(1) Vyšetřete bodovou a stejnoměrnou konvergenci pol. fní

$$f_n = \frac{nx}{1+n+x} \qquad x \in [0,1]$$

(a) fix x

$$\lim_{n\to\infty} \frac{nx \cdot 1}{n\left(\frac{1}{n}+1+\frac{x}{n}\right)} = x$$

$f = x$

Bodově konvergují k $f(x) = x$

(b) $$\lim_{n\to\infty} \sup_{x\in[0,1]} \left|\frac{nx}{1+n+x} - x\right|$$

$$\frac{\cancel{nx} - x - \cancel{nx} - x^2}{1+n+x} = \frac{-x(1+x)}{1+n+x} =: g(x)$$

$$g'(x) = \frac{(-1-2x)(1+n+x) + x + x^2}{(1+n+x)^2}$$

$$= \frac{-1-n-\cancel{x}-2x-2nx-2x^2+\cancel{x}+x^2}{(1+n+x)^2}$$

$$= \frac{-1-n-2x(1+n)-x^2}{(1+n+x)} \le 0 \quad \text{na } [0,1]$$

fce je klesající, minima dosahuje v $x=1$

→ přidám Abs. hodnotu a je to maximum:

$$\sup_{x\in[0,1]} \left|\frac{nx}{1+n+x} - x\right| = \left|\frac{n}{1+n+1} - 1\right| = \left|\frac{n-n-2}{n+2}\right| = \frac{2}{n+2}$$

$$\lim_{n\to\infty} \frac{2}{n+2} = 0 \ \checkmark \qquad f_n \rightrightarrows x \text{ na } [0,1]$$

(2) $\displaystyle\sum_{n=1}^{\infty} \operatorname{arctg} \frac{2x}{x^2+n^3}$ Vyšetřete st. konvergenci

$x \geq 0$

$\operatorname{arctg} \dfrac{2x}{x^2+n^3} \leq \dfrac{2x}{x^2+n^3}$ (vlastnosti arctg)

a pak $\dfrac{2x}{x^2+n^3} \leq \dfrac{1}{n^{3/2}}$

neb $2xn^{3/2} \leq x^2+n^3$

$0 \leq (x - n^{3/2})^2$

Tedy $\displaystyle\sum_{n=1}^{\infty} \operatorname{arctg} \frac{2x}{x^2+n^3} \leq \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \to$ KS.

Tedy i zadaná řada konv. stejnoměrně na $[0,\infty)$

(3a) Najděte poloměr konvergence a konvergenci v krajních bodech, a sečtěte řadu (lze sečíst i v krajních bodech?) Stejn. konv.

$$\sum_{k=1}^{\infty} k x^k$$

(a) $\rho = \dfrac{1}{\limsup \sqrt[k]{k}} = 1$ Poloměr $\rho = 1$

(a.1) $x = 1$ $\sum k$ D

(a.2) $x = (-1)$ $\sum k(-1)^k$ D $(k \not\to 0)$

Řada konverguje na $(-1, 1)$.

stejnoměrně na $[-q, q]$ $0 < q < 1$

(b) $f(x) := \sum_{k=1}^{\infty} x k x^{k-1} = x \underbrace{\sum_{k=1}^{\infty} k x^{k-1}}_{=: g(x)}$ → konverguje tamtéž

$$\int g(x)\,dx = \sum_{k=1}^{\infty} x^k = \frac{1}{1-x} - 1 = \frac{-1+x}{1-x}$$

$$g(x) = \left(\frac{1}{1-x} - 1\right)' = \frac{+1}{(1-x)^2}$$

Tedy $\underline{\underline{f(x) = \dfrac{x}{(1-x)^2}}}$ na $(-1, 1)$

(3b) Najděte poloměr konvergence

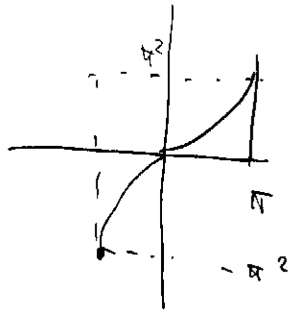
$$\sum_{n=0}^{\infty} \frac{3^n}{2^n} \frac{x^n}{n+1} \left(-1-(-1)^n\right)$$

$$\rho = \frac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{\left|\left(\frac{3}{2}\right)^n \frac{1}{n+1}\left(1+(-1)^n\right)\right|}} = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{\left(\frac{3}{2}\right)^n \frac{1}{n+1}\cdot 2}}$$

$$= \frac{1}{\lim\limits_{n\to\infty} \frac{3}{2}\sqrt[n]{\frac{2}{n+1}}} = \underline{\underline{\frac{2}{3}}}$$

(4)

$f(x) = x^2$ na $(0, \pi)$



Najděte sinovou FŘ:

$a_n = 0 \quad n \in \mathbb{N}_0$

$$b_n = \frac{1}{\pi}\int_0^\pi x^2 \sin nx \, dx + \frac{1}{\pi}\int_{-\pi}^0 (-x^2)\sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^\pi x^2 \sin nx \, dx$$

$$\int \underset{u}{x^2}\, \underset{v'}{\sin nx} = -\frac{x^2\cos nx}{n} + \int \underset{u}{2x}\, \underset{v'}{\frac{\cos nx}{n}}$$

$u' = 2x \quad v = -\frac{\cos nx}{n}$

$u' = 2 \quad v = \frac{\sin nx}{n^2}$

$$= -\frac{x^2}{n}\cos nx + \frac{2x \sin nx}{n^2} - \int \frac{2}{n^2}\sin nx$$

$$= -\frac{x^2}{n}\cos nx + \frac{2x}{n^2}\sin nx + \frac{2}{n^3}\cos nx$$

$$= \frac{2}{\pi}\left[-\frac{x^2}{n}\cos nx + \frac{2}{n^3}\cos nx + \frac{2x}{n^2}\sin nx\right]_0^\pi = \frac{2}{\pi}\left[-\frac{2}{n^3} + \frac{2}{n^3}(-1)^n - \frac{\pi^2}{n}(-1)^n\right]$$

n sudé: $\frac{-2}{\pi}\cdot\frac{\pi^2}{n} = \frac{-2\pi}{n}$

n liché: $\frac{2}{\pi}\left(-\frac{4}{n^3} - \frac{\pi^2}{n}\right)$

Řada: $$f(x) \approx \sum_{n=1}^{\infty} \frac{2}{\pi}\left(\frac{-2+2(-1)^n}{n^3} - \frac{(-1)^n\pi^2}{n}\right)\sin nx$$