$\boxed{\text{CVIČENÍ 5}}$

- $$\lim_{x\to16}\frac{\sqrt[4]{x}-2}{\sqrt{x}-4}=\lim_{x\to16}\frac{\sqrt[4]{x}-2}{(\sqrt[4]{x}-2)(\sqrt[4]{x}+2)}=\lim_{x\to16}\frac{1}{\sqrt[4]{x}+2}=\frac{1}{4}$$

- $$\lim_{x\to0}\frac{\sqrt{x+1}-1}{x}=\lim_{x\to0}\frac{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}{x(\sqrt{x+1}+1)}=\lim_{x\to0}\frac{x+1-1}{x(\sqrt{x+1}+1)}=$$

$$=\lim_{x\to0}\frac{1}{\sqrt{x+1}+1}=\frac{1}{2}$$

! pp. $1-x\geq0$

$$\lim_{x\to0}\frac{\sqrt{1-2x-x^2}-(1-x)}{x}=\lim_{x\to0}\frac{\sqrt{1-2x-x^2}-\sqrt{(1-x)^2}}{x}\cdot\frac{\sqrt{1-2x-x^2}+\sqrt{(1-x)^2}}{\sqrt{1-2x-x^2}+\sqrt{(1-x)^2}}=$$

$$=\lim_{x\to0}\frac{1-2x-x^2-1+2x-x^2}{x(\sqrt{1-2x-x^2}+\sqrt{(1-x)^2})}=\lim_{x\to0}\frac{-2x}{\sqrt{1-2x-x^2}+\sqrt{(1-x)^2}}=-\frac{2\cdot0}{1+1}=0$$

- $$\lim_{x\to0}\frac{\sqrt[3]{27+x}-\sqrt[3]{27-x}}{x+2\sqrt[3]{x^4}}\frac{(27+x)^{\frac{2}{3}}+(27+x)^{\frac{1}{3}}(27-x)^{\frac{1}{3}}+(27-x)^{\frac{2}{3}}}{\text{dtto}}=\lim_{x\to0}\frac{27+x-27+x}{x+2\sqrt[3]{x^4}}$$

$\boxed{\text{podmíněně}}$ proč?

$$\cdot\frac{1}{\text{dtto}}=\lim_{x\to0}\frac{2x}{x+2x^{4/3}}\cdot\frac{1}{\text{dtto}}=\lim_{x\to0}\frac{2}{1+2x^{1/3}}\cdot\frac{1}{3^2+3\cdot3+3^2}$$

$$=\frac{2}{1+2\cdot0}\cdot\frac{1}{3\cdot3^2}=\frac{2}{27}$$

- $$\lim_{x\to0}\frac{\sqrt{1+x}-\sqrt[3]{1-x}}{\sqrt[3]{1+x}-\sqrt{1-x}}=\lim_{x\to0}\frac{[(1+x)^3]^{\frac{1}{6}}-[(1-x)^2]^{\frac{1}{6}}}{[(1+x)^2]^{\frac{1}{6}}-[(1-x)^3]^{\frac{1}{6}}}\cdot\frac{[(1+x)^3]^{\frac{5}{6}}+[(1+x)^3]^{\frac{4}{6}}[(1-x)^2]^{\frac{1}{6}}+\dots+[(1-x)^2]^{\frac{5}{6}}}{\text{dtto}}$$

$$\cdot\frac{\text{dtto}'}{[(1+x)^2]^{\frac{5}{6}}+\dots+[(1-x)^3]^{\frac{5}{6}}}=\lim_{x\to0}\frac{1+3x+3x^2+x^3-(1-2x+x^2)}{1+2x+x^2-(1-3x+3x^2-x^3)}\ \frac{1}{\text{dtto}}\ \frac{\text{dtto}'}{1}=$$

$$=\lim_{x\to0}\frac{x^3+2x^2+5x}{x^3-2x^2+5x}\cdot\frac{6}{6}=\lim_{x\to0}\frac{x^2+2x+5}{x^2-2x+5}=1$$

$$\lim_{x\to 0}\frac{(1+x)^{\frac{1}{m}}-(1+x)^{\frac{1}{n}}}{x}=\lim_{x\to 0}\frac{[(1+x)^{n}]^{\frac{1}{mn}}-[(1+x)^{m}]^{\frac{1}{mn}}}{x}$$

$$=\left|\begin{array}{l}\sqrt[k]{x'}-\sqrt[k]{y'}=x'^{\frac{1}{k}}-y'^{\frac{1}{k}}\ /\cdot\big(\underbrace{x'^{\frac{k-1}{k}}+x'^{\frac{k-2}{k}}y'+\dots+y'^{\frac{k-2}{k}}x+y'^{\frac{k-1}{k}}}_{A:=}\big)\\[2ex] x'-y'=(x'^{\frac{1}{k}}-y'^{\frac{1}{k}})A\end{array}\right|=$$

$$=\lim_{x\to 0}\frac{(1+x)^{n}-(1+x)^{m}}{x\cdot A}=\lim_{x\to 0}\frac{1+nx+\dots+x^{n}-(1+mx+\dots+x^{m})}{x\cdot A}=$$

$$=\lim_{x\to 0}\frac{nx+\dots+x^{n}-mx-\dots-x^{m}}{x\cdot A}=\lim_{x\to 0}\frac{\left(n-m+x\binom{n}{2}+\dots+x^{n}-x\binom{m}{2}-\dots-x^{m}\right)}{A}$$

$$\left|\ A=\lim_{x\to 0}\left[(1+x)^{\frac{mn-1}{mn}}+(1+x)^{\frac{mn-2}{mn}}(1+x)^{\frac{1}{mn}}+\dots+(1+x)^{\frac{mn-1}{mn}}\right]=mn\ \right|$$

$$=\lim_{x\to 0}\frac{\left(n-m+x\binom{m}{2}+\dots\right)}{mn}=\frac{n-m}{mn}.$$

$$\lim_{x\to 0}\frac{(1+x)(1+2x)\cdots(1+nx)-1}{x} \underset{\uparrow}{=} \lim_{x\to 0}\frac{1+x+2x+\cdots+nx+\cancel{\ }+x^2A+\cdots+x^n n!-1}{x} = \lim_{x\to 0}\frac{\frac{1}{2}n(n+1)x+x^2A+\cdots+x^n n!}{x} =$$

musím vykrátit x, neb nejde VOAL

$$= \lim_{x\to 0}\left(\frac{1}{2}n(n+1)+Ax+x^{n-1}n!\right)=\frac{1}{2}n(n+1).$$

$$\lim_{x\to 1}\frac{x^{100}-2x+1}{x^{50}-2x+1} = \lim_{x\to 1}\frac{(x-1)(x^{99}+\cdots+x^1-1)}{(x-1)(x^{49}+\cdots+x-1)} =$$

$$=\lim_{x\to 1}\frac{x^{99}+\cdots+x^1-1}{x^{49}+\cdots+x^1-1}=\frac{98}{48}.$$

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binom

$$\lim_{x\to 0}\frac{(1+mx)^n-(1+nx)^m}{x^2} = \lim_{x\to 0}\frac{1+mnx+m^2\binom{n}{2}x^2+\cdots+m^nx^n-(1-mnx-n^2\binom{m}{2}x^2-\cdots-n^mx^m)}{x^2} = \lim_{x\to 0}\frac{m^2\binom{n}{2}x^2-n^2\binom{m}{2}x^2+A_3x^3+\cdots+A_mx^m}{x^2}$$

$\overbrace{\ m\geq n\ }$

$$=\lim_{x\to 0}\left(m^2\binom{n}{2}-n^2\binom{m}{2}+A_3x^3+\cdots+A_mx^m\right)=m^2\binom{n}{2}-n^2\binom{m}{2}$$

$$=\frac{1}{2}nm^2(n-1)-\frac{1}{2}mn^2(m-1)=\frac{1}{2}mn(mn-m-mn+n)=$$

$$=\frac{1}{2}mn(n-m).$$

Pro $m\leq n$: $\lim_{x\to 0}\frac{m^2\binom{n}{2}x^2-n^2\binom{m}{2}x^2+B_3x^3+\cdots+B_nx^n}{x^2}=$

$=\lim_{x\to 0}\left(m^2\binom{n}{2}-n^2\binom{m}{2}+B_3x^3+\cdots+B_nx^n\right)=$ dále stejně jako

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$$\lim_{x\to 1}\frac{x^{n+1}-(n+1)x+n}{(x-1)^2}=\lim_{x\to 1}\frac{x^{n+1}-nx-x+n}{(x-1)^2}=\lim_{x\to 1}\frac{x^{n+1}-x+n(1-x)}{(x-1)^2}$$

$$= \lim_{x\to 1} \frac{x(x^n-1)-n(x-1)}{(x-1)^2} = \lim_{x\to 1} \frac{x(x^{n-1}+\dots+x+1)-n}{(x-1)} =$$

$$= \lim_{x\to 1} \frac{(x^n+x^{n-1}+x^2+x-n)}{(x-1)} = \lim_{x\to 1} \frac{x^n-1+x^{n-1}-1+\dots+x-1}{(x-1)} =$$

$$= \lim_{x\to 1}(x^{n-1}+\dots+x+1) + \lim_{x\to 1}(x^{n-2}+\dots+x+1)+\dots+\lim 1 =$$

$$= n+(n-1)+\dots+1 = \frac{1}{2}n(n+1).$$

bylo indukce

Některé limity jsme počítali pomocí vět o limitě slož. fce:

V1: $\lim_{x\to a} g(x) = b \wedge \lim_{y\to b} f(y) = A \wedge \exists \delta>0\ \forall x\in P_\delta(a)\ g(x)\neq b$

$$\Rightarrow \lim_{x\to a}(f\circ g)(x) = A \left(= \lim_{y\to b} f(y) = \lim_{y\to \lim_{x\to a} g(x)} f(y)\right).$$

V2: $\lim_{x\to a} g(x) = b \wedge f$ spoj. v b $\Rightarrow \lim_{x\to a}(f\circ g) = f(b)\ (= A = \lim_{y\to b} f(y))$

Dle V2 stačí spočíst limitu vnitřní a dosadit do f, pokud f spojitá.

Př. Vzorně (s odůvodněním dle vět) spočtěte: $\lim_{x\to 0} \frac{\frac{2}{x^2}+1}{\sqrt{\frac{3}{x^4}-\frac{6}{x^2}+5}} =$

lemma o rovnosti v naprstencovém

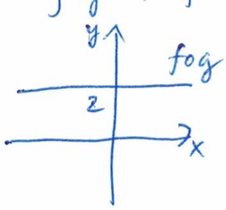
$$= \lim_{x\to 0}\frac{2+x^2}{\sqrt{3-6x^2+5x^4}} \overset{\text{VOAL (podíl)}}{=} \frac{\lim_{x\to 0}(2+x^2)}{\lim_{x\to 0}\sqrt{3-6x^2+5x^4}} = \frac{2}{\sqrt{\lim_{x\to 0}(3-6x^2+5x^4)}} = \frac{2}{\sqrt{3}}$$

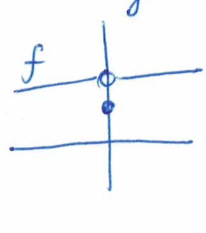
$\sqrt{x^4} = x^2$! Bez abs.

VOLSF
$\sqrt{\ }$ spojitá

Př.: $g(x) = 0$, $f(x) = \begin{cases} 3, & x\neq 0 \\ 2, & x = 0\end{cases}$ $\qquad \lim_{y\to 0} f(y) = 3$

$f(g(x)) = f(1) = 2$





Tj. $\lim_{x\to 0}(f\circ g)(x) = 2 \neq 3 = \lim_{y\to 0} f(y)$, $\quad 0 = \lim_{x\to 0} g(x) = b$

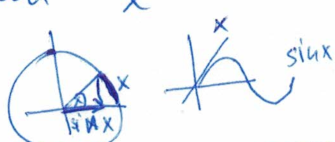
Zpět: $\lim\limits_{x\to 0} \dfrac{\sqrt[3]{8+x^2}-\sqrt[3]{8-x^2}}{x\left[(x+1)^2-(x-1)^2\right]} = \lim\limits_{x\to 0} \dfrac{\sqrt[3]{8+x^2}-\sqrt[3]{8-x^2}\cdot\left[(8+x^2)^{\frac{2}{3}}+\dots+(8-x^2)^{\frac{2}{3}}\right]}{x\left[4x\right]\cdot\left[\quad \text{dtto} \quad\right]}$ 5

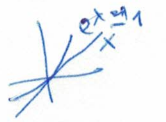
$$= \lim_{x\to 0} \frac{(8+x^2)-(8-x^2)}{4x^2}\cdot\frac{1}{[\ \text{dtto}\ ]} = \lim_{x\to 0}\frac{1}{2}\,\frac{1}{[\ \text{dtto}\ ]} =$$

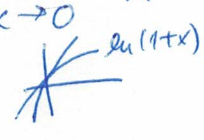
$$= \lim_{x\to 0}\frac{1}{2}\,\frac{1}{4+2\cdot 2+4} = \frac{1}{2\cdot 3\cdot 4} = \frac{1}{24}.$$

• Teď začneme některé příklady se „zákl." limitami. Dokážete je až později! (na přednášce).

$\lim \dfrac{\sin x}{x} = 1$, $\lim \dfrac{e^x-1}{x} = 1$, $\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{x} = 1$







Dále: $\boxed{\lim\limits_{x\to a} f(x)^{g(x)} = e^{\lim\limits_{x\to a}\{[\ln f(x)]\,g(x)\}}}$ (*) (Princip: spojitost $y=e^x$ (později) a „definice" $\beta^x := e^{x\ln\beta}$; odtud lim vlevo existuje $\Leftrightarrow$ limita napravo v (*).

Tento vzorec je nebývale užitečný!

! Př.: ~~$\lim\limits_{x\to 0}\sqrt{1-\dots}$~~ $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2} = \left|\ \text{Chci tam vidět } \dfrac{\sin x}{x}\ \right| =$

$$= \lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)}{x^2\,(1+\cos x)} = \lim_{x\to 0}\frac{1-\cos^2 x}{x^2(1+\cos x)} = \lim\frac{\sin^2 x}{x^2}\cdot\frac{1}{1+\cos x} = 1^2\cdot\frac{1}{1+1} = \underline{\underline{\frac{1}{2}}}\ \text{(do paměti)}.$$