int f(x)$ D pro $y<0$

Závěr: $J = (2,\infty)$

(2) Podmínky: (a) $\int_a^b \frac{x}{2+xy}$ $\exists$ pro $y \in (2,\infty)$ (viz výše)

(b) majoranta na $[a,b)$ vzhl. k $y \in J$:

: stačí najít na lib. intervalu $[p,\infty)$, $p \in (2,\infty)$

pak

$$g(x) = \begin{cases} \frac{x}{2} & x \in (0,1) \\ \frac{x}{2+xp} & x \in [1,\infty) \end{cases}$$

nez. na y

$f(x,y) \leq g(x) \quad \forall x \in [0,\infty),\ y \in (p,\infty)$

(3) tedy f je spoj. pro $y \in [p,\infty)$ pro $\forall p > 2$

a tedy je spoj na $\underline{[0,\infty) \times (2,\infty)}$

$\boxed{\text{Limita}}$

$$\lim_{y\to\infty} \int_0^\infty \frac{x\,dx}{2+x^y}$$

(1) Rozbor $f(x,y) = \dfrac{x}{2+x^y}$

$[a,b) = [0,\infty)$

$M = [10,\infty)$ (možno zvolit lib. okolí ∞)

(2) $\displaystyle\int_0^\infty \frac{x}{2+x^y}\,dx$ ⤳ pro $y \in M$ (viz Spojitost)

Majoranta

$$g(x) = \begin{cases} \frac{x}{2} & x \in (0,1) \\ \frac{x}{2+x^{10}} & x \in [1,\infty) \end{cases}$$

(3) Možno prohodit lim a $\int$

(4) $$\lim_{y\to\infty} \frac{x}{2+x^y} \begin{cases} 0 & x > 1 \\ \frac{1}{3} & x = 1 \\ \frac{x}{2} & x < 1 \end{cases}$$

(5) Výpočet

$$\lim \int = \int_0^\infty \lim_{y\to\infty} \frac{x}{2+x^y}\,dx = \int_0^1 \frac{x}{2}\,dx + \int_1^\infty 0 =$$

$= \underline{\underline{1/4}}$

$$F(\alpha) = \int_0^\infty \frac{1-e^{-\alpha x}}{xe^x}$$ Derivace I

(1) Rozbor

$$f = \frac{1-e^{\alpha x}}{xe^x}$$

$$[a,b) = [0,\infty)$$

$$J = ?$$

(A) pro jaká α $\int_0^\infty \frac{1-e^{-\alpha x}}{xe^x}\,dx$ K ?

(a) probl. body : $0, \infty$

0: u 0 lze spojitě dodefinovat ✓

(neb $\alpha = 0 \to f \equiv 0$

$\alpha \neq 0$: $\lim_{x\to 0} \frac{1-e^{-\alpha x}}{xe^x} = \frac{1-e^{-\alpha x}}{-\alpha x} \cdot \frac{-\alpha}{e^x} = -\alpha$

∞: pro $\alpha = 0$ $f \equiv 0$

$\alpha \neq 0$ LSK, $[a,b) = [10,\infty)$, $g(x) = \frac{e^{-\alpha x}}{xe^x}$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1-e^{-\alpha x}}{e^{-\alpha x}} = -1$$

pak $\int_{10}^\infty f(x)\,dx$ K $\Leftrightarrow$ $\int_{10}^\infty g(x)\,dx$ K

ale $\int g(x)$ $\Leftrightarrow$ $-\alpha - 1 < 0$ tedy $\boxed{-1 < \alpha}$

tedy $J = (-1,\infty)$

(2) Podm. • f je spoj na $(0,\infty) \times (-1,\infty)$

ale my chceme spoj na $[0,\infty)$ $\to$ nevadí, frí lze spojitě dodefinovat

• $\int f$ taktéž J (právě jsme ukázali)

• $\frac{\partial f}{\partial y} = \frac{+xe^{-\alpha x}}{xe^x} = +\frac{e^{-\alpha x}}{e^x}$ spojitá ✓

$$e^{(-\alpha-1)x} \leq \underbrace{e^{-(p+1)x}}_{\text{majoranta } g(x)} \quad \text{pro} \quad p \in (-1, \infty)$$

$\alpha \in [p, \infty)$

$x \in (0, \infty)$

(opravdu je konvergentní)

(3) Výpočet $F'(\alpha) = \frac{d}{dy} \int_0^\infty \frac{1-e^{-\alpha x}}{xe^x} dx = \int_0^\infty \frac{\partial}{\partial y} \frac{1-e^{-\alpha x}}{xe^x} dx$

$$= \int_0^\infty e^{(-\alpha-1)x} dx = \frac{1}{\alpha+1}$$

pak $F(\alpha) = \int \frac{1}{\alpha+1}$

$$F(\alpha) = \ln(\alpha+1) + k$$

kolik je k?

dosadíme $\alpha = 0$, pak

$$F(0) = \int_0^\infty 0 \, dx = 0$$

$$0 = \ln(1) + k$$

$$k = 0$$

tedy $\underline{\underline{F(\alpha) = \ln(\alpha+1)}} \quad \alpha \in (-1, \infty)$

$\boxed{\text{Derivace II}}$

$$F(\alpha,\beta)=\int_0^\infty \frac{\arctan\alpha x-\arctan\beta x}{x}$$

$\alpha=\beta$
$\vee\ (\alpha,\beta>0)$
$\vee\ (\alpha,\beta<0)$

$\boxed{\text{krok 1}}$ Fixuji β (i hromou, pro $\beta=0=\alpha$ je to $F\equiv 0$)

(1) Rozbor $f(x,\alpha)=\dfrac{\arctan\alpha x-\arctan\beta x}{x}\,dx$

$[a,b)=[0,\infty)$

$J=(0,\infty)$ pro $\beta>0$

$(-\infty,0)$ $\beta<0$

- probl. body: $0,\infty$

$0:\ \lim_{x\to0}\dfrac{\arctan\alpha x-\arctan\beta x}{x}=\underline{\underline{\alpha-\beta}}$

spoj. tě dodefinujeme

∞: LSK s $g(x)=\frac{1}{x^2}$

(a) $\alpha,\beta>0$ $\lim_{x\to\infty}\dfrac{f(x)}{g(x)}=\lim_{x\to\infty}\dfrac{\frac{(\arctan\alpha x-\frac{\pi}{2})+(\frac{\pi}{2}-\arctan\beta x)}{x}}{\frac{1}{x^2}}$

$=\lim_{x\to\infty}x(\arctan\alpha x-\frac{\pi}{2})+x(\frac{\pi}{2}-\arctan\beta x)$

$=\frac{1}{\alpha}+\frac{1}{\beta}\rightarrow\boxed{\int f(x)\ K}$

(b) $\alpha>0,\beta<0$ LSK s $g(x)=\frac{1}{x}$

$\lim_{x\to\infty}\dfrac{f(x)}{g(x)}=\lim_{x\to\infty}\arctan\alpha x-\arctan\beta x$

$=\frac{\pi}{2}+\frac{\pi}{2}=\frac{\pi}{4}\rightarrow\boxed{\int f(x)\ D}$

$\alpha=0,\beta\neq0$
$\alpha\neq0,\beta=0$ analog. (b)

pro $\alpha,\beta<0$ analog (a)

$\alpha<0,\beta>0$ (b)

(2) Podm. , f je spoj. na $[0, \infty) \times (0, \infty)$, necht $\beta > 0$

dodefinovali jsme

, $\int_0^\infty f(x, \alpha)$ existuje (viz výše)

, $\frac{\partial f}{\partial \alpha} = \frac{1}{1+\alpha^2 x^2} \cdot \frac{x}{x}$ spoj. na $[0,\infty) \times (0,\infty)$

, majoranta $g(x) = \frac{1}{1+p^2x^2}$ pro $\alpha \in [p, \infty)$ $p > 0$

(3) Výpočet

$$\frac{d}{d\alpha} \int_0^\infty f(x, \alpha)\, dx = \int_0^\infty \frac{\partial}{\partial \alpha} \frac{\arctan \alpha x - \arctan \beta x}{x}\, dx$$

$$= \int_0^\infty \frac{1}{1+\alpha^2x^2}\, dx =$$

$$= \left[\frac{1}{\alpha} \arctan \alpha x\right]_0^\infty = \frac{\pi}{2\alpha}$$

Pak $F(\alpha, \beta) = \frac{\pi}{2} \ln \alpha + c(\beta)$

pro $\alpha = \beta$:

$$F(\beta, \beta) = \int_0^\infty \frac{0}{x} = 0 \quad , \text{ pak}$$

$$0 = \frac{\pi}{2} \ln \beta + c(\beta)$$

$$\rightarrow \quad c(\beta) = -\frac{\pi}{2} \ln \beta$$

celkem $\underline{F(\alpha, \beta) = \frac{\pi}{2} \ln \frac{\alpha}{\beta}}$

(4) Závěr pro $\beta < 0, \alpha < 0$ je fce lichá $\rightarrow$ stačí dodat "–"