Limita posloupnosti - komplexní úloha XII

Určete limitu posloupnosti

$$\lim_{n\to\infty} \frac{2^n \sqrt{n^2+n} - n\sqrt{4^n+1}}{\sqrt[n]{2^{n^2}+1}} . = \frac{1}{2}$$

Řešení

$$\lim \frac{2^n\cdot\sqrt{n^2+n} - n\cdot\sqrt{4^n+1}}{\sqrt[n]{2^{n^2}+1}} \cdot \frac{2^n\cdot\sqrt{n^2+n} + n\cdot\sqrt{4^n+1}}{2^n\cdot\sqrt{n^2+n} + n\cdot\sqrt{4^n+1}} =$$

$$= \lim \frac{4^n(n^2+n) - n^2(4^n+1)}{2^n\cdot\sqrt[n]{1+\frac{1}{2^{n^2}}}\cdot\left(2^n\cdot n\cdot\sqrt{1+\frac{1}{n}} + n\cdot 2^n\cdot\sqrt{1+\frac{1}{4^n}}\right)} =$$

$$= \lim \frac{4^n n^2 + 4^n n - 4^n n^2 - n^2}{4^n\cdot n\cdot\sqrt[n]{1+\frac{1}{2^{n^2}}}\left(\sqrt{1+\frac{1}{n}} + \sqrt{1+\frac{1}{4^n}}\right)} = \lim \frac{1-\frac{n}{4^n}}{\sqrt[n]{1+\frac{1}{2^{n^2}}}\left(\sqrt{1+\frac{1}{n}} + \sqrt{1+\frac{1}{4^n}}\right)} \overset{VOAL}{=}$$

$$\overset{VOAL}{=} \frac{\lim\left(1-\frac{n}{4^n}\right)}{\lim\left[\sqrt[n]{1+\frac{1}{2^{n^2}}}\left(\sqrt{1+\frac{1}{n}} + \sqrt{1+\frac{1}{4^n}}\right)\right]} \overset{VOAL}{=} \frac{\lim 1 - \lim\frac{n}{4^n}}{\lim\sqrt[n]{1+\frac{1}{2^{n^2}}}\cdot\lim\left(\sqrt{1+\frac{1}{n}} + \sqrt{1+\frac{1}{4^n}}\right)} \overset{VO2P}{=}$$

$$\overset{VO2P}{=} \cancel{\frac{1-0}{\lim 1\cdot\lim\left(\sqrt{1+\frac{1}{n}} + \sqrt{1+\frac{1}{4^n}}\right)}} \overset{VOAL}{=} \frac{\lim 1 - \lim\frac{n}{4^n}}{\lim\sqrt[n]{1+\frac{1}{2^{n^2}}}\cdot\left(\lim\sqrt{1+\frac{1}{n}} + \lim\sqrt{1+\frac{1}{4^n}}\right)} \overset{\substack{VO2P\\VOML}}{=} \frac{1-0}{1(1+1)} = \frac{1}{2}$$

VOAL = Věta o aritmetice limit

VO2P = Věta o dvou policajtech: $1 = \lim\sqrt[n]{1} \le \lim\sqrt[n]{1+\frac{1}{2^{n^2}}} \le \lim\sqrt[n]{n} = 1 \Rightarrow \lim\sqrt[n]{1+\frac{1}{2^{n^2}}} = 1$

VOML = Věta o mocnině limity:

- $\lim\sqrt{1+\frac{1}{n}} = \sqrt{\lim\left(1+\underbrace{\frac{1}{n}}_{\to 0}\right)} = \sqrt{1} = 1$
- $\lim\sqrt{1+\frac{1}{4^n}} = \sqrt{\lim\left(1+\underbrace{\frac{1}{4^n}}_{\to 0}\right)} = \sqrt{1} = 1$