1. Nechť $f(x) = \sqrt{1-e^{-x^2}}\,\operatorname{arctan}\left(\frac{x}{x+1}\right)$.

Určete definiční obor D_f.

Spočtěte derivaci.

Napište obor, kde Váš výpočet platí.

BONUS: Spočtěte $f'(0)$, pokud existuje.

2. Určete $\int \frac{1}{\sin^2 x + 2\sin x \cos x + 2\cos^2 x}\,dx$ (Použijte $y = \operatorname{tg} x$)

Př.: složitější lepení

$$\int \frac{\sin^2 x}{1+\sin^2 x}\,dx = \left| \begin{array}{l} y = \tan x \\ \text{doma} \ " \\ " \ x \in (-\frac{\pi}{2}, \frac{\pi}{2}) + \pi k \end{array} \right| = \dots = \int \frac{y^2}{(1+y^2)(1+2y^2)}\,dy = \left| \begin{array}{l} \text{parc.} \\ \text{zlom.} \\ \text{doma} \end{array} \right| =$$

nebo zkusím řešit [imag kořeny stačí sečíst] :

$$\int \left(\frac{+1}{1+y^2} + \frac{-1}{1+2y^2} \right) dy = \left| \begin{array}{l} z = \sqrt{2}\, y \\ dy = \frac{dz}{\sqrt{2}} \end{array} \right|$$

$$= \operatorname{arctg} y - \frac{1}{\sqrt{2}} \operatorname{arctg} \sqrt{2}\, y + C \ \left| \begin{array}{l} C \text{ je závislé na } k \\ y \text{ taky, } x \text{ taky} \end{array} \right| =$$

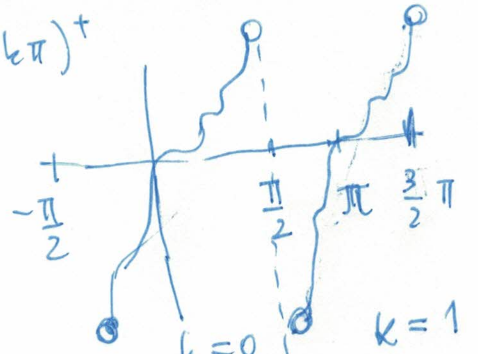
$$= x - \frac{1}{\sqrt{2}} \operatorname{arctg}(\sqrt{2} \tan x) + C_k, \quad x \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) + k\pi$$

Věta o nalepování:

$$F_k(x) = x - \frac{1}{\sqrt{2}} \operatorname{arctg}(\sqrt{2} \tan x) + C_k, \quad x \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) + k\pi$$

$$F_{k+1}(x) = x - \frac{1}{\sqrt{2}} \operatorname{arctg}(\sqrt{2} \tan x) + C_{k+1}, \quad x \in (\tfrac{\pi}{2}, \tfrac{3}{2}\pi) + k\pi$$

- $\displaystyle \lim_{x \to (\frac{\pi}{2} + k\pi)^-} F_k(x) = \frac{\pi}{2} + k\pi - \frac{1}{\sqrt{2}} \frac{\pi}{2} + C_k$
- $\displaystyle \lim_{x \to (\frac{\pi}{2} + k\pi)^+} F_{k+1}(x) = \frac{\pi}{2} + k\pi + \frac{1}{\sqrt{2}} \frac{\pi}{2} + C_{k+1}$



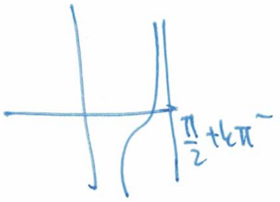
$k \in \mathbb{Z}$

$$F(x) = \begin{cases} F_k(x), & x \in (-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi) \\ \frac{\pi}{2} + k\pi - \frac{1}{\sqrt{2}} \frac{\pi}{2} + C_k \\ x - \frac{1}{\sqrt{2}} \operatorname{arctg}(\sqrt{2} \tan x) + C_{k+1} - \frac{\pi}{2} - k\pi - \frac{1}{\sqrt{2}} \frac{\pi}{2} - C_{k+1} + \\ \quad + \frac{\pi}{2} + k\pi + \frac{1}{\sqrt{2}} \frac{\pi}{2} + C_k \end{cases}$$

$$= x - \frac{1}{\sqrt{2}} \operatorname{arctg}(\sqrt{2} \tan x) + C_k + \frac{1}{\sqrt{2}} \frac{\pi}{2}$$

$x \to \left(\frac{\pi}{2}+k\pi\right)_-$: $\frac{\pi}{2}+k\pi-\frac{1}{\sqrt{2}}\frac{\pi}{2}+C_k$

Proč: $\lim\limits_{x\to\left(\frac{\pi}{2}+k\pi\right)_-} \operatorname{arctg}(\sqrt{2}\tan x) = \left| \begin{array}{l} y=\tan x \\ x\neq\frac{\pi}{2}+k\pi \end{array} \right|$

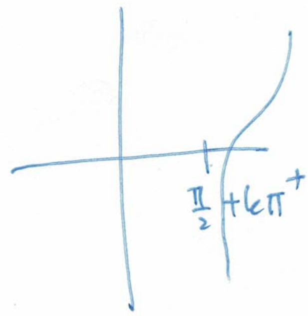


$\lim\limits_{y\to+\infty} \operatorname{arctg}(\sqrt{2}y) = \left| z=\sqrt{2}y \right| = \lim\limits_{z\to\infty}\operatorname{arctg}(z) = \frac{\pi}{2}$

víme $\left[\begin{array}{l} \text{nebo:} \\ z=\tan w \; : \lim\limits_{z\to\frac{\pi}{2}^-} z = \frac{\pi}{2} \\ w\to\frac{\pi}{2}^- \end{array}\right]$

Nalepovací teorém:

$x \to \left(\frac{\pi}{2}+k\pi\right)_+$: $\frac{\pi}{2}+k\pi-\frac{1}{\sqrt{2}}\left(-\frac{\pi}{2}\right)+C_k$



Proč: Obdobně ale

$\lim\limits_{x\to\left(\frac{\pi}{2}+k\pi\right)_+} \operatorname{arctg}(\sqrt{2}\tan x) = \lim\limits_{y\to-\infty}\operatorname{arctg}(\sqrt{2}y) = -\frac{\pi}{2}$

Vyjde tedy: $\frac{\pi}{2}+k\pi+\frac{1}{\sqrt{2}}\frac{\pi}{2}+C_k$

$F_2(x) - \lim F_2(x) - \lim F_1(x) = x-\frac{1}{\sqrt{2}}\operatorname{arctg}(\sqrt{2}\tan x)+C_k$

$-\frac{\pi}{2}-k\pi+\frac{1}{\sqrt{2}}\frac{\pi}{2}+C_k+\frac{\pi}{2}+k\pi-\frac{1}{\sqrt{2}}\frac{\pi}{2}-C_k=$

$= x-\frac{1}{\sqrt{2}}\operatorname{arctg}\sqrt{2}(\tan x)+C_k.$

$$C_k = k\frac{\pi}{\sqrt{2}}$$

$$F = \begin{cases} F_k(x) = x-\frac{1}{\sqrt{2}}\operatorname{arctg}\sqrt{2}(\tan x)+k\frac{\pi}{\sqrt{2}}+C, & x\in\left(-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi\right) \\ F_k\left(\frac{\pi}{2}+k\pi\right) := \lim\limits_{x\to\frac{\pi}{2}+k\pi} F_k(x). \end{cases}$$

(Nalepovací v. říká, že F má derivaci!)

Dále $x - \frac{1}{\sqrt{2}} \operatorname{arctg}(\sqrt{2}\tan x) + C_k + 2\frac{1}{\sqrt{2}}\frac{\pi}{2}$ $\boxed{\text{atd.}}$ (indukcí)

$$x - \frac{1}{\sqrt{2}} \operatorname{arctg}(\sqrt{2}\tan x) + \underline{\underline{k}}\frac{1}{\sqrt{2}}\frac{\pi}{2} + C .$$

$F(x)$ na $\left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right)$ je $\underline{x - \frac{1}{\sqrt{2}} \operatorname{arctan}(\sqrt{2}\tan x) +}$

$$\underline{+ \frac{k\pi}{2\sqrt{2}} + C}, \quad C \text{ lib.}$$

Odpovídá např. tomu, že $F(x)$ je určena $f(x)$ na intervalu (souvislá množina) až na konstantu.

(Na sj. disj. int. až na dvě konstanty :-))

$$\left[\begin{array}{l}\forall k \in \mathbb{N} \quad D_k = D_{k+1} \Rightarrow \exists D \ \forall k \ D_k = D \quad \text{a sice } D := D_1 \\ \quad 1.\ k=1 \quad D_1 = D_2 \quad \text{Ano } D_1 = D_2 \text{ podle předp.} \\ \quad 2.\ k \text{ vím} \\ \quad D_{k+1} = D_k \overset{\text{ind.př.}}{=} D, \quad D_{k+1} = D \text{ jak jsem měl dokázat}\end{array}\right]$$

zač.

Bez $\boxed{\text{atd}}$: ${}^{k}F(x) = \begin{cases} F_k(x) \\ \frac{\pi}{2} + k\pi - \frac{1}{\sqrt{2}}\frac{\pi}{2} + C_k \\ x - \frac{1}{\sqrt{2}}\operatorname{arctg}(\sqrt{2}\tan x) + C_k + \frac{1}{\sqrt{2}}\frac{\pi}{2} \end{cases}$

$${}^{k+1}F(x) = \begin{array}{l} x - \frac{1}{\sqrt{2}}\operatorname{arctg}(\sqrt{2}\tan x) + C_k + \frac{1}{\sqrt{2}}\frac{\pi}{2} \\ \frac{\pi}{2} + (k+1)\pi - \frac{1}{\sqrt{2}}\frac{\pi}{2} + C_k \\ x - \frac{1}{\sqrt{2}}\operatorname{arctg}(\sqrt{2}\tan x) + C_k + \frac{1}{\sqrt{2}}\frac{\pi}{2} + \frac{1}{\sqrt{2}}\frac{\pi}{2} \end{array}$$

$$\Rightarrow F(x) = x - \frac{1}{\sqrt{2}}\operatorname{arctg}(\sqrt{2}\tan x) + C_k + \frac{k}{\sqrt{2}}\frac{\pi}{2}$$

$C_k = C$ dle indukce výše.

Nevl. limity a limity v nevl. bodech

(A)

① $c \pm \infty = \pm\infty$, $c > -\infty$; $c - \infty = -\infty$, $c < \infty$, $c \in \mathbb{R}^* = \mathbb{R}\cup\{\pm\infty\}$! $+\infty - \infty$ NDF

② $c\cdot\pm\infty = \pm\infty$, $c>0$; $c\cdot\pm\infty = \mp\infty$, $c<0$, $c\in\mathbb{R}^* := \mathbb{R}\cup\{\pm\infty\}$! $0\cdot\infty$ NDF

spc $c \neq 0$

③ $\frac{1}{\pm\infty} = 0^\pm$ $\frac{1}{0^\pm} = \pm\infty$

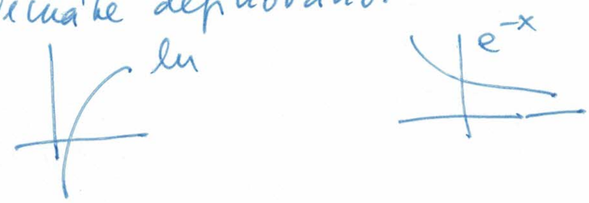
(odtud třeba $\frac{c}{+\infty}$, $c>0$, $c\in\mathbb{R}$; $\frac{c}{\frac{1}{0^+}} = c\cdot 0^+ = 0$.)

NDF: $\frac{1}{0}$!  $\lim \frac{1}{x} \nexists$ (Ani ndf $\frac{\pm\infty}{0}$!) Ale $\frac{+\infty}{+\infty} = +\infty\cdot\frac{1}{0^+} = +\infty$

Viz před. Dále třeba $\frac{c}{0^\pm}$, $c>0$ $= c\cdot\frac{1}{0^\pm} = c\cdot\pm\infty = \pm\infty$ (vč. $c=+\infty$)

④ Moc.: NDF 1^∞ převádím na exp. → V limitách občasná (ale typická!!) chyba je: $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} 1^x = 1$! (má být e) TZV ČÁST. DOSAZENÍ.

Pozn.: $(0^+)^{+\infty} = e^{(\ln 0^+)(+\infty)} = e^{-\infty\cdot+\infty} = e^{-\infty} = 0$. Velmi symbolický výpočet. Nemáme definováno.



(C) Růstová řada

$a^x \gg x^\alpha \gg \log x$ u $+\infty$. Tím myslíme

$a>1$ $\alpha>0$

$\lim_{x\to+\infty}\frac{a^x}{x^\alpha} = +\infty \wedge \lim_{x\to+\infty}\frac{x^\alpha}{\log x} = +\infty$. Co to je vlastně $\lim_{x\to\pm\infty}$?

(B) Limity v nevlastních bodech

$$\lim_{x\to\pm\infty} f(x) := \lim_{x\to0^\pm} f\left(\frac{1}{x}\right) \quad \left(= \lim_{y\to0^\pm} f\left(\frac{1}{y}\right)\right)$$

Nutně $f\circ\frac{1}{x}$ def. na $P_\delta^+(0)$ nebo na $P_\delta^-(0)$

$P_\delta^\pm(a) := \{x\in\mathbb{R} \mid x \gtrless a\} \cap P_\delta(a)$

levé a pravé prstencové okolí

Např. $P_\delta^-(a) = \{x\in\mathbb{R} \mid a-\delta<x<a\}$

(D) Podrobnější růstové řady:

$$e^{-x} \ll x^{-\beta} \ll x^{-\alpha} \ll \frac{1}{\log x} \ll 1 \ll \log x \ll x^{\alpha} \ll x^{\beta} \ll e^{x}$$

$0 < \alpha < \beta$ $\quad x \to +\infty$

$$x \to 0^+ : \quad x^{\beta} \ll x^{\alpha} \ll \frac{1}{\log \frac{1}{x}} \ll 1 \ll \log \frac{1}{x} \ll x^{-\alpha} \ll x^{-\beta}$$

$$\frac{1}{\log\frac{1}{x}} = \left(\frac{1}{-\log x}\right) \qquad \log\frac{1}{x} = (-\log x)$$

Mocniny na exp: princip $0.1^x \to 0$, ale $10^x \to \infty$.

(E) Limity posloupností $\lim\limits_{n \to +\infty} a_n = A \Leftrightarrow \forall \varepsilon > 0 \; \exists n_0 \; \forall n \geq n_0$ $|a_n - A| < \varepsilon$. Analogicky jako pro fce $\lim\limits_{n \to \infty} a_n = \pm\infty$.

Heine: $f: \mathbb{R} \to \mathbb{R}^*$, $a \in \mathbb{R}^*$, $\forall \delta: D_f \cap P_\delta(a) \neq \emptyset$, $A \in \mathbb{R}^*$. $\lim\limits_{x \to a} f(x) = A \Leftrightarrow \forall (x_n) \subseteq D_f \setminus \{a\}$ $x_n \to a$ je $\lim\limits_{n \to \infty} f(x_n) = A$

Př.: Limity v nevlast. bodech

1. Základní $\lim\limits_{x \to +\infty} \dfrac{x^3 + 2x + 1}{x^5 + 3x + 7} = \lim\limits_{x \to 0^+} \dfrac{\frac{1}{x^3} + \frac{2}{x} + 1}{\frac{1}{x^5} + \frac{3}{x} + 7} =$ ODSTR. ZLOMKŮ

$$= \lim_{x \to 0^+} \frac{x^2 + 2x^4 + x^5}{1 + 3x^4 + 7x^5} = \frac{0}{1} = 0$$

Základní $\lim\limits_{x \to +\infty} \dfrac{x^4 + 2x + 2}{13x^4 + x + 7} = \lim\limits_{x \to 0^+} \dfrac{\frac{1}{x^4} + \frac{2}{x} + 2}{\frac{13}{x^4} + \frac{1}{x} + 7} = \lim\limits_{x \to 0^+}$ $\neq \frac{2}{7}$! ugbrž

$$\frac{1 + 2x^3 + 2x^4}{13 + x^3 + 7x^4} = \frac{1 + 0 + 0}{13 + 0 + 0} = \frac{1}{13}$$

Zákl. $\lim\limits_{x \to +\infty} \dfrac{x^4 + x^2}{x^2 + x^3}$ analogicky: $\lim\limits_{x \to 0^+} \dfrac{\frac{1}{x^4} + \frac{1}{x^2}}{\frac{1}{x^2} + \frac{1}{x^3}} = \lim\limits_{x \to 0^+} \dfrac{1 + x^2}{x^2 + x} =$

$$= \text{"}\frac{1}{0^+}\text{"} = +\infty$$

"Aritmetika limit"

Př.: Obdobně spočteme:

$$\lim_{x\to+\infty} \frac{2x^2+1}{\sqrt{3x^4-6x^2+5}} = \lim_{x\to0^+} \frac{\frac{2}{x^2}+1}{\sqrt{\frac{3}{x^4}-\frac{6}{x^2}+5}} =$$

$$= \lim_{x\to0^+} \frac{2+x^2}{\sqrt{3-6x^2+5x^4}} = \frac{2}{\sqrt{3}}.$$

Př.: $$\lim_{x\to\infty} x\left(\sqrt{x^2+1}-\sqrt{x^2-1}\right) = \lim_{x\to0^+} \frac{\sqrt{\frac{1}{x^2}+1}-\sqrt{\frac{1}{x^2}-1}}{x} =$$

$$= \lim_{x\to0^+} \frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{x^2} \overset{\frac{0(+)}{0(+)}\ \text{NDF}}{=} \lim_{x\to0^+} \frac{\left(\sqrt{1+x^2}-\sqrt{1-x^2}\right)\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)}{x^2\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)}$$

$$= \lim_{x\to0^+} \frac{2x^2}{x^2\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)} = \lim_{x\to0^+} \frac{2}{\sqrt{1+x^2}+\sqrt{1-x^2}} = \frac{2}{2} = 1$$

L'Hospitalovo pravidlo

Předpoklad

1. f', g' vlastní, $g' \neq 0$ na $P_\delta(a)$ $(\exists \delta)$, $(a-\delta, a+\delta)\setminus\{a\}$
2. $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$
   nebo
   $\lim_{x\to a} f(x) = \pm\infty$, $\lim_{x\to a} g(x) = \pm\infty$.
3. Existuje $\lim_{x\to a} \frac{f'(x)}{g'(x)}$.

Pak $\exists \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

Pozn.: $\frac{0}{\infty}$ lze také spočítat l'Hosp.

Př.: $\lim\limits_{x\to+\infty} \sqrt[x]{x} = \lim\limits_{x\to+\infty} x^{\frac{1}{x}} = \lim\limits_{x\to+\infty} \underline{e^{\frac{1}{x}\ln x}} = \lim\limits_{x\to 0^+} e^{x\ln\frac{1}{x}} = \lim\limits_{x\to 0^+} e^{\overline{-x\ln x}}$

$\left[\lim\limits_{x\to 0^+}(-\underline{x\ln x}) = -0 \right| = \lim\limits_{x\to 0^+} e^{-x\ln x} = \underline{e^{-0^+} = 1.}\Big]$

růstová řada

Podrobněji: $0^+ \leftarrow \dfrac{+x}{\frac{1}{-\ln x}} = -x\ln x$ $\left[x \ll \dfrac{1}{-\ln x} \Rightarrow \dfrac{\frac{1}{-\ln x}}{x} \to +\infty \quad \dfrac{x}{\frac{1}{-\ln x}} \to 0^+ \right]$

Př.: Růstové řady

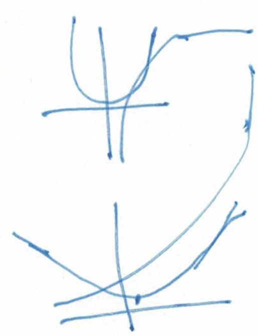
$n \geq 0$ $\lim\limits_{x\to\infty} \dfrac{x^n}{e^x} = \left|\dfrac{\infty}{\infty}\right| \overset{\text{l'Hosp.}}{=} \lim\limits_{x\to\infty}\dfrac{n x^{n-1}}{e^x} = \dots = \lim\limits_{x\to\infty}\dfrac{n!}{e^x} = 0$

$\left(= n! \lim\limits_{x\to\infty} e^{-x} = n!\ 0 = 0\right)$

Pro $n < 0$ Varit m. limit

Př.: $\lim\limits_{x\to\infty}\dfrac{x^n}{\ln x} = \lim \dfrac{n x^{n-1}}{\frac{1}{x}} = \lim\limits_{x\to\infty} n x^n = +\infty$, $n > 0$

Zde je dobré: do paměti „obrázy":



(l'Hosp. je odůvodňuje rychlost růstu)

L'Hospital (h, s necite) l-hvalě!

Př.: $\lim\limits_{x\to 0} \frac{\operatorname{tg} x - x}{x - \sin x} \overset{l'H}{\underset{\frac{0}{0}\ \text{Lze}}{=}} \lim\limits_{x\to 0} \frac{\frac{1}{\cos^2 x} - 1}{1 - \cos x} = \lim\limits_{x\to 0} \frac{1 - \cos^2 x}{(1-\cos x)\cos^2 x} =$

$= \lim\limits_{x\to 0} \frac{1+\cos x}{\cos^2 x} = \frac{2}{1} = 2$

Př.: $\lim\limits_{x\to 0} \frac{x(e^x+1) - 2(e^x-1)}{x^3} \overset{l'H}{\underset{\frac{0}{0}}{=}} \lim\limits_{x\to 0} \frac{e^x + 1 + xe^x - 2e^x}{3x^2} =$

$= \lim\limits_{x\to 0} \frac{xe^x - e^x + 1}{3x^2} \overset{l'H}{=} \lim\limits_{x\to 0} \frac{e^x + xe^x - e^x}{6x} = \lim\limits_{x\to 0} \frac{e^x + xe^x}{6} = \frac{1}{6}.$

Př.: $\lim\limits_{x\to 0} \frac{1-\cos^2 x}{x^2 \sin x^2} \overset{lH}{=} \lim\limits_{x\to 0} \frac{2x\sin x^2}{2x\sin x^2 + x^2 2x\cos x^2} = \lim\limits_{x\to 0} \frac{\sin x^2}{\sin x^2 + x^2\cos x^2} =$

$$= \lim_{x\to 0} \frac{\frac{\sin x^2}{x^2}}{\frac{\sin x^2}{x^2} + \cos x^2} \overset{VOS}{=} \frac{1}{1+1} = \frac{1}{2}.$$

(VOSem na $\frac{\sin y}{y}$)

Př.: $\lim\limits_{x\to 0^+} x^x = \lim\limits_{x\to 0^+} e^{x\ln x} = e^{\lim\limits_{x\to 0} x\ln x} = e^0 = 1$

Růst.

Místo růstu lze i $\lim\limits_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim\limits_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim\limits_{x\to 0^+} x = 0.$

Př.: $\lim\limits_{x\to \frac{\pi}{4}} (\operatorname{tg} x)^{\operatorname{tg} 2x} = e^{\lim\limits_{x\to\frac{\pi}{4}} \operatorname{tg} 2x \ln(\operatorname{tg} x)}$

$$\lim_{x\to\frac{\pi}{4}} \frac{\ln(\operatorname{tg} x)}{\frac{1}{\operatorname{tg} 2x}} = \lim_{x\to\frac{\pi}{4}} \frac{\frac{1}{\operatorname{tg} x}\,\frac{1}{\cos^2 x}}{\frac{-1}{\operatorname{tg}^2 2x}\,\frac{1}{\cos^2 2x}\cdot 2} =$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{2\sin^2 2x}{\operatorname{tg} x\cos^2 x} = -\frac{1}{2\cdot 1\left(\frac{\sqrt{2}}{2}\right)^2} = -1 \Rightarrow \lim_{x\to\frac{\pi}{4}} (\operatorname{tg} x)^{\operatorname{tg} 2x} = \frac{1}{e}$$

Symboly o, O, ∼

1. „$f(x) = o(g(x)),\ x \to a$" $\equiv \lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = 0$

$x^n = o(x^{n-1}),\ x \to 0$? $\quad \lim\limits_{x\to 0} \dfrac{x^n}{x^{n-1}} = \lim\limits_{x\to 0} x = 0$

$x^n = o(x^{n+1}),\ x \to \infty$? $\quad \lim\limits_{x\to\infty} \dfrac{x^n}{x^{n+1}} = \lim\limits_{x\to\infty} \dfrac{1}{x} = 0 \left(= \lim\limits_{y\to 0} \dfrac{1}{\frac{1}{y}} = \lim\limits_{y\to 0} y = 0 \right)$

2. „$f(x) = O(g(x)),\ x \to a$" $\equiv \exists C > 0\ \exists \delta > 0\ \forall x \in P_\delta(a)\ |f(x)| \le C\,|g(x)|$

(stačí $\lim\limits_{x\to a} \left|\dfrac{f(x)}{g(x)}\right| \in \mathbb{R}\setminus\{0\}$, není ale nutné.)

3. „$f(x) \sim g(x),\ x \to a$" $\equiv \lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = 1$ asympt. ekv. ♯

Někdy: „$f(x) \sim g(x),\ x \to a$" $\equiv \lim\limits_{x\to a} \dfrac{f(x)}{g(x)} \in \mathbb{R}\setminus\{0\}$ ← & slabě asym. ekv.

„$f(x) \simeq g(x),\ x \to a$" $\equiv \lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = 1$" ♯

Př.: $e^x - \cos x \sim x$, $x \to 0$ Dk.!

$$\lim_{x\to 0}\frac{e^x-\cos x}{x} = \lim_{x\to 0}\frac{e^x+\sin x}{1} = 1$$

Př.: $e^x - \cos x = o(x^a)$, $x \to 0$ $\forall a < 1$

$$\lim_{x\to 0}\frac{e^x-\cos x}{x^a} = \lim_{x\to 0}\frac{e^x-\cos x}{x}\, x^{1-a} = 1 \cdot \lim_{x\to 0} x^{1-a} = 1\cdot 0 = 0.$$

Př. Limita posloupnosti

1. $$\lim_{n\to\infty}\frac{\sqrt{n^3-2n^2+1}+\sqrt[3]{n^4+1}}{\sqrt{n^6-6n^5+2}+\sqrt[5]{n^7+n^3+1}} \overset{\text{„triv." Heine nebo def}}{=} \lim_{x\to\infty}\frac{\sqrt{x^3-2x^2+1}+\sqrt[3]{x^4+1}}{\sqrt{x^6-6x^5+1}+\sqrt[5]{x^7+x^3+1}}$$ [illegible]

$\boxed{\frac{3}{2}}$ $\boxed{\frac{4}{3}}$ $\boxed{3}$ $\boxed{7/5}$ $0?$

$$= \lim_{x\to 0^+}\frac{\sqrt{\frac{1}{x^3}-\frac{2}{x^2}+1}+\sqrt[3]{\frac{1}{x^4}+1}}{\sqrt{\frac{1}{x^6}-\frac{6}{x^5}+1}+\sqrt[5]{\frac{1}{x^7}+\frac{1}{x^3}+1}}\cdot\frac{x^3}{x^3} =$$

$$= \lim_{x\to 0^+}\frac{\sqrt{x^3-2x^4+x^6}+\sqrt{x^5+x^9}}{\sqrt{1-6x+x^6}+\sqrt{x^8+x^{12}+x^{15}}} = \frac{0+0}{1+0} = 0$$

2. $\lim_{n\to\infty}\frac{a^n}{n!}$, $a\in\mathbb{R}$ $\quad \lim_{n\to\infty}\frac{a^n}{n!} = \lim_{n\to\infty}\frac{a}{n}\cdot\frac{a}{n-1}\cdots\frac{a}{1}$, Zvol ε. $\frac{a}{n}<\varepsilon$ $\forall n \geq n_0$

Najdi $n_0\in\mathbb{N}$, že $\frac{a}{n_0}<\varepsilon$

$$= \lim_{n\to\infty}\Big(\underbrace{\frac{a}{1}\cdots\frac{a}{n_0-1}}\cdot\underbrace{\frac{a}{n_0+1}\cdots\frac{a}{n}}\Big) \leq \lim_{n\to\infty}\Big(\frac{a}{1}\cdots\frac{a}{n_0-1}\Big)\cdot\varepsilon^{n-n_0} = 0$$

triv. Heine

3. $$\lim_{n\to\infty}\sqrt[n]{n} = \lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{x\to\infty} x^{\frac{1}{x}} = e^{\lim_{x\to\infty}(\ln x)\frac{1}{x}} = e^0 = 1$$

$\left(\frac{\ln x}{x}\to 0, x\to\infty\right.$ růstová ř.

nebo l'Hosp.$)$

4. $$\lim_{n\to\infty}\Big[\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}\Big] = \lim_{n\to\infty}\Big[\Big(\frac{1}{1}-\frac{1}{2}\Big)+\Big(\frac{1}{2}-\frac{1}{3}\Big)+\cdots+\Big(\frac{1}{n}-\frac{1}{n+1}\Big)\Big] =$$

$$= \lim_{n\to\infty} \left[\frac{1}{1} - \frac{1}{n+1} \right] = 1.$$

Př Symboly o, O, ~

1) $\operatorname{arctg} x = O(1), x \to \infty$, tj. $\exists C \in \mathbb{R}$ $|f(x)| \leq C|g(x)|$

$|f(x)| \leq C$

$|\operatorname{arctg} x| \leq \frac{\pi}{2}$

2) $x^2 e^{-x} = o(x^a),\ a<0 \iff \lim\limits_{x\to\infty} \frac{x^{2-a}}{e^x}$ $\left| 2-a>0 \right|$
$x \to \infty$

$$= \frac{(2-a)x^{1-a}}{e^x} \underset{\substack{\uparrow \\ \text{nestovonradou.}}}{=} 0$$

Nebo derivuji $(2-a)$-krát : $\frac{(2-a)(1-a)\cdots 1}{e^x} \overset{l'}{\underset{\frac{\infty}{\infty}}{=}}$

$\frac{0}{e^x} = 0.$

3) $\sqrt{x+\sqrt{x+\sqrt{x}}} \sim \sqrt{x},\ x \to \infty$

$$\lim_{x\to\infty} \sqrt{\frac{x+\sqrt{x+\sqrt{x}}}{x}} = \lim_{x\to\infty} \sqrt{1+\frac{\sqrt{x+\sqrt{x}}}{x}} = \lim_{x\to\infty} \sqrt{1+\sqrt{\frac{1}{x}+\underset{\sqrt{\frac{x}{x^4}}}{\frac{\sqrt{x}}{x^2}}}} =$$

$$= \lim_{x\to\infty} \sqrt{1+\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x^3}}}} = \lim_{y\to 0} \sqrt{1+\sqrt{y+\sqrt{y^3}}} = 1.$$