Lecture notes - Model Theory (Math 411) Autumn 2002.

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1 Notation and review.

Let me begin by briefly discussing many-sorted structures. Although in most of the course I will be working with the traditional 1-sorted structures, everything is valid in the more general context.

By a many-sorted language L (or rather a language for many-sorted structures), we mean a set $\mathcal{S}$ of "sorts", a set $\mathcal{R}$ of sorted relation symbols, and a set $\mathcal{F}$ of sorted function symbols. What this means is that each $R \in \mathcal{R}$ comes together with a certain finite sequence $(S_1, .., S_n)$ of sorts where $n \geq 1$, and each $f \in \mathcal{F}$ comes together with a sequence $(S_1, .., S_n, S)$ of sorts where $n \geq 0$. By the cardinality $|L|$ of L we mean $|\mathcal{S}| + |\mathcal{R}| + |\mathcal{F}$.

By an L-structure M we mean a family $(S(M) : S \in \mathcal{S})$ of nonempty sets, together with, for each $R \in \mathcal{R}$ of sort $(S_1, .., S_n)$ a subset $R(M)$ of $S_1(M) \times ... \times S_n(M)$, and for each $f \in \mathcal{F}$ of sort $(S_1, .., S_n, S)$ a function $f(M) : S_1(M) \times .. \times S_n(M) \to S(M)$. So an L-structure is just an *interpretation* of the formal language L. The $S(M)$ are the sorts of the structure, the $R(M)$ the basic relations of the structure and the $f(M)$ the basic functions of the structure. By the cardinality $|M|$ of M we mean the sum of the cardinalities of the sets $S(M)$.

A one-sorted language is a (many-sorted) language with only one sort, that is $\mathcal{S}$ is a singleton, say $\{S\}$. In that case the sort of a relation symbol is just a natural number $n \geq 1$ and likewise for the sort of a function symbol. If M is an L-structure then $S(M)$ is called the underlying set (or universe) of

M and we often notationally identify it with M (just as we often notationally identify a group G with its underlying set) if there is no ambiguity.

The many-sorted point of view is quite natural, because a whole "category" of objects can be now viewed as a single structure. An example is the category of "algebraic varieties over $\mathbf{Q}$". By an algebraic variety over $\mathbf{Q}$ we mean (for now) a subset X of some $\mathbf{C}^n$ which is the common zero-set of a finite number of polynomials $P_i(x_1, .., x_n)$ in $\mathbf{Q}[x_1, .., x_n]$. By a morphism (over $\mathbf{Q}$) between $X \subseteq \mathbf{C}^n$ and $Y \subseteq \mathbf{C}^m$, we mean a map $f : X \to Y$ given by m polynomials $Q_1(x_1, .., x_n), ..., Q_m(x_1, .., x_n)$ over $\mathbf{Q}$. The many-sorted structure $Var_{\mathbf{Q}}$ has as sorts the algebraic varieties over $\mathbf{Q}$, as basic relations the algebraic subvarieties (over $\mathbf{Q}$) of products of the sorts, and as basic functions morphisms from products of sorts to sorts.

Of course there is another natural associated one-sorted structure, the structure $(\mathbf{C}, +, \cdot, -, 0, 1)$. This is the structure with universe $\mathbf{C}$, two basic 2-ary functions, addition and multiplication, one basic unary function $(-)$ and two basic 0-ary functions (or constants), 0 and 1.

Note that $Var_{\mathbf{Q}}$ and $(\mathbf{C}, +, \cdot, -, 0, 1)$ are not only different structures, but structures for different languages. But we hope later to explain a sense in which they are the "same". (They are "bi-interpretable".)

Given a many-sorted language L we build up in the usual manner terms and first order formulas of L from the relation and function symbols of L together with $\wedge$, $\vee$, $\neg$, $\to$, $\forall$, $\exists$ and a countable set of variables for each sort $S \in \mathcal{S}$, as well as the symbol $=_S$ for each $S \in \mathcal{S}$. Note that every variable comes equipped with a sort. If ϕ is an L-formula we may write $\phi(x_1, .., x_n)$ to mean that the free variables in ϕ are among $x_1, .., x_n$. An L-sentence is an L-formula with no free variables. If $\phi(x_1, .., x_n)$ is an L-formula where x_i is of sort S_i, M is an L-structure and $a_i \in S_i(M)$ for $i = 1, .., n$ we define (inductively)
$M \models \phi[x_1/a_1, .., x_n/a_n]$
in the usual way. ("$(a_1, ., a_n)$ satisfies $\phi(x_1, .., x_n)$ in M", or "$\phi(x_1, .., x_n)$ is true of $(a_1, .., a_n)$ in M".) When there is no ambiguity we just write $M \models \phi[a_1, .., a_n]$. An important fact is that exactly one of $M \models \phi[a_1, .., a_n]$, $M \models (\neg\phi)[a_1, .., a_n]$ holds.

(By the way the $=_S$ symbols are interpreted as equality, namely if x, y are variables of sort S, M is an L-structure, and $a, b \in S(M)$ we define $M \models x =_S y[x/a, y/b]$ to hold if $a = b$.)

If σ is an L-sentence we say "M is a model of σ" for $M \models \sigma$ and this is where the expression "model theory" comes from.

Definition 1.1 *Let L be a language and M an L-structure, Σ a set of L-sentences, and σ an L-sentence.*
(i) We say that $M \models \Sigma$ (M is a model of Σ) if $M \models \sigma$ for all $\sigma \in \Sigma$.
(ii) $\Sigma \models \sigma$ (Σ logically implies σ) means that every model of Σ is a model of σ.
(iii) By an L-theory we mean a set of L-sentences closed under $\models$. (L-theories are often debtoed by T.)
(iv) We say that Σ is consistent if Σ has a model.
(v) We say that Σ is complete if Σ is consistent and for every σ, either $\sigma \in \Sigma$ or $\neg\sigma \in \Sigma$.
(vi) Let $\mathcal{K}$ be a class of L-structures. By $Th(\mathcal{K})$ we mean $\{\sigma : M \models \sigma$ for all $M \in \mathcal{K}\}$. If $\mathcal{K} = \{M\}$ we write $Th(M)$ in place of $Th(\mathcal{K})$.
(vii) Let $\mathcal{K}$ be a class of L-structures. We say that $\mathcal{K}$ is an elementary class if there is a set Σ of L-sentences such that $\mathcal{K} = \{M : M \models \Sigma\}$.
(viii) We write $M \equiv N$ (M is elementarily equivalent to N)if M and N are models of the same L-sentences, that is, if $Th(M) = Th(N)$.
(ix) Let T be an L-theory. We say that Σ axiomatizes T if $\Sigma \subseteq T$ and $\Sigma \models \sigma$ for all $\sigma \in T$.

Exercise 1.2 *(i) Let Σ be a set of L-sentences and $\Sigma' = \{\sigma : \Sigma \models \sigma\}$. Then Σ' is a theory.*
(ii) $Th(M)$ is a complete theory.
(iii) Let $\mathcal{K}$ be a class of L-structures. Then $\mathcal{K}$ is an elementary class iff $\mathcal{K} = \{M : M \models Th(\mathcal{K})\}$.
(iv) T is complete iff all models of T are elementarily equivalent

Example 1.3 *Let L be the 1-sorted language containing a single binary function symbol f say, and a single constant sybol e say. Any group can be considered to be an L-structure.*
(i) The class of groups is elementary.
(ii) The class of torsion-free divisible abelian groups is an elementary class.
(iii) (needs compactness theorem) The class of torsion abelian groups is NOT elementary.
(iv) (needs more) The class of simple nonabelian groups is NOT an elementary class.

Example 1.4 *Let L be the language "of unitary rings" that is L is again one-sorted, contains two binary function symbolds $+$ and $\cdot$ say, one unary funtion symbol $-$, and two constant symbols $0, 1$. Any ring can be naturally viewed as an L-structure.*
(i) The class of finite fields is NOT an elementary class. (Needs compactness.)
(ii) Let $\mathcal{K}$ be the class of finite fields. By definition a pseudofinite field is an infinite model of $Th(\mathcal{K})$. Then the class of pseudofinite fields IS an elementary class. (Why?)

We now consider relations between L-structures. Let us fix a (many-sorted) language L. The notion of an *isomorphism* between two L-structures is pretty obvious and clearly we are only interested in L-structures up to isomorphism. We have already defined elementary equvalence of L-structures and it is important that this is generally weaker than isomorphism. However:

Exercise 1.5 *Let M and N be elementarily equivalent L-structures. Suppose that for every sort symbol S of L, $S(M)$ is finite. Then M and N are isomorphic.*

Definition 1.6 *(i) Let M, N be L-structures. We say that M is a substructure of N if for each sort symbol S, $S(M) \subseteq S(N)$, for each relation symbol R of L of sort $S_1 \times ... \times S_n$, $R(M) = R(N) \cap (S_1(M) \times ... \times S_n(M))$ and for each function symbol f of L of sort $(S_1, .., S_n, S)$, and choice $a_i \in S_i(M)$ for $i = 1, .., n$, $f(M)(a_1, ., a_n) = f(N)(a_1, .., a_n)$.*
(ii) We say that M is an elementary substructure of N if M is a substructure of N, and for all L-formulas $\phi(x_1, ., x_n)$ and $a_1, ., a_n$ in the sorts of M corresponding to variables $x_1, .., x_n$ respectively, we have $M \models \phi[a_1, .., a_n]$ iff $N \models \phi[a_1, .., a_n]$.
(iii) An (elementary) embedding of M into N is an isomorphism of M with a (elementary) substructure of N.

Exercise 1.7 *(i) Let L be the language of unitary rings mentioned earlier. Let F, K be fields. Then F is a subfield of K iff F is a substructure of K.*
(ii) Let L be 1-sorted and consist of a single unary function symbol. Let S be the succesor function on $\mathbf{Z}$. Then $(\mathbf{Z}, S)$ and $\mathbf{N}, S)$ are naturally L-structures. The first is a substructure of the second, but not an elementary

substructure.
(iii) Let M be a substructure of N. Then for any quantifier-free L-formula $\phi(x_1,..,x_n)$ and $a_i \in M$ of suitable sort, $M \models \phi[a_1,..,a_n]$ iff $N \models \phi[a_1,..,a_n]$.

Exercise 1.8 *(Tarski-Vaught test.) Let M be a substructure of N. Then M is an elementary substructure if M if and only if for each L-formula $\phi(x_1,..,x_n,x)$ and $a_1,..,a_n \in M$ (in the appropriate sorts), if $N \models (\exists x(\phi))[a_1,..,a_n]$ then for some b of the appropriate sort in M, $N \models \phi[a_1,..,a_n,b]$.*

Let us introduce a bit of notation. Let L be a language as usual, and M an L-structure. Let L_M be L together with a set of new (sorted) constant symbols c_m for m ranging over elements of M (i.e. of elements of sorts of M). Then we can "expand" canonically M to an L_M-structure M' say by defining $c_m(M') = m$. We often write M' as $(M,m)_{m\in M}$. For $\phi(x_1,..,x_n)$ an L-formula, and $m_1,..,m_n \in M$ of appropriate sorts, let $\phi(c_{m_1},..,c_{m_n})$ be the L_M-formula which results from substituting c_{m_i} for x_i in ϕ. In this context we have:

Remark 1.9 *(i) $M \models \phi[m_1,..,m_n]$ iff $(M,m)_{m\in M} \models \phi(c_{m_1},..,c_{m_n})$.*

Note that we can do exactly the same thing, if we add constants for a subset A of M (that is each element of A is in some $S(M)$), to obtain a language L_A and an "expansion" $(M,a)_{a\in A}$ of M to an L_A-structure.

With this notation we have:

Lemma 1.10 *Let M be a substructure of N. Then M is an elementary substructure of N iff and only if $(M,m)_{m\in M}$ and $(N,m)_{m\in M}$ are elementarily equivalent as L_M-structures.*

Sometimes we completely abuse notation by writing $M \models \phi(m_1,..,m_n)$ in place of the equivalent conditions in Remark 1.9.

Here is a useful slight strengthening of 1.8 which we state for convenience in the 1-sorted context.

Exercise 1.11 *(Assume L to be 1-sorted.) Let M be an L-structure, and A a subset of the underlying set (or universe) of M. Then A is the universe of an elementary substructure of M if and only if for each L_A-formula $\phi(x)$, if $M \models \exists x(\phi(x))$ then there is $b \in A$ such that $M \models \phi(b)$.*

By a *chain* of L-structures we mean a sequence $(M_i : i \in I)$ (of L-structures) where $I = (I, <)$ is an ordered set and $i < j$ implies that M_i is a substructure of M_j. Note that the union of the underlying sets (or sorts) of the M_i is naturally equipped with an L-structure, which is an extension of each M_i. We call this L-structure $\cup_i M_i$. An elelementary chain of L-structures is a chain as above but with the additional feature that $i < j$ implies M_i is an elementary substructure of M_j.

Lemma 1.12 *Let $(M_i : i \in I)$ be an elementary chain, and $M = \cup_i M_i$. Then M_i is an elementary substructure of M for all i.*

Proof. We prove by induction on the complexity of the L-formula $\phi(x_1, .., x_n)$ that for all i and $a_1, .., a_n \in M_i$, $M_i \models \phi(a_1, ., a_n)$ iff $M \models \phi(a_1, .., a_n)$. For ϕ quantifier-free this is Exercise 1.7 (iii). If ϕ is built from formulas we already dealt with, by $\neg$, $\rightarrow$, $\vee$, or $\wedge$, it is easy.
Suppose ϕ is $\exists x(\psi(x, x_1, .., x_n))$. and suppose that $M_i \models \phi(a_1, .., a_n)$, then $M_i \psi(b, a_1, .., a_n)$ for some $b \in M_i$, so $M \models \psi(b, a_1, .., a_n)$ by induction hypothesis and so $M \models \phi(a_1, .., a_n)$. Conversely, suppse $M \models \phi(a_1, .., a_n)$, so $M \models \psi(b, a_1, .., a_n)$ for some $b \in M$. Then $b \in M_j$ for some $i < j$. By induction hypothesis $M_j \models \psi(b, a_1, .., a_n)$, so $M_j \models \phi(a_1, .., a_n)$ so by our assumptions $M_i \models \phi(a_1, .., a_n)$.

Definition 1.13 *Let Σ be a set of L-sentences. We say that Σ is finitely consistent, if every finite subset of Σ is consistent (namely every finite subset of Σ has a model).*

From now on we assume L to be one-sorted for convenience.

Exercise 1.14 *Suppose Σ is a finitely consistent set of L-sentences. Then*
(i) For any L-sentence σ either $\Sigma \cup \sigma$ or $\Sigma \cup \neg\sigma$ is finitely consustent.
(ii) Suppose $\exists x \phi(x) \in \Sigma$ and c is a constant symbol not appearing in Σ. Then $\Sigma \cup \phi(c)$ is finitely consistent.

Theorem 1.15 *(Compactness) Let Σ be a set of L-sentences. Then Σ is finitely consistent if and only if Σ is consistent.*

Note that right to left is trivial.

We will give a couple of sketch proofs of left to right.
Proof A. (essentially Henkin.) Assume for simplicity that the language L is

countable and has no function symbols. Assume Σ to be finitely consistent. Let $\{c_i : i < \omega\}$ be a countable set of new constant symbols. Let L' be L together with these new symbols. Let $(\sigma_i : i < \omega)$ be an enumeration of all L'-sentences. We build an increasing sequence $\Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 ...$ of sets of L-sentences with the following properties:
(i) $\Sigma_0 = \Sigma$,
(ii) Each Σ_i is finitely consistent and consists of Σ together with at most finitely many additional sentences,
(iii) If (a) $\Sigma_i \cup \{\sigma_i\}$ is finitely consustent then $\sigma_i \in \Sigma_{i+1}$. Otherwise $\neg\sigma_i \in \Sigma_{i+1}$.
(iv) If (a) holds in (iii) and σ_i has the form $\exists x\phi(x)$ then for some new constant c_j, $\phi(c_j) \in \Sigma_{i+1}$.

The construction of the Σ_i is by induction, using Exercise 1.13.
Let Σ' be the union of the Σ_i. Then we have:
(v) Σ' is a finitely consistent set of L'-sentences,
(vi) For each L'-sentence σ, exactly one of $\sigma, \neg\sigma$ is in Σ', and
(vii) if $\exists x\phi(x) \in \Sigma'$ then $\phi(c) \in \Sigma'$ for some constant symbol c of L'.

Now we build an L'-structure. Let $\sim$ be the following relation on $\{c_i : i < \omega\}$:
$c_i \sim c_j$ if $c_i = c_j \in \Sigma'$.
Check that $\sim$ is an equivalence relation.
Let M be the L'-structure whose elements are the $\sim$ classes, and such that for an n-place relation symbol R of L, $((c_{i_1}/\sim), ..., (c_{i_n}/\sim)) \in R(M)$ if $R(c_{i_1}, .., c_{i_n}) \in \Sigma'$.
Check this is well-defined.
Finally check that for each L'-sentence σ, $M \models \sigma$ iff $\sigma \in \Sigma'$. (By induction on the complexity of σ.) In particular $M \models \Sigma'$. Let M_0 be the L-reduct of M, that is M considered as an L-structure. So M_0 is a model of Σ, and Σ is consistent.
(Where in the proof do we use finite consistency of Σ'?)

Proof B. This proof is by ultraproducts which we will define subsequently. Let I be the set of finite subsets of Σ. For each $i \in I$ let M_i be a model of i. For each $\sigma \in \Sigma$, let $X_\sigma = \{i \in I : \sigma \in i\}$. Let $\mathcal{F}_0 = \{X_\sigma : \sigma \in \Sigma\}$. Then $\mathcal{F}_0$ is a collection of subsets of I, i.e. a subset of $P(I)$.
Let $\mathcal{F}$ be the filter on I generated by $\mathcal{F}_0$, namely the closure of $\mathcal{F}_0$ under supersets and finite intersections.

Claim. $\mathcal{F}$ is a nontrivial filter on I, that is $\emptyset \notin \mathcal{F}$.
Proof. It is enough to show that for $\sigma_1, .., \sigma_n \in \Sigma$, $X_{\sigma_1} \cap ... \cap X_{\sigma_n} \neq \emptyset$, which is clear since it contains $\{\sigma_1, .., \sigma_n\}$.

By Zorn's Lemma we can extend $\mathcal{F}$ to an *ultrafilter* $\mathcal{U}$ on I. This means that $\mathcal{U}$ is a maximal nontrivial filter on I, equivalent $\mathcal{U}$ is a nontrivial filter on I such that for any $X \subseteq I$, either X or the complement of X is in $\mathcal{U}$. Now let M be the *ultraproduct* $(\prod M_i)/\mathcal{U}$ (to be expained below).
Los' Theorem states that for any L-sentence σ, $M \models \sigma$ if and only if $\{i \in I : M_i \models \sigma\} \in \mathcal{U}$.
Claim. M is a model of Σ.
Proof. Let $\sigma \in \Sigma$. Then for each $i \in X_\sigma$, $M_i \models \sigma$ (by choice), so $\{i \in I : M_i \models \sigma\} \supseteq X_\sigma$ so is in $\mathcal{F}$ so also in $\mathcal{U}$. By Los' Theorem, $M \models \sigma$.

Let us define ultraproducts and state Los' Theorem. Let I be a set, and $\mathcal{U}$ an ultrafilter on I. We will say that a subset X of I is large if $X \in \mathcal{U}$. By $\prod_{i \in I} M_i$ we mean for now just the set of sequences $s = (s(i) : i \in I)$ such that $s(i) \in M_i$ for each i. Define $\sim$ on this set by $s \sim t$ if $\{i : s(i) = t(i)\}$ is large. $M = (\prod_i M_i)/\mathcal{U}$ will be the L-structure whose underlying set is $\{s/\sim : s \in \prod_i M_i)$, and such that for R an n-ary relation symbol $(s_1/\sim, .., s_n/\sim) \in R(M)$ if $\{i \in I : (s_1(i), .., s_n(i)) \in R(M_i)\}$ is large, and for an n-ary function symbol f, $f(M)(s_1/\sim, .., s_n/\sim) = t/\sim$ where $t(i) = f(M_i)(s_1(i), .., s_n(i))$. One must prove that this is well-defined.

Theorem 1.16 *(Los.) Let $\phi(x_1, .., x_n)$ be an L-formula, and $s_1/\sim, ., s_n/\sim$ elements of $(\prod_i M_i)/\mathcal{U}$. Then*
$(\prod_i M_i)/\mathcal{U} \models \phi(s_1/\sim, .., s_n/\sim)$ if and only if $\{i \in I : M_i \models \phi(s_1(i), .., s_n(i))\}$ is large.

The theorem is proved by induction on the complexity of ϕ (with maybe a prior lemma on terms). In the proof B above of the compactness theorem we used the special case of the Theorem where σ is a sentence.

Lemma 1.17 *Let $\mathcal{K}$ be a class of L-structures. Then $\mathcal{K}$ is an elementary class if and only if $\mathcal{K}$ is closed under elementary equivalence and ultraproducts.*

Proof. Left to right is immediate (using Theorem 1.16). For right to left. Let $T = Th(\mathcal{K})$. We must prove that every model M of T is in $\mathcal{K}$. Let M

be a model of T. Let $\Sigma = Th(M)$. Let I be the set of finite subsets of Σ. Note that for each $i \in I$ there is $M_i \in \mathcal{K}$ which is a model of i. (Otherwise $\neg \wedge i$ is a sentence in T so in Σ, contradiction.) Let $\mathcal{U}$ be the ultrafilter on I produced as in the proof B of the compactness theorem. Then $(\prod_i M_i)/\mathcal{U}$ is a model of Σ so elementarily equivelent to M. By the hypotheses, $M \in \mathcal{K}$.

Let us give some corollaries and reformulations of the compactness theorem.

Proposition 1.18 *Let Σ be a set of L-sentences and suppose that for every $n < \omega$, every finite subset of Σ has a model of cardinality at least n. Then Σ has an infinite model*

Proof. Let σ_n be the L-sentence expressing that there are at least n elements (for $n < \omega$). Then By hypothesis $\Sigma \cup \{\sigma_n : n < \omega\}$ is finitely consistent, so consistent by the compactness theorem.

Proposition 1.19 *Let Σ and Γ be sets of L-sentences. Suppose that for every model M of Σ there is $\gamma \in \Gamma$ such that $M \models \gamma$. (We could write here $\Sigma \models \vee\Gamma$.) Then there is are finite subsets Σ' of Σ and Γ' of Γ such that $\Sigma' \models \vee\Gamma'$.*

Proof. The hypothesis says that $\Sigma \cup \{\neg\gamma : \gamma \in \Gamma\}$ is inconsistent. By the compactness theorem a finite subset is inconsistent, yielding the desired conclusion.

Proposition 1.20 *Let $\mathcal{T}$ be the set of complete (consistent) L-theories. Define $X \subseteq \mathcal{T}$ to be closed if $X = \{T : \Sigma \subseteq T\}$ for some set Σ of L-sentences. Then this makes $\mathcal{T}$ into a topological space which is compact (Hausdorff) and totally disconnected.*

Proof. First as a word of explanation, a topological space is said to be totally disconnected if it has a basis of clopens.

First we'll show that $\mathcal{T}$ *is* a topological space. Clearly an intersection of closed sets is closed. The whole space is closed (by taking $\Sigma = \emptyset$) and the empty set is closed (by taking Σ to be $\{\sigma \wedge \neg\sigma\}$ for some sentence σ). $\mathcal{T}$ is Hausdorff because if T_1 and T_2 are distinct complete L-theories then there is σ such that $\sigma \in T_1$ and $\neg\sigma \in T_2$. σ and $\neg\sigma$ yield disjoint clopen subsets of $\mathcal{T}$. For σ a sentence $X_\sigma =_{def} \{T : \sigma \in T\}$ is clopen, and the X_σ form a basis for $\mathcal{T}$ so $\mathcal{T}$ is totally disconnected. Finally compactness of $\mathcal{T}$ is precisely the compactness theorem: Suppose that $\mathcal{T}$ is covered by open sets U_i so by basic open sets X_{σ_i}, for $i \in I$. This means that every complete theory T contains some σ_i, and thus no complete theory contains $\neg\sigma_i$ for all i. So $\{\neg\sigma_i : i \in I\}$ is inconsistent (has no model) thus by the compactness theorem sme finite subset, say $\{\neg\sigma_1, .., \neg\sigma_n\}$ is inconsistent. But then every complete theory T contains one of the $\sigma_1, .., \sigma_n$ so $X_{\sigma_1}, .., X_{\sigma_n}$ cover $\mathcal{T}$.

Remark 1.21 *Compact (Hausdorff) totally disconnected spaces are precisely the Boolean spaces. The set of clopen subsets of such a space S form (under the natural operations) a Boolean algebra B. The set of ultrafilters on B form a Boolean space homeomorphic to S.*

There are other interpretations of the compactness theorem, including "Deligne's Theorem": A coherent topos has enough points.

I will point out now how the compactness theorem for sets of sentences also holds for sets of formulas. Let L be a language as before. We defined the cardinality of L to be the number of (nonlogical) symbols in L. We built up L-formulas from these symbols together with the logical symbols (including a countably infinite supply of variables). Note that the cardinality of the set of L-formulas is $|L| + \omega$. Let us now allow ourselves as many variables as we want. We then get possibly more L-formulas, but still the same number of L-sentences up to logical equivalence. (Why?) Let us write $\Sigma(x_i)_{i \in I}$ to mean that Σ is a set of L-formulas with free variables among $\{x_i : i \in I\}$. By a model of $\Sigma(x_i)_{i \in I}$ we mean an L-structure M together with a set $\{a_i : i \in I\}$ of elements of M such that for each formula $\phi(x_{i_1}, .., x_{i_n}) \in \Sigma$, $M \models \phi(a_{i_1}, .., a_{i_n})$. We'll say that $\Sigma(x_i)_{i \in I}$ is consistent if it has a model, and finitely consistent if every finite subset has a model.

Remark 1.22 $\Sigma(x_i)_{i \in I}$ *is finitely consistent, if for each finite subset Σ' of Σ, and finite sequence $\bar{x}$ of variables from among the x_i which includes the free varoables occuring in Σ', the L-sentence $\exists \bar{x}(\wedge \Sigma'(\bar{x}))$ is consistent.*

Exercise 1.23 *Let $\Sigma(x_i)_{i \in I}$ be a set of L-formulas. Let $\{c_i : i \in I\}$ be a set of new constant symbols, and let L' be the language got by adjoining these to L. Let $\Sigma(c_i)_{i \in I}$ be the set of L'-sentences obtained by substituting the c_i for the x_i in Σ. Then $\sigma(x_i)_{i \in I}$ is consustent if and only if $\Sigma(c_i)_{i \in I}$ is consistent (as a set of L'-sentences.*

This we get:

Proposition 1.24 *$\Sigma(x_i)_{i\in I}$ is consistent if and only if it is finitely consistent.*

Definition 1.25 *Let T be an L-theory. Let $(x_i : i \in I)$ be a sequence of variables. By an I-type of T we mean a set $\Sigma(x_i)_{i\in I}$ of L-formulas such that $T \cup \Sigma$ is consistent.*

Lemma 1.26 *Suppose $T = Th(M)$. Let $\Sigma(x_i)_{i\in I}$ be a set of L-formulas. Then $\Sigma(x_i)_{i\in I}$ is an I-type of T if and only if for every finite subset $\Sigma'(\bar{x})$ of Σ, $M \models \exists\bar{x}(\wedge\Sigma'(\bar{x}))$.*

Proof. By 1.24, Σ is an I-type of T if and only if $T \cup \Sigma$ is finitely consistent if and only if for each finite subset $\Sigma'(\bar{x})$ of Σ, $T \cup \Sigma'(\bar{x})$ is consistent if and only if for each such Σ', $T \cup \exists\bar{x}(\wedge\Sigma'(\bar{x}))$ is consistent if and only for each such Σ', $M \models \exists\bar{x}(\wedge\Sigma'(\bar{x}))$.

We will come back to types in the next section. We complete this section with the Lowenheim-Skolem theorems and some applications.

Proposition 1.27 *(Downward L-S.) Let M be an L-structure of cardinality κ. Let A be a subset of the universe of M and let λ be a cardinal such that $|L| + \omega + |A| \leq \lambda \leq \kappa$. Then There is an elementary substructure N of M such that A is a subset of the universe of N and N has cardinality λ.*

Proof. Without loss of generality $|A| = \lambda$. Note that the cardinality of the set of L_A-formulas is λ. For each L_A-formula $\phi(x)$ such that $M \models \exists x(\phi(x))$, pick $b_\phi \in M$ such that $M \models \phi(b_\phi)$ and let $A_1 = A \cup \{b_\phi\}_\phi$. Then $|A_1| = \lambda$. Now construct A_2 from A_1 in the same way that A_1 was constructed from A. Continue to get $A \subseteq A_1 \subseteq A_2 \subseteq A_3$....., and let B be the union. Then B is a subset of the universe of M of cardinality λ and for each L_B-formula $\phi(x)$ such that $M \models \exists x\phi(x)$, there is $b \in B$ such that $M \models \phi(b)$. By 1.11, B is the universe of an elementary substructure of M.

Definition 1.28 *Let $L \subset L'$ be say one-sorted languages. Let M' be an L'-structure. By $M'|L$ (the L-reduct of M') we mean the L-structure whose universe is the same as that of M and such that for all relation symbols*

$R \in L$, $R(M'|L) = R(M')$ and for all functions symbols $f \in L$, $f(M'|L) = f(M')$. We call M' an expansion of $M'|L$ to an L'-structure

(If L and L' are many sorted languages and $L \subseteq L'$ then maybe sort sort symbols of L' are not in L. It is then natural to consider the L-reduct of an L'-structure, to be the L-structure M such that $S(M) = S(M')$ for all sort symbols in L, etc.)

Definition 1.29 *Let M be an L-stucture. Let $D_c(M)$, the complete diagram of M be $Th(M, m)_{m \in M}$ in the language L_M.*

Lemma 1.30 *Let M, N be L-structures. Then M can be elementarily embedded in N if and only if N can be expanded to a model of $D_c(M)$.*

Proof. Suppose first that there is an elementary embedding f of M into N. Expand N to an L_M-structure by interpreting c_m as $f(m)$. Check that this expansion N' say is a model of $D_c(M)$.

Conversely, if n can be expanded to a model N' say of $D_c(M)$, let us denote $f(m)$ as the interpretation of c_m in M', and then check that f is an elemetary embedding of M into N.

Proposition 1.31 *(Upward L-S.) Let M be an infinite L-structure. Let $\lambda \geq |M| + |L|$. Then M has an elementary extension of cardinality λ.*

Proof. Let $(c_i : i < \lambda)$ be a sequence of new constants. Let L' be L_M together with these new constant symbols. Let $\Sigma = D_c(M) \cup \{c_i \neq c_j : i < j < \lambda\}$. Σ is a set of L'-sentences. We claim that Σ is finitely consistent. This is immediate as any finite subset Σ' of Σ involves a finite part of $D_c(M)$ and only finitely many of the constant symbols c_i. So $(M, m)_{m \in M}$ together with distinct interpretations of these finitely many c_i will be a model of Σ'. By the compactness theorem, Σ has a model N' say. N' has cardinality at least λ. Let N the be L-reduct of N'. By 1.30 we may assume that N is an elementary extension of M. By 1.27 we may find an elementary substructure of N of cardinality exactly λ which contais the universe of M and so is also an elementary extension of M.

Corollary 1.32 *Let T be an L-theory, and suppose that T has an infinite model. Then for any infinite cardinal $\lambda \geq |L|$, T has a model of cardinality λ.*

Here is another important application of 1.30.

Proposition 1.33 *Suppose that M and N are elementarily equivalent L-structures. Then there is an L-structure M_1 and elementary embedding f of M into M_1 and g of N into M_1. (Also we could choose one of f, g to be the identity map.)*

Proof. There is no harm in assuming the universes of M and N to be disjoint. Let $L' = L_M \cup L_N$. (So the new constants for elements of M are assumed disjoint from the new constants for elements of N. By 1.30 it suffices to prove that $D_c(M) \cup D_c(N)$ is consistent (as a set of L'-sentences). We'll show it to be finitely consistent (and use compactness). Let Σ_1 be a finite subset of $D_c(M)$ and Σ_2 a finite subset of $D_c(M)$. Let $c_{n_1}, .., c_{n_k}$ be the new constants appearing in Σ_2. So we can write Σ_2 as $\Sigma_3(c_{n_1}, .., c_{n_k})$ where $\Sigma_3(x_1, .., x_k)$ is a finite set of L-formulas. Note that $N \models \Sigma_3(n_1, .., n_k)$ and thus $N \models \exists x_1, .., x_k(\wedge\Sigma_3(x_1, .., x_k))$, and so also $M \models \exists x_1...x_k(\wedge\Sigma_3(x_1, .., x_k))$. Let $a_1, .., a_k \in M$ be such that $M \models \Sigma_3(a_1, .., a_k)$. Then clearly, if we interpret c_{n_i} as a_i (and the other c_n as anything you want) we expand M to an L_N-structure which is a model of Σ_2. Of course M can also be expanded (tautologically) to an L_M-structure which is a mdel of Σ_1. So $\Sigma_1 \cup \Sigma_2$ is consistent.

Definition 1.34 *Let T be an L-theory and κ a (possibly finite) cardinal. T is said to be κ-categorical if T has exactly one model of cardinality κ (up to isomorphism).*

Proposition 1.35 *Suppose that T has only infinite models, and is κ-categorical for some cardinal $\kappa \geq |L|$. Then T is complete.*

Proof. Let M, N be models of T. Let $T_1 = Th(M)$ and $T_2 = Th(N)$. By Corollary 1.31, T_1 has a model M_1 of cardinality κ and T_2 has a model N_2 of cardinality κ. Then M_1 and N_1 are models of T of cardinality κ so are isomorphic by assumption. It follows that $T_1 = T_2$ and $M \equiv N$. Thus T is complete.

Example 1.36 *Let L be the empty language and T the theory axiomatized by $\{\sigma_n : n < \omega\}$ where σ_n "says" there are at least n elements. Then T is "totally categorical" (categorical in all infinite cardinalities) and so complete.*

Proof. A model of T is just an infinite set. An isomorphism between models is simply a bijection between their universes. So T is κ-categorical for any infinite cardinal.

Example 1.37 *Let $L = \{+, 0\}$. Let T be the theory of torsion-free divisible nontrivial abelian groups. Then T is categorical in every uncountable cardinality, and complete.*

Proof. I leave it to you to axiomatize T. The models of T are precisely the nontrivial torsion-free divisible abelian groups. Any such model is a $\mathbf{Q}$-vector space of dimension ≥ 1. An isomorphism between two such models is precisely an isomorphism of $\mathbf{Q}$-vector spaces. Any nontrivial $\mathbf{Q}$-vector space V is infinite, and its isomorphism type is determined by its dimension (the cardinality of a $\mathbf{Q}$-basis. If V has cardinality $\lambda > \omega$, then clearly $dim(V) = \kappa$. (Why?) Thus T is κ-categorical for any uncountable κ.

Example 1.38 *Let G be an infinite group of cardinality κ say. Let $L = \{\lambda_g : g \in G\}$ where the λ_g are unary function symbols. Let T be the theory of free G-sets. Then T is categorical in every cardinality $> \kappa$, and complete.*

Proof. First what do we mean by a free G-set? First of all, a G-set is a set X together with mapping from $G \times X$ to X ($(g, x) \to g \cdot x$) such that $1.x = x$ for all $x \in X$ and $(gh) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G$ and $x \in X$. X is said to be a free G-set if it is a G-set and for $g \neq 1$ in G and $x \in X$, $g \cdot x \neq x$. Any G-set X can be viewed naturally as an L-structure, and we can write axioms for free G-sets in the language L. If X is a free G-set then X is the disjoint union of G-orbits where a G-orbit is $G \cdot a = \{g \cdot a : g \in G\}$. Any such G-orbit is in bijection with G via $g \to g \cdot a$ (for a fixed a in the G-orbit). If X, Y are free G-sets with the same number of orbits then X and Y are isomorphic (as G-sets and as L-structures) as follows: Let $(a_i : i \in I)$ and $(b_i : i \in I)$ be representatives of the G-orbits in X, Y respectively. Define a bijection between X and Y by $(g \cdot a_i)$ goes to $g \cdot b_i$ for each $g \in G$ and $i \in I$. Then this is an isomorphism. Now if X is a G-set of cardinality $\lambda > \kappa$ then X has precisely λ-many G-orbits. It follows that T is λ-categorical.

Example 1.39 *Let $L = \{<\}$ and let T be the theory of dense linear orderings without endpoints. Then T ω-categorical and complete.*

Proof. The axioms are obvious. Any model is infinite. A well-known back-and forth argument shows that any two countable models of T are isomorphic so T is ω-categorical, so complete. T has 2^κ models of cardinality κ for every uncountable κ.

Example 1.40 *Let $L = \{R\}$ where R is a binary relation symbol. T says (informally) that R is irreflexive, symmetric and that for any distinct $x_1, .., x_n$, $y_1, ., y_n$ there is z such that $R(z, x_i)$ and $\neg R(z, y_i)$ for all $i = 1, .., n$. Then T is ω-categorical and complete.*

Proof. Again the axiomatization of T is clear. Clearly T has only infinite models. One issue is why T has a model. An important general construction of Fraisse produces a countable model. For now we just assume T to be consistent. Now if M, N are countable models we can show by a back-and-forth argument that M and N are isomorphic. T is ω-categorical, so complete. However again T has 2^κ models of cardinality κ for any uncountable κ. The countable model of T is called the random graph.

2 Types, ω-saturation and quantifier elimination

We have seen in the previous section that categoricity can be used to prove completeness of a theory. However this method does not always work as there are complete theories which are not categorical in any uncountable cardinality (such as what). The method of (relative) quantifier elimination often enables one to prove completeness of particular theories and we'll talk about this now. Moreover quantifier elimination tells one about the definable sets in models of a theory, and that is also very important. We continue our convention of working with one-sorted structures although everything holds in the many-sorted case.

Definition 2.1 *Let M be an L-structure, and $X \subset M^n$. We say that X is $\emptyset$-definable in M if there is an L-formula $\phi(x_1, .., x_n)$ such that $X = \{(a_1, .., a_n) \in M^n : M \models \phi(a_1, .., a_n)\}$ ($= \phi(M)$.) If A is a subset of the universe of M we say that X is A-definable in M if we can find an L_A-formula (or if you want an L_M-formula where only constants for elements of*

A appear) such that $X = \phi(M)$ again. A definable set in M is by definition an L_M-definable set in M.

Understanding a structure M from the model-theoretic point of view involves at least understanding its definable sets.

Definition 2.2 *Let T be a theory in language L. We say that T has quantifier elimination if for any L-formula $\phi(x_1, .., x_n)$ ($n \geq 1$) there is a quantifier-free L-formula $\psi(x_1, .., x_n)$ such that $T \models \forall \bar{x}(\phi(\bar{x}) \leftrightarrow \psi(\bar{x}))$.*

Remark 2.3 *(i) Suppose that T has quantifier-elimination and L contains at least one constant symbol. Then for any L-sentence σ there is a quantifier-free L-sentence τ such that $T \models \sigma \leftrightarrow \tau$.*
(ii) Suppose T has quantifier-elimination and M is a model of T. Then every definable set in M is defined by a quantifier-free L_M-formula. Every $\emptyset$-definable set in M is defined by a quantifier-free L-formula.

Definition 2.4 *Let T be an L-theory. Let $L' = L$ together with some new relation symbols and function symbols. For each new n-ary relation symbol R, let $\phi_R(x_1, .., x_n)$ be an L-formula. For each new $n$ary function symbol f, let $\psi_f(\bar{x}, y)$ be an L-formula such that $T \models (\forall \bar{x})(\exists^{=1} y)(\psi_f(\bar{x}, y))$. Let T' be axiomatized by T together with $\forall \bar{x}(R(x_1, ., x_n) \leftrightarrow \phi_R(x_1, .., x_n))$, and $\forall \bar{x} \forall y(\psi_f(\bar{x}, y) \leftrightarrow f(\bar{x}) = y))$, for all new R and f. We call any T' obtained this way a definitional expansion of T.*

Remark 2.5 *Note that a definitional expansion T' of T is to all intents and purposes the same as T: any model of T ca be expanded to a unique model of T', etc. On the other hand for any theory T we can find a definitional expansion of T which has quantifier-elimination: for each L-formula $\phi(x_1, .., x_n)$ adjoin a new n-ary relation symbol R_ϕ and the axiom $\forall \bar{x}(R_\phi(\bar{x}) \leftrightarrow \phi(\bar{x}))$. This definitional expansion is called the Morleyization of T.*

We will give a useful method for proving quantifier elimination (when it's true) using back-and-forth systems of partial isomorphisms in ω-saturated models.

Let us first recall types. We saw in 1.25 the definition of an I-type of a theory. By an n-type we mean an I-type where $I = \{1, .., n\}$. $\bar{x}$ will usually denote an n-tuple of variables $(x_1, .., x_n)$ for some $1 \leq n < \omega$.

Definition 2.6 *(i) Let $\Sigma(x_i)_{i\in I}$ be a collection of L-formulas and M an L-structure. We say that Σ is realized in M if there is a tuple $(a_i)_{i\in I}$ of elements of M such that $M \models \Sigma(a_i)_{i\in I}$ (and we also say then that $(a_i)_{i\in I}$ realizes Σ in M). We say that Σ is omitted in M if its is not realized in M.*
(ii) By a complete n-type of T we mean an n-type $\Sigma(\bar{x})$ of T such that for each L-formula $\phi(\bar{x})$ either $\phi(\bar{x})$ or $\neg\phi(\bar{x})$ is in Σ. Complete types are often denoted by $p, q, ...$
(iii) Let M be an L-structure and $\bar{a}$ be an n-tuple from M. By $tp_M(\bar{a})$ (the type of $\bar{a}$ in M) we mean $\{\phi(\bar{x}) \in L : M \models \phi(\bar{a})\}$.

Remark 2.7 *(i) By definition, an I-type of T is realized in some model of T.*
(ii) Let $\Sigma(\bar{x})$ be a set of L-formulas. Then Σ is a complete n-type of T if and only if $\Sigma(\bar{x}) = tp_M(\bar{a})$ for some model M of T and n-tuple $\bar{a}$ from M.

Definition 2.8 *The L-structure M is said to be ω-saturated if whenever A is a finite subset of (the universe of) M and $\Sigma(\bar{x})$ is a set of L_A-formulas which is finitely satisfiable in M (that is every finite subset of Σ is realized in M), then Σ is realized in M.*
(ii) For κ an arbitrary infinite cardinal, we define κ-saturated in the same way except that A has cardinality $< \kappa$.

Lemma 2.9 *The following are equuivalent:*
(i) M is ω-saturated.
(ii) For any finite subset A of M and n-type Σ of $Th(M, a)_{a\in A}$, Σ is realized in M.
(iii) For any finite subset A of M and complete n-type $p(\bar{x})$ of $Th(M, a)_{a\in A}$, p is realized in M.

Proof. The lemma follows from a couple of observations. First, a set $\Sigma(\bar{x})$ of L_A-formulas is a n-type of $Th(M, a)_{a\in A}$ if and only if Σ is finitely satisfiable in M (by 1.26). Second, any n-type of a theory extends to a complete m-type of that theory.

Remark 2.10 *(i) If the structure M is understood and A is a subset of M then by a type over A we mean a type of $Th(M, a)_{a\in A}$. To make things more explicit we may say : a type over A in the sense of M. Note that if N is an elementary extension of M then a type over A in the sense of M is the same*

thing as a type over A in the sense of N.
(ii) With the notation of (i), any type over A is realized in an elementary extension of M.

Proof of (ii). If $\Sigma(\bar{x})$ is a type over A then it is easy to see that it is also an n-type of $D_c(M)$, and so is realized in a model of $D_c(M)$ and thus in an elementary extension of M.

A related notion to ω-saturation is ω-homogeneity.

Definition 2.11 *M is ω-homogeneous if for any n, whenever $\bar{a}, \bar{b}$ are n-tuples from M and $tp_M(\bar{a}) = tp_M(\bar{b})$ then for any $c \in M$ there is $d \in M$ such that $tp_M(\bar{a}c) = tp_M(\bar{b}d)$.*

Lemma 2.12 *Any L-structure M has an ω-saturated elementary extension N. Moreover if L and M are at most countable then we can choose N to be of cardinality at most 2^ω.*

Proof. By compactness (a variant of Remark 2.10(ii)) we can find an elementary extension M_1 of M realizing all types over finite subsets of M. Continue with M_1 in place of M. We build an elementary chain $M \subset M_1 \subset M_2...$ and let N be the union, an elementary extension of M. N is ω-saturated, as any finite subset of N is contained in some M_i. Under the additional cardinality hypotheses, note that there are at most continuum many types over finite subsets of M so we can choose M_1 to have cardiality at most 2^ω (by downward Lowenheim-Skolem). Likewise there are at most continuum many types over finite subsets of M_1 etc.

Although it is not easy to recogize when a model of a theory is ω-saturated, we can nevertheless often deduce properties from ω-saturation:

Example 2.13 *Let T be the theory of torsion-free divisible abelian groups (from 1.35). Let $(V, +, 0)$ be an ω-saturated model of T. Then $dim_{\mathbf{Q}}(V)$ is infinite.*

Proof. Without loss of generality we will assume that our language also contains symbols for scalar multiplication by each element of $\mathbf{Q}$. (Why can we assume this?). For each n consider the set $\Sigma_n(x_1, .., x_n)$ of L-formulas: $\{r_1x_1 + ...r_nx_n \neq 0 : r_1, .., r_n \in \mathbf{Q}$ not all zero$\}$. Then Σ_n is finitely satisfiable in V (why?), so realized, by ω-saturation.

Lemma 2.14 *Let M be a structure and $\bar{a}, \bar{b}$ two k-tuples in M such that $tp_M(\bar{a}) = tp_M(\bar{b})$. Let $\Sigma(\bar{x})$ be an n-type over a. Let $\Sigma'(\bar{x}) = \{\phi(x_1, .., x_n, b_1, .., b_k) : \phi(x_1, .., x_n) \in L$ and $\phi(x_1, .., x_n, a_1, .., a_k) \in \Sigma\}$. Then $\Sigma'(\bar{x})$ is an n-type over $\bar{b}$. Moreover Σ' is complete (as an n-type over $\bar{b}$) if $\Sigma(\bar{x})$ is complete as an n-type over $\bar{a}$.*

Proof. We have to show that Σ' is finitely satisfiable in M. Choose a finite subset, without loss of generality a single formula $\phi(x_1, .., x_n, b_1, .., b_k)$. So $\phi(x_1, .., x_n, a_1, .., a_k) \in \Sigma$ so $M \models \exists x_1, .., x_n \phi(x_1, .., x_n, a_1, .., a_k)$. Let $\psi(y_1, .., y_k)$ be the L-formula $\exists x_1, ..., x_n \phi(x_1, ., x_n, y_1, .., y_k)$. So $\psi(y_1, .., y_k) \in tp_M(\bar{a})$ so also in $tp_M(\bar{b})$. Thus $M \models \psi(b_1, .., b_n)$.

Remark 2.15 *Here are some useful bits of notation.*
(i)In the above lemma let us write Σ as $\Sigma_{\bar{a}}$ to emphasize the dependence on $\bar{a}$. Then we write Σ' as $\Sigma_{\bar{b}}$. (So Σ' is just the result of replacing the parameters for $\bar{a}$ by $\bar{b}$.)
(ii) Let A be a subset of M and $\bar{b}$ a tuple from M. By $tp_M(\bar{b}/A)$ we mean the type of $\bar{b}$ in the structure $(M, a)_{a \in A}$.

Lemma 2.16 *Any ω-saturated structure is ω-homogeneous.*

Proof. Let M be ω-saturated. Let $\bar{a}, \bar{b}$ be tuples from M with the same (complete) type in M. Let $c \in M$. Let $p_a(x) = tp_M(c/a)$. By Lemma 2.14, $p_b(x)$ is a complete type (over b in M). By ω-saturation of M we can realize $p_b(x)$ by some d.
Claim. $tp_M(\bar{a}, c) = tp_M(\bar{b}, d)$.
Proof. Let $\phi(x, \bar{y})$ be an L-formula. Then $M \models \phi(c, \bar{a})$ iff $\phi(x, \bar{a}) \in p_a(x)$ iff $\phi(x, \bar{b}) \in p_b(x)$ iff $M \models \phi(d, \bar{b})$.

Corollary 2.17 *Any structure has an ω-homogeneous elementary extension.*

We can do better if L is countable.

Proposition 2.18 *Let L be countable and M a countable L-structure. Then M has a countable ω-homogeneous elementary extension.*

Proof. Let $\mathcal{P}$ be the set of all types of the form $p_{\bar{b}}(x)$ where $\bar{a}$ and $\bar{b}$ are finite tuples from M with the same type in M and where $p_{\bar{a}}(x) = tp_M(c/\bar{a})$ for some $c \in M$. Note that $\mathcal{P}$ is a countable set of types, so there is a countable elementary extension M_1 of M which realizes all types in $\mathcal{P}$. Now do the same thing, with M_1 in place of M. Continue to get a countable elementary chain M_i for $i < \omega$. If N is the union, then N is ω-homogenous by construction.

Lemma 2.18 can be used to relate types and automorphisms.

Lemma 2.19 *Let M be a countable ω-homogeneous structure. Let $\bar{a}, \bar{b}$ be finite tuples from M with the same type in M. Then there is an automorphism f of M such that $f(\bar{a}) = \bar{b}$.*

Proof. (Back-and-forth.) List the elements of M as $(e_i : i < \omega)$. Let us define c_i and d_i from M by induction, such that $tp_M(\bar{a}, c_1, .., c_n) = tp_M(\bar{b}, d_1, .., d_n)$ for all n. At stage $2n$, let c_{2n} be e_n, and let d_{2n} be such that $tp_M(\bar{a}, c_0, .., c_{2n-1}, c_{2n}) = tp_M(\bar{b}, d_0, .., d_{2n-1}, d_{2n})$ (by induction hypothesis and homogeneity of M). At stage $2n+1$ let $d_{2n+1} = e_n$ and let c_{2n+1} be such that $tp_M(\bar{a}, c_0, .., c_{2n}, c_{2n+1}) = tp_M(\bar{b}, d_0, .., d_{2n}, d_{2n+1})$. Then the map f which takes c_n to d_n is by construction well-defined (why?), a permutation of the universe of M (why?) and as for each formula $\phi(x_1, .., x_m)$ of L, $M \models \phi(c_1, .., c_m)$ iff $M \models \phi(d_1, .., d_m)$, f is an automorphism of M. Finally $f(\bar{a}) = \bar{b}$ (why?)

Corollary 2.20 *Let M be an L-structure, and $\bar{a}, \bar{b}$ n-tuples from M. Then $tp_M(\bar{a}) = tp_M(\bar{b})$ iff there is an elementary extension N of M and an automorphism f of N such that $f(\bar{a}) = \bar{b}$.*

Proof. Right to left is immediate (???).
Left to right: Add a new function symbol f. Note that we can express "f is an L-automorphism" by set of sentences in $L \cup \{f\}$. Let $L' = L_M \cup \{f\}$, and let Σ be $D_c(M) \cup$ "f is an L-automorphism" $\cup \{f(\bar{a}) = \bar{b}\}$. It is enough to show that Σ is finitely consistent. Let Σ_0 be a finite subset of Σ. Σ_0 involves only symbols from a finite sublanguage L_0 of L and finitely many elements of M. Let M_0 be a countable elementary substructure of $M|L_0$ containing those elements as well as $\bar{a}$ and $\bar{b}$. Let N_0 be a countable ω-homogeneous elementary extension of M_0. Then by Lemma 2.19 there is an automorphism of N_0 taking $\bar{a}$ to $\bar{b}$. So Σ_0 has a model.

Definition 2.21 *(i) By a quantifier-free n-type of T we mean a set $\Sigma(x_1, .., x_n)$ of quantifier-free L-formulas such that $T \cup \Sigma$ is consistent.*
(ii) A complete quantifier-free n-type of T is a quantifier-free n-type Σ of T

such that for each quantifier-free formula $\phi(x_1,..,x_n)$, ϕ or $\neg\phi$ is in Σ.
(iii) Let M be an L-structure and $\bar{a}$ an n-tuple from M. Then by $qftp_M(\bar{a})$ (the quantifier-free type of $\bar{a}$ in M) we mean $\{\phi(x_1,..,x_n) : \phi \in L$ quantifier-free and $M \models \phi(\bar{a})$.

Remark 2.22 *(i) $\Sigma(\bar{x})$ is a complete quantifier-free type of T if and only if $\Sigma = qftp_M(\bar{a})$ for some model M of T and tuple $\bar{a}$ from M.*
(ii) The quantifier-free type of $\bar{a}$ in M is precisely the set of quantifier-free formulas in $tp_M(\bar{a})$.

Before getting into the details of quantifier elimination, let us make some remarks on the topology on type spaces.

Remark 2.23 *(i) Let T be a theory and let $S_n(T)$ be the set of complete n-types of T. Put a topology on $S_n(T)$ be calling a subset X closed if there is a set $\Sigma(\bar{x})$ of L-formulas ($\bar{x} = (x_1,..,x_n)$) such that $X = \{p(\bar{x} \in S_n(T) : \Sigma \subseteq p\}$. Write $X = X_\Sigma$. Note that this IS a topology, and under it $S_n(T)$ becomes a compact Hausdorff space with a basis of clopens (namely X_ϕ) for $\phi(x_1,..,x_n)$ an L-formula. (Exercise.)*
(ii) We can do exactly the same thing, but now with quantifier-free types. Let $S_n^{qf}(T)$ be the set of complete quantifier-free n-types of T. Again give it a topology where the closed sets are of the form X_Σ where Σ is a set of quantifier-free L-formulas. Again $S_n(T)$ becomes a compact Hausdorff, totally disconnected space.
(iii) In the context of (i), let $\phi(\bar{x})$ and $\psi(\bar{x})$ be L-formulas. Then $T \models \forall\bar{x}(\phi(\bar{x}) \leftrightarrow \psi(\bar{x})$ if and only if $X_\phi = X_\psi$. Likewise in the context of (ii), when ϕ and ψ are quantifier-free formulas.

Definition 2.24 *We will say that T has property (Q) if for any two models M, N of T, $n < \omega$ and n-tuples $\bar{a}$ from M, $\bar{b}$ from N, IF $qftp_M(\bar{a}) = qftp_N(\bar{b})$ THEN $tp_M(\bar{a}) = tp_N(\bar{b})$.*

Proposition 2.25 *T has quantifier-elimination if and only if T has property (Q).*

Proof. Left to right is obvious.
We will give two proofs of the right to left direction. (In fact they will be the same proof.)

PROOF A. Assume that T has (Q). Fix $n \geq 1$. Define $f : S_n(T) \to S_N^{qf}(T)$, by $f(p)$ = the set of quantifier-free formulas in p. Note that $f(p)$ IS a complete quantifier-free n-type of T, and that f is surjective and continuous. (For continuity, if $\phi(\bar{x})$ is a quantifier-free formula of T, then $\{p \in S_n(T) : \phi \in f(p)\}$ is clearly $\{p \in S_n(T) : \phi \in p\}$.) Now the assumption that T has property (Q) means precisely that f is also injective. So f is a continuous bijection between compact Hausdorff spaces. It is well-known (exercise) that f must therefore be a homeomorphism. Let $\phi(\bar{x})$ be an L-formula. So the image of X_ϕ under f mut also be clopen hence of the form X_ψ for ψ quantifier-free. But it then follows that (in the space $S_n(T)$) $X_\phi = X_\psi$ whereby $\phi(\bar{x})$ and $\psi(\bar{x})$ are equivalent modulo T.

PROOF B. Assume T has property (Q). Let $\phi(\bar{x})$ be an arbitrary L-formula with free variables $\bar{x} = (x_1, .., x_n)$. We seek a quantifier-free formula equivalent to ϕ modulo T.
Let $\Sigma(\bar{x})$ be the set of quantifier-free formulas $\psi(\bar{x})$ such that $T \models \forall \bar{x}(\phi(\bar{x}) \to \psi(\bar{x}))$.
Claim. $T \cup \Sigma(\bar{x}) \models \phi(\bar{x})$. (That is, for any model M of T and realization $\bar{a}$ of Σ in M, also $M \models \phi(\bar{a})$.)
Proof of claim. Suppose not. So there is a model M of T and $\bar{a}$ in M such that $M \models \Sigma(\bar{a})$ and $M \models \neg\phi(\bar{a})$. Let $\Gamma(\bar{x}) = qftp_M(\bar{a})$. Then $T \cup \{\Gamma(\bar{x})\} \cup \{\phi(\bar{x})\}$ is consistent. (Otherwise by compactness $T \cup \{\phi(\bar{x})\} \models \neg\gamma(\bar{x})$ for some $\gamma \in \Gamma$ whereby $\neg\gamma \in \Sigma$, a contradiction.)
So there is a model N of T and $\bar{b}$ from N such that $qftp_N(\bar{b}) = \Gamma(= qftp_M(\bar{a}))$ and $N \models \phi(\bar{b})$. This contradicts (Q) and proves the claim.

By the claim and compactness (and the closure of Σ under conjunctions) there is $\psi(\bar{x}) \in \Sigma$ such that $T \models \forall \bar{x}(\psi(\bar{x}) \to \phi(\bar{x}))$. So ψ works.

Definition 2.26 *Let M and N be L-structures.*
By a finite partial isomorphism between M and N we mean a pair $(\bar{a}, \bar{b})$ of finite (nonempty) tuples $\bar{a}$ from M and $\bar{b}$ from N. Such that $qftp_M(\bar{a}) = qftp_N(\bar{b})$.

Remark 2.27 *A finite partial isomorphism between M and N is the same thing as an isomorphism (of L-structures) between a finitely generated substructure of M and a finitely generated substructure of N.*

Explanation. Suppose first that $(\bar{a}, \bar{b})$ is as in the definition above. For each L-term $\tau(\bar{x})$, define $f(\tau(\bar{a})) = \tau(\bar{b})$. Then f is an isomorphism between $< \bar{a} >$

and $< \bar{b} >$ (where $< - >$ denotes substructure generated by). On the other hand if f is an isomorphism between substructures A of M and B of N and A is generated by the finite tuple $\bar{a}$, and $\bar{b} = f(\bar{a})$ then $qftp_M(\bar{a}) = qftp_N(\bar{b})$. So we often denote a finite partial isomorphism by f.

Definition 2.28 *Let M and N be L-structures and let I be a set of partial isomorphisms between M and N. We say that I has the back-and-forth property if whenever $(\bar{a}, \bar{b}) \in I$ and $c \in M$ then there is $d \in N$ such that $(\bar{a}c, \bar{b}d) \in I$, and dually.*

Proposition 2.29 *The following are equivalent:*
(i) T has quantifier-elimination,
(ii) whenever M and N are ω-saturated models of T, and I is the set of all finite partial isomorphisms between M and N, then I has the back-and-forth property.

Proof. (i) implies (ii). Assume T has QE. So T has property (Q). Let M, N be ω-saturated models of T, and I the set of all finite partial isomorphisms beteween M and N. So $(\bar{a}, \bar{b}) \in I$ iff $tp_M(\bar{a}) = tp_N(\bar{b})$. Let $tp_M(\bar{a}) = tp_N(\bar{b})$ and $c \in M$. Let $p_{\bar{a}}(x) = tp_M(c/\bar{a})$. Then as in 2.14, $p_{\bar{b}}(x)$ is a complete type over $\bar{b}$ in the sense of N, so by ω-saturation, it is realized in N by d say. Then $tp_M(\bar{a}c) = tp_N(\bar{b}d)$.
(ii) implies (i). We assume (ii) and prove that T has property (Q).
Let M, N be models of T, $\bar{a} \in M$ and $\bar{b} \in$, and $qftp_M(\bar{a}) = qftp_N(\bar{b})$. Let M', N' be ω-saturated elementary extensions of M, N respectively (by 2.12). Let I be the set of finite partial isomorphisms between M' and N'. Note that $(\bar{a}, \bar{b}) \in I$. So to complete the proof it is enough to prove (changing notation):
Claim. For any $(\bar{a}, \bar{b}) \in I$, $tp_{M'}(\bar{a}) = tp_{N'}(\bar{b})$.
Proof of claim. We prove by induction o the complexity of $\phi(x_1, .., x_n)$ (n varying) that for any pair $\bar{a}, \bar{b}) \in I$ of n-tuples, $M \models \phi(\bar{a})$ iff $N \models \phi(\bar{b})$. This is immediate for quantifier-free formulas, and the inductive step where we use $\wedge, \vee, \neg$ or $\rightarrow$ is clear. So suppose now tht ϕ has the form $\exists y(\psi(x_1, .., x_n, y))$. Suppose $(\bar{a}, \bar{b}) \in I$, and $M \models \phi(\bar{a})$. So there is $c \in M$ such that $M \models \psi(\bar{a}, c)$. As I has the back-and-forth property , there is $d \in N$ such that $(\bar{a}c, \bar{b}d) \in I$. By inductive assumption, $N \models \psi(\bar{b}, d)$, and thus $N \models \phi(\bar{b})$. (The dual case is the same).
The claim is proved. We have shown that T has (Q) and so QE (by 2.25).

Proposition 2.30 *Suppose that for any two ω-saturated models M, N of T, the set of finite partial isomorphisms between M and N has the back-and-forth property AND is nonempty. Then T is complete.*

Proof. Let M, N be arbitrary models of T. Let M', N' be ω-saturated elementary extensions of M, N respectively. By assuption, let $(\bar{a}, \bar{b}) \in I$ (the set of finite partial isomorphisms between M' and N'). By 2.29, T has quantifier-elimination, and thus $tp_M(\bar{a}) = tp_N(\bar{b})$. In particular M' and N' are elementarily equivalent. So M and N are elementarily equivalent. Thus T is complete.

Before passing to examples and applications of quantifier-elimination, it will be convenient to discuss notions around model-completeness and existentially closed structures.

Definition 2.31 *(i) A theory T will be called universal if T can be axiomatized by universal sentences.*
(ii) T will be called $\forall\exists$ if T can be axiomatized by $\forall\exists$ sentences.
(iii) $T_\forall$ is the set of universal consequences of T.
(iv) Let M be an L-structure. By $D(M)$ (the diagram of M) we mean the set of quantifier-free sentences of L_M true in M (namely the quantifier-free part of the complete diagram $D_c(M)$ of M).

Exercise 2.32 *(i) Let M and N be an L-structures. Then M can be embedded in N if and only if N can be expanded to a model of $D(M)$.*
(ii) The models of $T_\forall$ are precisely the substructures of models of T.
(iii) T is universal iff any substructure of a model of T is a model of T.
(iv) T is $\forall\exists$ iff the class of models of T is closed under unions of chains.

Definition 2.33 *Let T be a universal theory. A model M of T is said to be existentially closed (in the class of models of T) if whenever $M \subseteq N$ is a model of T and σ is an existential L_M-formula such that $N \models \sigma$, then $M \models \sigma$.*

Lemma 2.34 *Let T be universal.*
(i) For any model M of T there is an existentially closed model N of T such that $M \subseteq N$ and N has cardinality at most $|M| + |L| + \omega$.
(ii) Suppose T' is an $\forall\exists$ theory and $T = T'_\forall$. Then any existentially closed

model of T is a model of T'.
(iii) Let M_1, M_2 be models of T. Let $\bar{a}, \bar{b}$ be finite tuples from M_1, M_2 respectively, and suppose that $etp_{M_1}(\bar{a}) \subseteq etp_{M_2}(\bar{b})$ (where $etp(-)$ denotes the existential type). Then there is a model N of T and embeddings $f : M_1 \to N$ and $g : M_2 \to N$ such that $f(a) = g(b)$.
(iv) Let

Proof. (i) First by a union of chains argument, produce a model M^* of T (of the right cardinality) containing M such that whenever σ is an existential L_M-sentence true in some extension of M^* to a model of T, then σ is true in M^*. Now construct M^{**}, carry on ω times and take the union.
(ii) As M is a model of $T'_\forall$ there is by Exercise 2.32, an extension N of M which is a model of T'. Let σ be an $\forall\exists$ sentence of L which is true in N. σ has the form $\forall\bar{x}\phi(\bar{x})$ where ϕ is existential. Pick $\bar{a}$ from M. Then $N \models \phi(\bar{a})$ so $M \models \phi(\bar{a})$.
(iii) Consider $L_{M_1} \cup L_{M_2}$ where we only identify $\bar{a}$ and $\bar{b}$ (by constants $\bar{c}$ say). In this language consider $\Sigma = T \cup D(M_1) \cup D(M_2)$. We claim that Σ is consistent. Let $\phi(\bar{c}, \bar{m}_1) \in D(M_1)$ and $\psi(\bar{c}, \bar{m}_2) \in D(M_2)$. Then $\exists\bar{y}\phi(\bar{x}, \bar{y}) \in etp_{M_1}(\bar{a})$ so is also in $etp_{M_2}(\bar{b})$. So we can find $\bar{m}' \in M_2$ such that $M_2 \models \phi(\bar{b}, \bar{m}') \wedge \psi(\bar{b}, \bar{m}_2)$. So Σ is consistent. Let N_0 be a model of Σ, and let N be the L-reduct of N_0. Then we have embeddings f of M_1 into N and g of M_2 into N, such that $f(\bar{a}) = g(\bar{b})$.

Corollary 2.35 *Suppose T is universal. If M is an ec model of the universal theory T and $\bar{a}$ is an n-tuple from M, then $etp_M(\bar{a})$ is maximal among existential types realized in models of T. That is, whenever $N \models T$, $\bar{b}$ is an n-tuple from N and $etp_M(\bar{a}) \subseteq etp_N(\bar{b})$, then $etp_M)\bar{a}) = etp_N(\bar{b})$.*

Proof. By Lemma 2.34 we may assume that both M and N are substructures of some model N' of T and that $\bar{a} = \bar{b}$. But then $etp_N(\bar{a}) \subseteq etp_{N'}(\bar{a}) \subseteq etp_M(\bar{a})$ where the last inclusion is because of M being existentially closed.

Corollary 2.36 *Let T be universal. Let $\phi(\bar{x})$ be an existential L-formula. Then there is a set $\Psi(\bar{x})$ of universal L-formulas, such that for any exitentially closed model M of T, $M \models (\forall\bar{x})(\phi(\bar{x}) \leftrightarrow \wedge\Psi(\bar{x}))$*

Proof. Let $\mathcal{P}(\bar{x}) = \{etp_M(\bar{a}) : M$ an ec model of T, $\bar{a}$ a tuple from M, and $M \models \neg\phi(\bar{a})\}$. By the above remark, for each $p(\bar{x}) \in \mathcal{P}$, $T \cup p(\bar{x}) \cup \{\phi(\bar{x})\}$

is inconsistent, hence there is $\psi_p(\bar{x}) \in p$ such that $T \cup \{\psi_p(\bar{x})\} \cup \{\phi(\bar{x})\}$ is inconsistent. Let Ψ be the set of negations of the ψ_p's.

Definition 2.37 *Let T be an (arbitrary) theory. T is said to be model-complete if whenever $M \subseteq N$ are models of T then M is an elementary substructure of N.*

Remark 2.38 *(i) If T has QE then T is model-complete. (The converse fails.)*
(ii) T is model-complete if and only if for any model M of T, $T \cup D(M)$ is complete (as an L_M-theory).
(iii) If T is model-complete then the class of models of T is closed under unions of chains, hence T is $\forall\exists$.

Proposition 2.39 *Let T be any theory. The following are equivalent:*
(i) T is model-complete.
(ii) The models of T are precisely the ec models of $T_\forall$.
(iii) Every model of T is an ec model of $T_\forall$.
(iv) for every L-formula $\phi(\bar{x})$ there is an existential L-formula $\psi(\bar{x})$ such that $T \models \forall\bar{x}(\phi(\bar{x}) \leftrightarrow \psi(\bar{x}))$.

Proof. (i) implies (ii). Assume T is model-complete. So T is $\forall\exists$. If M is an ec model of $T_\forall$ then by Lemma 2.34 (ii), M is a model of T. Conversely, let M be a model of T. Let $M \subseteq N$ where N is a model of $T_\forall$. Let σ be an existential sentence of L_M true in N. Let N' be an model of T extending N. Clearly $N' \models \sigma$ too. By model-completeness of T, M is an elementary substructure of N' so $M \models \sigma$. Thus M is an ec model of $T_\forall$.
(ii) implies (iii) is trivial.
(iii) implies (iv). By Corollary 2.36 and compactness every existential L-formula is equivalent modulo T to a universal formula. It follows that every L-formula is equivalent modulo T to both a universal formula and an existential formula.
(iv) implies (i). Let $M \subseteq N$ be models of T, $\phi(\bar{x})$ an L-formula, and $\bar{a}$ a tuple from M. Let $\psi_1(\bar{x})$ be an existential formula equivalent to ϕ mod T and ψ_2 a universal formula equivalent to ϕ mod T. Then $M \models \phi(\bar{a})$ implies $M \models \psi_1(\bar{a})$ implies $N \models \psi_1(\bar{a})$ implies $N \models \phi(\bar{a})$. Conversely, $N \models \phi(\bar{a})$ implies $N \models \psi_2(\bar{a})$ implies $M \models \psi_2(\bar{a})$ implies $M \models \phi(\bar{a})$.

Here is a useful criterion for a theory to be model-complete.

Proposition 2.40 *The following are equivalent:*
(i) T is model-complete.
(ii) Whenever $M \subseteq N$ are models of T, then there is an elementary extension N' of M and an embedding $f : N \to N'$ such that $f|M$ is the identity.

Proof. (i) implies (ii) is immediate because N is already an elementary extension of M.
(ii) implies (i). Assume (ii). We will show that any model M of T is an existentially closed model of $T_\forall$, and then we can apply Proposition 2.39. So let M be a model of T. Let $M \subseteq N$ where N is a model of $T_\forall$. So there is $N \subseteq N_1$ where N_1 is a model of T. By (ii) there is an embedding f over M of N_1 into an elementary extension N' of M. In particular $f|N$ is an embedding over M of N into N'. Without loss of generality N is already a substructure of N'. Let σ be an existential L_M-sentence such that $N \models \sigma$. Then clearly $N' \models \sigma$. As N' is an elementary extension of M, $M \models \sigma$.

Note also

Lemma 2.41 *Suppose that T is model-complete and that there is a model M_0 of T which is embeddable in every model of T. Then T is complete.*

Proof. Let M be a model of T. So we may assume that M_0 is a substructure of M, whereby M_0 is an elementary substructure of M and $Th(M) = Th(M_0)$.

Finally, for cultural reasons, we tie up the above material with the notion of "model-companion".

Definition 2.42 *Let T be an arbitrary theory in a language L. We say that T' is a model-companion of T if $T'_\forall = T_\forall$ and T' is model-complete.*

Remark 2.43 *To say that $T'_\forall = T_\forall$ means precisely that every model of T is a substructure of a model of T' and every model of T' is a substructure of a model of T. Note that T' is a model-companion of T iff T' is a model companion of $T_\forall$.*

Proposition 2.44 *(T arbitrary.)*
(i) T has a model-companion if and only if the class of ec models of $T_\forall$ is an elementary class.
(ii) The model companion of T, if it exists, is unique and equals the theory of ec models of $T_\forall$.

Proof. (i) Suppose T has a model companion T'. By 2.39, the models of T' are precisely the ec models of $T'_\forall = T_\forall$. Conversely suppose the class of ec models of $T_\forall$ is elementary, and let T' be the theory of this class. Note then that $T'_\forall = T_\forall$, so again by 2.39 T' is model complete, so is a model companion of T.
(ii) is contained in the first part of the proof of (i).

Definition 2.45 *Let $\mathcal{K}$ be a class of L-structures. We say that $\mathcal{K}$ has the amalgamation property, if whenever $M_0 \subseteq M_1$ and $M_0 \subseteq M_2$ are all in $\mathcal{K}$ then there is $N \in \mathcal{K}$ and embeddings $f_i : M_i \to N$ for $i = 1, 2$ such that f_1 and f_2 agree on M_0. A theory T is said to have the amalgamation property if its class of models does.*

Proposition 2.46 *Suppose that T is a universal theory with the amalgamation property. Suppose T' is a model-companion for T'. Then T' has QE.*

Proof. By 2.44, the class of ec models of T is elementary and T' is its theory.
Claim. Suppose M_1, M_2 are models of T', and $\bar{a}, \bar{b}$ are tuples in M_1, M_2 respectively with the same quantifier-free types. Then $tp_{M_1}(\bar{a}) = tp_{M_2}(\bar{b})$.
Proof of claim. Let M_0 be the substructure of M_1 generated by $\bar{a}$ and M'_0 the substructure of M_1 generated by $\bar{b}$. Then the "map" taking a_i to b_i yields an isomorphisms between M_0 and M'_0. So there is no harm in assuming that $M_0 = M'_0$ (so $\bar{a} = \bar{b}$). As all the M_i are models of $T'_\forall = T_\forall$, and $T_\forall$ has the amalgamation property, we may assume that M_1 and M_2 are common substructures of a model N of $T'_\forall$ and thus (by extending N), common substructures of a model N' of T'. But then M_i is an elementary substructure of N' for $i = 1, 2$ whereby $tp_{M_1}(\bar{a}) = tp_{N'}(\bar{a}) = tp_{M_2}(\bar{a})$, proving the claim.

Quantifier-elimination for T' now follows from Proposition 2.29.

3 Examples and applications of quantifier elimination.

Example 3.1 *Let T be the theory of infinite sets in the empty language. Then T has quantifier-elimination (and is complete). T is the model companion of the empty theory.*

Proof. Let M, N be models of T. Let $(\bar{a}, \bar{b})$ be a finite partial isomorphism between M and N. So $a_i = a_j$ iff $b_i = b_j$. Let $c \in M$. then as N is infinite there is $d \in N$ such that for each i, $a_i = c$ iff $b_i = d$. So $(\bar{a}c, \bar{b}d)$ is a finite partial isomorphism between M and N. Similarly given $d \in N$ we can find $c \in M$ etc. By 2.29 T has quantifier elimination. We already know T to be complete but we could also use 2.30 to prove this.

Example 3.2 *Let T be the theory of dense linear orderings without first or last element (in the language $L = \{<\}$). Then T has quantifier-elimination and is complete. T is the model companion of the theory of linear orderings.*

Proof. Again the set of finite partial isomorphisms between any two models of T is nonempty and has the back-and-forth property. Note that $T_\forall$ is precisely thee theory of linear orderings.

Example 3.3 *Let T be the theory of discrete linear orderings without first or last element, in the language $L = \{<\}$. Then T is complete but not model-complete.*

Proof. We leave it as an exercise to show that T is not model complete.

For each $n < \omega$ let S_n and T_n be new unary function symbols. Let L' be L together with these new symbols. Let T' be the definitional expansion of T obtained by defining $S_n = y$ iff y is the "nth successor" of x and $T_n = y$ if y is the "nth predecessor of x". A back-and-forth argument in ω-saturated models of T' shows that T' has quantifier-elimination and is complete (using 2.29 and 2.30). In particular T is complete.

Remark 3.4 *Let T and T' be as in Example 3.3. Notice that each term $\tau(x)$ of T' is equivalent mod T' to some S_n or T_n. Now each $_n$ and T_i is defined by an L-formula which is a conjunction of an existential formula and a universal formla. As T' has QE, each L-formula is equivalent mod T to a Boolean combination of existential L-formulas (vut NOT to a single existential L-formula, as T is not model-complete).*

Example 3.5 *Let T be the theory of the random graph from Example 1.40. Then T has quantifier-elimination. T is the model companion of the theory of irreflexive, symmetric graphs.*

Proof. Quantifier elimination is obtained by back-and-forth. For the rest we must show that every irreflexive, symmetric graph embeds in a model of T. Left to you.

Example 3.6 *Let $L = \{+, -, \cdot, 0, 1\}$ be the language of rings. Let ACF be the theory of algebraically closed fields. For p a prime or zero, let ACF_p be the theory of algebraically closed fields of characteristic p. Then ACF has quantifier-elimination, and each ACF_p is complete. ACF is the model companion of the theory of fields.*

Proof and explanation. Let us start with some algebra. Let K be any field. Then there is a canonical homorphism ϕ from the ring $\mathbf{Z}$ into K: $\phi(n) = 1 + 1 + .. + 1$ (n-times). $Im(\phi)$ is an integral domain. So $ker(\phi)$ is either $\{0\}$ or $p\mathbf{Z}$ for some prime p. In the first case ϕ is an embedding and extends to an embedding of the field $\mathbf{Q}$ into K. In this case we say that K has characteristic 0, and call the canonical copy of $\mathbf{Q}$ the prime field. In the second case ϕ induces an isomorphism between $\mathbf{Z}/p\mathbf{Z} = \mathbf{F}_p$ (the field with p elements) and a subfield of K. We say that K has characteristic p, and again call $Im(\phi)$ the prime field. We can express having characteristic p by a single L-sentence and having characteristic 0 by infinitely many L-sentence.

The field K is said to be algebraically closed if any monic polynomial equation $x^n + a_{n-1}x^{n-1} + ...a_1x + a_0 = 0$ with $n \geq 1$ and $a_i \in K$ has a solution in K.

This can be expressed by infinitely many L-sentences (one for each n). So ACF as well as ACF_0 and ACF_p are L-theories. (For consistency we use the fact that any field has an algebraic closure.) It is worth remarking that any algebraically closed field is infinite.

We will prove that ACF has QE by using 2.29. Note that if K is a field then a substructure of K is precisely a subring. If K, L are fields and $f : R \cong S$ is an isomorphism of subrings, then f extends uniquely to an isomorphism $f' : Fracr(R) \cong Frac(S)$ between the fraction or quotient fields of R and S. So we may consider a finite partial isomorphisms between models of ACF to be an isomorphism of finitely generated subfields.

Let K_1 and K_2 be ω-saturated models of ACF. Suppose $f : k_1 \cong k_2$ is an isomorphism of finitely generated subfields of K_1 and K_2. Let $a \in K_1$ we must extend f to the field $k_1(a)$. If $a \in k_1$ already there is nothing to do.
Case 1. a satisfies some (nonzero) polynomial equation over k_1.
Let $h(x) = x^n + b_{n-1}x^{n-1} + .. + b_0$ be the minimal polynomial of a over k_1,

namely the monic polynomial in $k_1[x]$ of smallest degree such that $h(a) = 0$. It is well-known that h is irreducible in $k_1[x]$ and so the ideal $(h(x))$ it generates is prime. Let $h'(x) = f(h(x))$. (That is h' is the result of applying f to the coefficients of h.) Then $h'(x)$ is irreducible in $k_2[x]$. As K_2 is algebraically closed $h'(x) = 0$ has a solution say b in K_2. Extend f to f' on $k_1[a]$ by putting $f'(a) = b$. Then f' is an isomorphism between $k_1[a]$ and $k_2[b]$. $(k_1[x]/(h(x))$ is isomorphic to $k_1[a]$ by taking x to a. Likewise for $k_2[x]/(h'(x))$ and $k_2[b]$. But also $k_1[x]/(h(x))$ and $k_2[x]/(h'(x))$ are isomorphic via f.) In fact $k_1[a]$ is alreasy a subfield of K_1.
Case 2. a satisfies no nonzero polynomial equation over k_1. That is, a is transcendental over k_1.
It suffices to find $b \in K_2$ which is transcendental over k_2. Note that k_2 is $Frac(R)$ for some finitely generated ring R. Any nonzero polynomial $P(x)$ over k_2 can be rewritten as a nonzero polynomial over R, and thus as a polynomial whose coefficients are L-terms in the finitely many generators of R. So $\Sigma(x) = \{h(x) \neq 0 : h$ a nonzero polynomial over $R\}$, can be considered as a set of formulas with parameters from a fixed finite set of generators of R. But K_2 is infinite, and each $h(x)$ has at most $deg(h)$ many zeros in K_2. Thus $\Sigma(x)$ is finitely satisfiable in K_2, so by ω-saturation is realized in K_2 by some b. So b is transcendental over k_2. So f extends to f' on $k_1[a]$ by putting $f'(a) = b$. $(k_1[a] \cong k_1[x] \cong k_2[x] \cong k_2[b]$.)

By 2.29 this shows that ACF has quantifier-elimination. Fix p to be a prime or 0. Let K_1, K_2 be models of ACF_p. Then the prime fields are isomorphic, so there is at least one finite partial isomorphism between K_1 and K_2. By 4.30 ACF_p is complete.

As ACF has QE it is model-complete. Note that any field embeds in an algebraically closed field. Thus ACF is the model companion of the theory of fields.

Corollary 3.7 *(Hilbert's Nullstellensatz.) Let K be an algebraically closed field. Let $\bar{P}(\bar{x}) = 0$ be a finite system of polynomial equations in $(x_1, .., x_n)$ with coefficients in K. Suppose that this system has a common solution in some field extending K. Then it has a common solution in K.*

Proof. Suppose L is a field extending K which contains a solution of $\bar{P}(\bar{x}) = 0$. We may assume L to be algebraically closed too (why?). The L_K-sentence $\exists\bar{x}(\bar{P}(\bar{x}) = 0)$ is true in L so true in K. (K is an elementary substructure of L, by model-completeness of ACF.)

Corollary 3.8 *(Lefschetz principle.) Let σ be a sentence in the language of rings. Then the following are equivalent:*
(i) σ is true in the complex field.
(ii) σ is true in some algebraically closed field of characteristic 0,
(iii) σ is true in all algebraically closed fields of characteristic 0.
(iv) For infinitely many primes p, σ is true in some (all) algebraically closed fields of characteristic p.
(v) For all but finitely many primes p, σ is true in some (all) algebraically closed fields of characteristic p.

Proof. This is often seen as a consequence of the completeness of ACF_0 and the ACF_p's, but it can be better seen as coming from quantifier-elimination of ACF. So as ACF has QE (and L contains some constant symbols) there is a quantifier-free sentence τ such that $ACF \models \sigma \leftrightarrow \tau$. Now τ is a finite disjunction of finite conjunctions of sentences $t_1 = t_2$, $t_1 \neq t_2$ where t_i are closed terms of L. These just concern the arithmetic of the prime field, so everything is clear.(???)

Corollary 3.9 *(Ax) Let $V \subset \mathbf{C}^n$ be a variety (namely the common zero-set of finitely many polynomial equations over $\mathbf{C}$ in indeterminates $x_1, .., x_n$). Let $f : V \to V$ be an injective polynomial map. That is, the coordinates of f are given by polynomials $Q_i(x_1, .., x_n)$ with coefficients in $\mathbf{C}$ (and f is $1-1$). Then f is surjective.*

Proof. Suppose V is the common zero set of $P_i(x_1, .., x_n, \bar{a}_i) = 0$, $(i = 1, .., m)$ where P is over $\mathbf{Z}$ and a_i is a finite tuple from $\mathbf{C}$. Similarly let f be given by $(Q_1(\bar{x}, \bar{b}_1), ..., Q_n(\bar{x}, \bar{b}_n))$ where the b_i are finite tuples from $\mathbf{C}$, and the Q_i are over $\mathbf{Z}$. Let $\bar{c}$ be a finite tuple including all the a_i and b_i. Let us suppose for a contradiction that f is $1-1$ but not surjective. Let $\phi(\bar{y})$ be the L-formula expressing that $\bar{Q}(\bar{x}, \bar{y})$ defines an injective but not surjective map from the set defined by $\bar{P}(\bar{x}, \bar{y}) = 0$ to itself. Let σ be the L-sentence $\exists \bar{y}(\phi(\bar{y}))$. So σ is true in the field of complex numbers. By Corollary 3.8, σ is true in $\overline{\mathbf{F}}_p$ for some prime p. Let K be this field. So we can find some finite tuple d from K and a variety $W \subseteq K^n$ and a polynomial map $g : W \to W$ all defined with coefficients d such that g is $1-1$ but not onto. K is a (directed) union of the finite fields $\mathbf{F}_{p^n}$. So the coefficients d are contained in some $\mathbf{F}_q$. Now W is the union of the $W(\mathbf{F}_{q'})$ for $q' \geq q$. and each such set is finite. Note that g takes $W(\mathbf{F}_{q'})$ into itself and therefore onto itself (by counting). Thus $g : W \to W$ is surjective, a contradiction.

Example 3.10 *The theory RCF of real closed fields in the language of rings, is complete, model-complete, and is the model companion of the theory of formally real fields. RCF does not have quantifier-elimination. The theory $RCOF$ of real closed ordered fields, in the language of ordered rings is complete with quantifier-elimination (and is the model companion of the theory of ordered fields).*

Explanation and proof. The general idea is that real closed fields are to the reals as algebraically closed fields of characteristic zero are to the complexes.

Let L_r be the language of rings, and $L_{or} = L_r \cup \{<\}$ be the language of ordered rings.

We will need some algebraic notions and facts which we will now discuss. Among other things Lang's Algebra is a good source.

The field F is said to be formally real if -1 is not a sum of squares. Note that a formally real field has characteristic zero. Also note that the class of formally real fields (as L_r-structures) is an elementary class. the field F is said to be real closed if F is formally real and has no proper formally real algebraic extension.

Fact 3.11 *F is real closed iff -1 is not a square in F and $F[\sqrt{-1}]$ is algebraically closed.*

So note that the class of real closed fields is also elementary. (Why?) RCF is the corresponding theory. Also the field of real numbers is real closed. Note that it follows from the algebraic closedness of $F[\sqrt{-1}]$ that any polynomial $f(x)$ over a real closed field F splits (over F) into irreducible factors of degree 1 and 2. In particular any odd degree polynomial over F has a root in F.

By an ordered field we mean a field F equipped with a total ordering $<$ such that
(i) $0 < 1$,
(ii) $x < y \to x + z < y + z$,
(iii) $x < y \wedge z > 0 \to xz < yz$.

Note that the ordering on F has to be dense with no first or last element (why?).

By a real closed ordered field we mean an ordered field $(F, <)$ such that F is real closed.

So the class of ordered fields (as well as the class of real closed ordered fields) in the language L_{or} is elementary.

Fact 3.12 *(i) The field F is formally real if and only if there is an ordering on F making F into an ordered field. Moreover $a \in F$ is not a sum of squares iff there is an ordering $<$ of F under which $a < 0$.*
(ii) Suppose that F is a real closed field. Then F has a unique ordering (as an ordered field), namely that where the positive elements are the squares.
(iii) Let $(F, <)$ be an ordered field. Then F is real closed if and only if $(F, <)$ has the sign-change property for polynomials: namely whenever $f(x)$ is a polynomial over F, $a, b \in F$ and $f(a) < 0$ and $f(b) > 0$ then there is c between a and b (in the sense of the ordering) such that $f(c) = 0$.

By virtue of (iii) we get nice axioms for the class of real closed ordered fields. In any case we denote by $RCOF$ the theory of real closed ordered fields.

By a real closure of a formally real field we mean a real closed field which contains F and is algebraic over F. (Such a thing exists by definition).

Fact 3.13 *(Artin-Schreier) Let $(F, <)$ be an ordered field. Then there is a real closure R of F such that $<$ is induced by the (unique) ordering of R. Moreover any two such real closures of F are isomorphic over F.*

To prove quantifier elimination for $RCOF$ we will need the following:

Lemma 3.14 *Suppose that $(K, <)$ and $(K', <)$ are real closed order fields. Suppose $(F, <)$ is a real closed (ordered) subfield of $(K, <)$ and likewise for F'. Suppose that $f : F \to F'$ is an isomorphism between k and k' (as ordered fields). Suppose $b \in K$, $b' \in K'$ and for each $a \in F$, $b < a$ iff $b' < f(a)$. Then f extends to an isomorphism g between the the ordered rings $F[b]$ and $F'[b]$ by putting $g(b) = b'$.*

Proof. We have to show that for any polynomial $p(x) \in F[x]$, $p(b) > 0$ iff $p'(b') > 0$ (Where $p' = f(p)$). We may assume that $p(x)$ is monic, and irreducible in $F[x]$ so p has degree ≤ 2. If $p(x)$ has degree 1, then $p(x) = x - a$ for some $a \in F$. So $p(b) > 0$ iff $b > a$ iff $b' > f(a)$ iff $p'(b') > 0$ (using our assumptions). Otherwise $p(x)$ is degree 2. Note that F is relatively algebraically closed in K, so $p(x)$ is also irreducible in $K[x]$. By the sign change property, $p(x)$ has constant sign on K. Likewise $p'(x)$ has constant sign on K'. Suppose that $p(b) > 0$. Let $a \in F$. So also $p(a) > 0$. As f is an isomorphism, $p'(f(a)) > 0$. Thus $p'(b') > 0$.

Proof of QE for RCOF. We will prove that the class of partial isomorphisms between countable substructures of ω_1-saturated models of $RCOF$ has the back-and forth property. This clearly implies that the class of partial isomorphisms between finitely generated substructures of ω-saturated models of $RCOF$ has the back-and-forth property, and so we can apply 2.29.
[Let us give a brief explanation of this. Suppose we have a theory T and we know that the set of partial isomorphism between countable substructures of any two ω_1-saturated models of T has the back-and-forth property. Now let M, N be ω-saturated models of T. Let M', N' be ω_1-saturated elementary extensions of M, N respectively (which exist by modifying the proof of Lemma 2.12 or by 4.7 below). Let a, b be finite tuples from M, N respectively such that $tp_M(a) = tp_N(b)$, and let $c \in M$. Then $tp_{M'}(a) = tp_{N'}(b)$ so there is (by assumption) some $d \in N'$ such that $tp_{M'}(ac) = tp_{N'}(bd)$. As N is ω-saturated, we can realize $tp_{N'}(d/b)$ in N by d'. Now $tp_M(ac) = tp_N(bd')$. OK.]

So let $(K, <), (K', <)$ be ω_1-saturated real closed ordered fields. Let F, F' be countable substructures of K, K' respectively and f an isomorphism between them (so an isomorphism of ordered rings). f extends to an isomorphism between the ordered fields generated by F, F' and then, by Fact 3.13, to an isomorphism between the real closures of F, F' in K, K' respectively. As these are still countable, we may assume that F and F' are real closed. Let $b \in K$. As f is an isomorphism between $(F, <)$ and $(F', <)$, we may, by ω_1-saturation of K' (and denseness of the ordering) find $b' \in K'$ such that for all $a \in F$, $a < b$ iff $f(a) < b'$. By Lemma 3.14, f extends to an isomorphism between $(F[b], <)$ and $(F'[b'], <)$.

This yields QE for $RCOF$. Note that the ring $\mathbf{Z}$ of integers has a unique ordering on it (making it an ordered ring). So $(\mathbf{Z}, <)$ is a (finitely generated) substructure of every real closed ordered field. By 2.30, $RCOF$ is complete. (So $RCOF = Th(\mathbf{R}, +, -, 0, 1, <)$.)

Finally we look at RCF, the theory of real closed fields in the language L_r. By 3.12(ii) $RCOF$ is a definitional expansion of RCF, so RCF is complete too. In fact $x < y$ is defined by the existential formula $\exists z(y - x = z^2)$. But the negation of $x < y$ is equivalent (in $RCOF$) to $y < x \vee y = x$ which is also defined by an existential formula. So $x < y$ is equivalent, in $RCOF$, to both an existential L_r-formula and a universal L_r-formula. It follows from QE for $RCOF$ that every L_r-formula is equivalent in RCF to an existential L_r-formula. By 2.39, RCF is model-complete. Note that any formally real field is a substructure of a real closed field, and so RCF is the model companion of of the theory of formally real fields.

Finally we note that RCF does NOT have quantifier-elimination. In the model $\mathbf{R}$, both π and $-\pi$ have the same quantifier-free L_r-type (they are both transcendental), but different types (one is a square, one is not).

Corollary 3.15 *(Hilbert's 17th problem) Let $(R, <)$ be a real closed field (such as the field of reals), and let $f(x_1, .., x_n) \in R(x_1, .., x_n)$ be a rational function which is postive semidefinite, that is, for all $\bar{a} \in R^n$, $f(\bar{a}) \geq 0$. Then f is a sum of squares in $R(\bar{x})$.*

Proof. Suppose not. Then by 3.12(i), there is an ordering $<$ on the field $R(\bar{x})$ such that $f < 0$. Notice that $<$ restricts to the given ordering on R (why?). So $(R, <)$ is a substructure of $(R(\bar{x}), <)$. The formula $\exists \bar{x}(f(\bar{x}) < 0)$ is true in $R(\bar{x}), <)$ so also in $(R, <)$ (as the latter is existentially closed in the class of substructures of models of $RCOF$), a contradiction.

Example 3.16 *Consider the theory T of groups in the language $\{\cdot, 1, ^{-1}\}$. (Note that T is universal.) Then T has no model-companion.*

Proof. We will use the following well-known fact:
(*) Let G be a group and $a, b \in G$. Then there is a group extension $H \supseteq G$ and $h \in H$ such that $h^{-1}ah = b$ if and only if a and b have the same order.

It follows that if G is an ec model of T (that is, an existentially closed group), then for any a, b in G, $G \models \exists x(x^{-1}ax = b)$ iff a and b have same order.

Now assume for the sake of a contradiction that T has a model companion T^* (whose models will thus be precisely the ec groups). For each $n < \omega$ let $\psi_n(x, y)$ be a formula expressing that the order of x is n if and only if the order of y is n. Let $\phi(x, y)$ be the formula $\exists z(z^{-1}xz = y)$. So $\phi(x, y)$ is equivalent to $\wedge_n \psi_n(x,y)$ in all models of T^*. By compactness there is n such that $\phi(x,y)$ is equivalent to $\wedge_{i<n}\psi_i(x,y)$ in all models of T^*. Let $k \neq l$ be greater than n. Let G be a group containing an element a of order k and an element b of order l. (For example G could be the direct product of $\mathbf{Z}/k\mathbf{Z}$ and $\mathbf{Z}/l\mathbf{Z}$.) Let H be an existentially closed group extending G. So a, b still have orders k, l respectively in H. But then $H \models \wedge_{i<n}\psi_i(a,b)$ whence $H \models \phi(a,b)$ so a and b are conjugate in H which is impossible as conjugation is a group isomorphism and a and b have different orders.

See Hodges "Building models by games" for more on ec groups.

We finally mention (with no details) a couple of topical examples.

Example 3.17 *Fields with operators.*

A differential field is a field F equipped with a derivation ∂. This means that ∂ is an additive homomorphism and $\partial(xy) = x\partial(y) + \partial(x)y$. An example is $(\mathbf{C}(t), d/dt)$. We consider differential fields as structures in the language of rings together with a function symbol ∂. Let DF_0 be the (obvious) theory of differential fields of characteristic zero. DF_0 is $\forall\exists$ (because the theory of fields is). By definition a differentially closed field (of characteristic zero) is an existentially closed model of DF_0 (that is an ec structure for the DF_0 or equivalently $(DF_0)_\forall$). It is a fact that DF_0 DOES HAVE a model companion, DCF_0 (the theory of differentially closed fields of characteristic zero), and that moreover DCF_0 has quantifier-elimination and is complete.

Let FA be the theory of fields equipped with an automorphism (in the language of rings together with a new function symbol σ). FA is $\forall\exists$. FA DOES have a model-companion which is called $ACFA$. The models of $ACFA$ are precisely the existentially closed models of FA. Any model of $ACFA$ is, as a field, algebraically closed. $ACFA$ does NOT have quantifier-elimination. $ACFA$ is not complete. If (K,σ) is a model of $ACFA$ then $Th(K,\sigma)$ is determined by the isomorphism type of $(K_0, \sigma|K_0)$ where K_0 is the algebraic closure of the prime field of K.

4 Saturation, homogeneity, countable models, ω-categoricity.

In this section we complete our exposition of the elements of "basic" model theory.

As usual we fix a language L and we consider L-structures. We will often be considering types of possible infinite tuples in structures. We recall the notation. Let M be an L-structures. Let I be some index set (often an ordinal). Let A be a subset of the universe of M and $(b_i)_{i\in I}$ a set of elements of M, indexed by I. Let x_i $(i \in I)$ be distinct variables . Then by $tp_M((b_i)_{i\in I}/A)$ we mean the set of L_A-formulas, $\phi(x_{i_1},..,x_{i_n})$ (with free variables from among the $x_i, i \in I$) such that $M \models \phi(b_{i_1},..,b_{i_n})$. If $A = \emptyset$ we may omit it and just write $tp_M((b_i)_{i\in I})$.

Definition 4.1 *Let M, N be L-structures. By a partial elementary map between M and N we mean a map f from a subset $A = dom(f)$ of (the universe of) M into (the universe of) N such that for any L-formula $\phi(x_1,..,x_n)$ and $a_1,..,a_n \in A$, $M \models \phi(a_1,..,a_n)$ iff $N \models \phi(f(a_1),..,a_n)$.*

We will freely use the next remark.

Remark 4.2 *Let f be a map from a subset A of M into N. Let $(a_i)_{i\in I}$ be some indexing of A. (For example it could be $(a_\alpha : \alpha < \beta)$ for some ordinal β. Let $b_i = f(a_i)$. Then f is a a (partial) elementary map between M and N if and only if $tp_M((a_i)_{i\in I}) = tp_N(b_i)_{i\in I})$.*

Definition 4.3 *(i) Let M be an L-structure, and A a subset of the universe of M. Let $n < \omega$. Then $S_n(A, M)$ is the set of complete n-types of $Th(M, a)_{a\in A}$. Likewise for $S_I(A, M)$ where I is a possibly infinite index set. If I is an ordinal β we write $S_\beta(A, M)$. We sometimes omit M if it is understood. Note that $S_n(\emptyset, M) = S_n(Th(M))$ from 2.23.*
(ii) Let κ be an infinite cardinal. We say that M is κ-saturated if whenever $A \subseteq M$ has cardinality $< \kappa$, then any $p \in S_n(A)$ is realized in M. We say that M is saturated if it is $|M|$-saturated.
(iii) We say that M is κ-homogeneous, if whenever $\beta < \kappa$ and a, b are β-tuples from M with $tp_M(a) = tp_M(b)$, and c is an element of M then there is d such that $tp_M(a, c) = tp_M(b, d)$. We say that M is homogeneous if it is

$|M|$-homogeneous.
(iv), We say that M is strongly κ-homogeneous if whenever a, b are as in (hypothesis of) (iii) then there is an automorphism f of M such that $f(a) = b$. M is strongly homogeneous if it is strongly $|M|$-homogeneous.

Exercise 4.4 *(i) M is κ-saturated iff for any $A \subseteq M$ and $p \in S_1(A)$, p is realized in M if and only if whenever $\beta < \kappa$ and $A \subseteq M$, any $p \in S_\beta(A)$ is realized in M.*
(ii) In (iii) of the definition above, we can allow c to be a γ-tuple for any $\gamma < \kappa$.
(iii) If M is κ-saturated then M is κ-homogeneous.
(iv) If M is homogeneous then M is strongly homogeneous.
(v) If M is saturated then M is strongly homogeneous.

Proposition 4.5 *Suppose M, N are elementarily equivalent saturated structures of the same cardinality. Then $M \cong N$.*

Proof. Let $|M| = |N| = \kappa$. List $M = (a_i : i < \kappa)$ and $N = (b_i : i < \kappa)$. We define inductively $c_i \in M$ and $d_i \in N$ for $i < \kappa$ such that for all $\alpha < \kappa$, $tp_M(c_i : i < \alpha) = tp_N(d_i : i < \alpha)$. Suppose we have done this for all $\alpha < \beta$. Write $\beta = \gamma + n$ for γ limit. Suppose $n = 2m$. Put $c_\beta = a_{\gamma+m}$. Let $p(x) = tp_M(c_\beta / c_{<\beta})$. Let $q(x)$ be the result of replacing the parameters c_i by d_i (for $i < \beta$). Then as, by induction hypothesis, $tp_M(c_{<\beta}) = tp_N(d_{<\beta})$, $q(x) \in S_1(d_{<\beta}, N)$ and so is realized in N by some d_β. So $tp_M(c_{\leq\beta}) = tp_N(d_{\leq\beta})$.

If $n = 2m + 1$, put $d_\beta = b_{\gamma+m}$ and find $c_\beta \in M$ as before.

By construction this yields a partial elementary map f from M to N with $dom(f) = M$ and $Ran(f) = N$. So f is an isomorphism.

Proposition 4.6 *Suppose M and N are elementarily equivalent, $|M| \leq \kappa$ and N is κ-saturated. Then there is an elementary embedding of M into N.*

Proof. Like that of Proposition 4.5.

Gien M and κ it is easy to find an elementary extension N of M which is κ-saturated. The difficult issue is to choose N of "small" cardinality. We will investigate these questions in the next few results. Let us first recall the notion of "cofinality". If κ is a cardinal, the cofinality of κ, $cf(\kappa)$, is the least ordinal δ such that there is an increasing sequence $(\alpha_i : i < \delta)$ of ordinals such that $lim_i \alpha_i = \kappa$. $cf(\kappa)$ is known to be a cardinal. κ is said to be a regular cardinal if $cf(\kappa) = \kappa$. It is a fact that any successor cardinal is regular.

Proposition 4.7 *Let M be an L-structure and τ a cardinal such that $\tau \geq |L| + \omega$, and $|M| \leq 2^\tau$. Then M has an elementary extension N which has cardinality at most 2^τ and is τ^+-saturated.*

Proof. We may assume that $|M| = 2^\tau$.
First we construct an elementary continuous chain $(M_i : i < \tau^+)$ of models of cardinality 2^τ such that $M_0 = M$ and for each ordinal $\alpha < \tau^+$, subset A of M_α of cardinality $\leq \tau$ and complete n-type Σ over A, Σ is realized in $M_{\alpha+1}$. (By "continuity" of the chain we mean that at limit stages take unions.) Note that the union of a chain of length at most τ^+ of models of cardinality 2^τ has cardinality 2^τ. So it suffices to prove:
Claim. Any model (without loss M) of cardinality 2^τ has an elementary extension of cardinality 2^τ realizing all types over subsets of M of cardinality at most τ.
Proof of claim. The number of subsets of M of cardinality at most τ is $\leq 2^\tau$. The number of complete types over any such set is at most 2^τ. So by compactness and Lowenheim-Skolem we can find the required model.

So from the claim the construction can be carried out. Let N be the union of the M_i. So N is an elementary extension of M (and of each M_i) and has cardinality 2^τ. We claim that N is τ^+-saturated. Let A be a subset of N of cardinality τ and p a complete n-type over A. Then (by regularity of τ^+, $A \subset M_\alpha$ for some $\alpha < \tau^+$. (Otherwise for arbitrarily large $\alpha < \tau^+$ there is something in A which is in $M_{\alpha+1} \setminus M_\alpha$. But then $cf(\tau^+) \leq |A| = \tau$, a contradiction.) So p is realized in $M_{\alpha+1}$ by construction, so also in N.

Exercise 4.8 *Show that the previous proposition is best possible. Let M be the structure with a single binary relation R, such that the univese of M is the set of finite subsets of ω and R is interpreted as inclusion. Show that any τ^+ saturated model of $Th(M)$ has cardinality at least 2^τ.*

Corollary 4.9 *Assume GCH ($2^\tau = \tau^+$ for every infinite cardinal τ). Let T be any L-theory. Then for ay regular cardinal $\lambda > |L + \omega|$, T has a saturated model of cardinality λ.*

Proof. When λ is a successor cardinal, this is given by 4.7. Assume now λ to be limit. By 4.7, we may construct an elementary (continuous) chain $(M_\mu : \mu < \lambda)$ of saturated models of T such $|M_\mu| = \mu^+$. Let N be the union. Then N has cardinality λ. If A is a subset of M of cardinality $< \lambda$, then by regularity of λ, A is contained in some M_μ. We may assume that $|A| < \mu^+$, so by saturation of M_μ every n-type over A is already realized in M_μ.

Exercise 4.10 *An infinite cardinal λ is said to be inaccessible if $2^\mu < \lambda$ whenever $\mu < \lambda$. Prove that if T is an L-theory and $\lambda > |L| + \omega$ with λ inaccessible. Then T has a saturated model of cardinality λ.*

Proposition 4.11 *Let κ be any cardinal. Then any structure M has an elementary extension which is κ-saturated and strongly κ-homogeneous.*

Proof. It is easy to construct (using 4.7 if you wish) an elementary continuous chain $(M_\alpha : \alpha < \kappa^+)$ such that
(i) $M_0 = M$,
(ii) $M_{\alpha+1}$ is $|M_\alpha|^+$-saturated.
Let N be the union.
Claim 1. N is κ^+-saturated.
Proof. As κ^+ is regular, any subset A of N of cardinality $< \kappa^+$ must be contained in some M_α. But then $|A| \leq |M_\alpha|$ so any n-type over A is realized in $M_{\alpha+1}$ by (ii).

Claim 2. N is strongly κ^+-homogeneous.
Proof. Let a, b be say λ-tuples from N where $\lambda < \kappa^+$, and $tp_N(a) = tp_N(b)$. Let $f : a \to b$ be the corresponding partial elementary map. We must extend f to an automorphism of N. As above there is α such that both a and b are included in M_α. Note that (M_α, a) is elementarily equivalent to $(M_{\alpha+1}, b)$, and that $(M_{\alpha+1}, b)$ is still $|M_\alpha|^+$-saturated. So by 4.6 there is an elementary embedding f_α of M_α in $M_{\alpha+1}$ taking a to b (so extending f). Likewise we can find an elementary embedding $f_{\alpha+1}$ of $M_{\alpha+1}$ in $M_{\alpha+2}$ extending f_α. Continue in this way, taking unions at limit ordinals, to obtain an automorphism of N extending f. (???)

The previous proposition is quite useful methodologically. For example, suppose we are given a complete theory T and we are interested in understanding the models of T of cardinality $< \kappa$. Let N be a model of T given by the previous proposition. Then any model of T of cardinality $< \kappa$ is isomorphic to an elementary substructure of N (by 4.6). Moreover any two tuples from N of length $< \kappa$ have the same type in N iff they are in the same orbit under $Aut(N)$. Of course, for T to have outright saturated models of arbitrarily large cardinality would be even better, but this in general requires some set-theoretic hypothesis.

Finally in this discussion of saturation and homogeneity, we give a rather surprising result about homogeneous models.

Lemma 4.12 *Suppose that M and N are L-structures, and κ is a cardinal, such that*
(i) for any finite tuple a from M, $tp_M(a)$ is realized in N, and conversely.
(ii) N is κ-homogeneous.
Then for any tuple a from M of length $< \kappa$, $tp_M(a)$ is realized in N.

Proof. We prove it by induction on the length of the tuple a. Suppose length a is i. For $j < i$ let $a|i$ be the first j elements of a. We will build b in N inductively. That is construct b^j for $j < i$ such that $tp_M(a|j) = tp_N(b^j)$ and $j < j'$ implies $b^{j'}$ is a extension of b^j. For j finite we can start and continue by our hypotheses. For j limit, take union of what we have so far. What about the case of an infinite successor ordinal. So we have $tp_M(a|j) = tp_N(b^j)$, and we want to extend b^j to b^{j+1} such that $tp_M(a|j+1) = tp_N(b^{j+1})$. We can reorder the tuple $a|j+1$ as a β-sequence c say where $\beta < j+1$. By induction hypothesis we can find a β-tuple d from N such that $tp_M(c) = tp_N(d)$. Now by κ-homogeneity of N we can extend b^j to b^{j+1} as required.

Exercise 4.13 *Let T be a complete theory, κ a cardinal, and M, N homogeneous models of T of cardinality κ. Suppose that M and N realizes the same types in $S_n(T)$ for all n. Then M and N are isomorphic.*

We will now specialize to countable models. Classification theory for countable models is quite different in spirit from that for uncountable models. But we will discuss some elementary results in which useful notions like prime models and cardinalities of type spaces are introduced.

Countable means of cardinality at most ω. From now on when we say that a theory T is countable we mean that the language L of T has cardinality at most ω.

Lemma 4.14 *Suppose T is countable and complete. Then the following are equivalent:*
(i) $S_n(T)$ is countable for all $n < \omega$.
(ii) For any model M of T and finite subset $A \subset M$, $S_1(A, M)$ is countable.
(iii) For any model M of T and finite $A \subset M$, $S_n(A, M)$ is countable for all $n < \omega$.

Proof. (i) implies (ii). Suppose for a contradiction that $S_1(A, M)$ is uncountable (for some $M \models T$ and finite $A \subset M$). So there is an elementary extension N of M and $b_i \in N$ for $i < \omega_1$ such that $tp_N(b_i/A) \neq tp_N(b_j/A)$ for $i \neq j$. Suppose $|A| = n$. Let $\bar{a}$ be an enumeration of A. Then clearly $tp_N(\bar{a}, b_i) \neq tp_N(\bar{a}, b_j)$ for $i \neq j$, whereby $S_{n+1}(T)$ is uncountable.
(ii) implies (iii). By induction on n. Assume for a contradiction that $S_{n+1}(A, M)$ is uncountable for some $M \models T$ and finite $A \subset M$. So again we have $n+1$-tuples b_i for $i < \omega_1$ in some elementary extension N of M such that the $tp_N(b_i/A)$ are all different. By 4.11 we may assume that N is strongly ω-homogeneous (this is not really necessary). Let c_i be the n-tuple consisting of the first n elements of b_i, and d_i the last element of b_i. By induction hypothesis there are only countably many types over A among the $tp_N(c_i/A)$. So relabelling, we may assume that $tp_N(c_i/A) = tp_N(c_j/A)$ for all i, j. By the strong homogeneity assumption, for each i, let f_i be an automorphism of N which fixes A and takes c_i to c_0. Let $d'_i = f_i(d_i)$. Then $tp_N(c_0 d'_i/A)$ are all different, so writing C_0 as the set enumerated by c_0, $tp_N(d'_i/A \cup C_0)$ are all different, contradicting (ii).
(iii) implies (i) is immediate.

Proposition 4.15 *Let T be countable and and complete. Then the following are equivalent:*
(i) $S_n(T)$ is countable for all $n < \omega$,
(ii) T has a countable ω-saturated model.

Proof. (ii) implies (i). Let M be a (in fact the) countable ω-saturated model of T. Then M realizes all types in $S_n(T)$ for all n (as M is weakly saturated). So by countability of M we get (i).
(i) implies (ii). Let M_0 be any countable model of T. By our hypothesis, Lemma 4.14 and Lowenheim-Skolem, we can find a countable elementary extension M_1 of M_0 such that for every finite $A \subset M_0$ and $p \in S_n(A, M_0)$, p is realized in M_1. Continue to build an elementary chain of countable

models $M_0 \subseteq M_1 \subseteq M_2 \ldots$. Let M the the union. Then M is countable and ω-saturated.

Remark 4.16 *A countable complete theory T is often said to be "small" if it satisfies the equivalent conditions of 4.15. By Proposition 4.5 any two countable saturated models of T are isomorphic. By 4.6, any countable model of T embeds in any ω-saturated model of T. So for small theories there is a "biggest" countable model, the countable saturated one. We will discuss below when there is a "smallest" countable model.*

Definition 4.17 *Let T be a complete theory. A model M of T is said to be a prime model of T if for any model N of T there is an elementary embedding of M in N.*

Note that a prime model of T, if it exists, has cardinality at most $|L| + \omega$. A natural question to ask is whether any two prime models of T must be isomorphic. This is false in general, but as we will see it is true for countable theories. We will need the "omitting types" theorem.

Definition 4.18 *Let T be a theory, and $\Sigma(x_1, .., x_n)$ an n-type of T. We will say that Σ is principal if there is an L-formula $\phi(x_1, .., x_n)$ such that*
(i) ϕ is consistent with T, and
(ii) for every $\psi \in \Sigma$, $T \models \forall \bar{x}(\phi(\bar{x}) \rightarrow \psi(\bar{x}))$

Remark 4.19 *(i) Suppose p is a complete n-type of T. Then p is principal iff p is an isolated point in the space $S_n(T)$.*
(ii) If T is complete and Σ is a principal type of T then Σ is realized in every model of T.

Proposition 4.20 *Suppose that T is a countable theory and Σ is a nonprincipal n-type of T. Then T has a (countable) model which omits Σ.*

Proof. (Note in passing that if $\Sigma(x_1, .., x_n)$ is a set of L-formulas which is not an n-type of T then every model of T omits Σ.)

For notational ease we prove the Proposition when $n = 1$. The proof will be a generalized Henkin construction. Let us add a countable set $\{c_0, c_1, ...\}$ of new constants to L to get L'. We will build a complete L' theory T' extending T with the properties

(i) for each L'-formula $\phi(x)$, if $\exists x\phi(x) \in T'$ then $\phi(c_m) \in T'$ for some m, and
(ii) for each m there is $\psi(x) \in \Sigma(x)$ such that $\neg\psi(c_m) \in T$ for all m.

Suppose have found such a theory T'. Let M' be a model of T'. By Tarkski-Vaught the subset of M' consisting of interpretations of the new constants c_i is the universe of a (countable) elementary substructure M'' of M' which clearly omits $\Sigma(x)$. The L-reduct of M'' satisfies the theorem.

So how to build T'? Let $\sigma_0, \sigma_1, ...$ be a list of all L'-sentences. We will build consistent sets of L'-sentences $T = S_0 \subseteq S_1 \subseteq .....S_n..$ $(n < \omega)$, such that each S_{i+1} is obtained by adding finitely many sentences to S_i, as follows: Suppose we are given S_i. If $S_i \cup \sigma_i$ is consistent, put $S'_i = S_i \cup \{\sigma_i\}$. Otherwise put $S'_i = S_i \cup \{\neg\sigma_i\}$.

Now suppose we added σ_i and σ_i was of the form $\exists x\phi(x)$. Then add $\phi(c_j)$ to S'_i for some c_j not appearing in S'_i to get S''_i. Otherwise put $S''_i = S'_i$. In any case S''_i is consistent. S''_i has the form $T \cup \{\psi_1, .., \psi_s\}$. Let $\bar{c}$ be the tuple of new constants occuring in $\psi_1, .., \psi_s$. Write $\wedge_{j=1,..,s}\psi_j$ as $\psi(\bar{c})$ where $\psi(x_1, .., x_n)$ is an L-formula. We may assume that c_i is included in $\bar{c}$.
Claim. For some $\delta(x) \in \Sigma(x)$, $S''_i \cup \{\neg\delta(c_i)\}$ is consistent.
Proof of claim. Otherwise $S''_i \models \delta(c_i)$ for all $\delta(x) \in \Sigma(x)$. So $T \cup \psi(\bar{c}) \models \delta(c_i)$ for all $\delta \in \Sigma$. Let $\chi(x_i)$ be the L-formula $\exists x_1...x_{i-1}x_{i+1}...x_n(\psi(x_1, .., x_n))$. So one sees that $T \models \forall x_i(\chi(x_i) \rightarrow \delta(x_i))$ for all $\delta \in \Sigma$. As $\chi(x_i)$ is consistent with T, this shows Σ is principal, a contradiction.

Put $S_{i+1} = S''_i \cup \{\neg\delta(c_i)\}$ for some $\delta \in \Sigma$ given by the claim.

If we take T' to be the union of the S_i, then T' satisfies (i) and (ii) as at the beginning of the proof.

Exercise 4.21 *Let T be countable and for each $i < \omega$ let Σ_i be a nonprincipal n_i-type of T (for some finite n_i). Then T has a countable model omitting all the Σ_i.*

Definition 4.22 *(i) An L-structure M is said to be atomic if for each n and every n-tuple $\bar{a}$ in M, $tp_M(\bar{a})$ is a principal type of $Th(M)$ (equivalently an isolated type in $S_n(Th(M))$.*
(ii) Let T be a complete theory. A formula $\phi(x_1, ., x_n)$ of L is said to be complete, or an atom, or an atomic formula, if $\phi(x_1, .., x_n)$ isolates a complete type in $S_n(T)$. (Equivalently, if ϕ is consistent with T and for each $\psi(x_1, .., x_n)$ of L, $T \models \forall\bar{x}(\phi(\bar{x}) \rightarrow \psi(\bar{x}))$ or $T \models \forall\bar{x}(\phi(\bar{x}) \rightarrow \neg\psi(\bar{x}))$.)

Remark 4.23 *Let T be complete. For each n let $\Phi_n(x_1,..,x_n) = \{\neg\phi(x_1,..,x_n) : \phi(x_1,..,x_n)$ is a complete n-formula of $T\}$. Then a model M of T is atomic if and only if M omits each Φ_n.*

Proposition 4.24 *Let T be countable and complete. The following are equivalent:*
(i) T has an atomic model,
(ii) T has a countable atomic model,
(iii) For each n the isolated types are dense in $S_n(T)$.

Proof. (i) implies (ii) is clear (by taking a countable elementary substructure).
(ii) implies (iii). Let $\phi(x_1,..,x_n)$ be any L-formula consistent with T. Let M be a (countable) atomic model of T. So ϕ is realized in M by some n-tuple $\bar{a}$. But $tp_M(\bar{a})$ is isolated (and contains ϕ). So the clopen subset of $S_n(T)$ determined by ϕ contains an isolated point.
(iii) implies (i). Let $\Phi_n(x_1,..,x_n)$ be as in Remark 4.23. We claim that each Φ_n is (if consistent with T) a nonprincipal n-type of T. For if $\phi(x_1,..,x_n)$ is consistent with T, then by (iii) there is a complete n-formula $\psi(x_1,..,x_n)$ consistent with ϕ and T (in fact which even implies ϕ mod T). But $\neg\psi \in \Phi_n$. Thus by Exercise 4.21, T has a model omitting each Φ_n. By Remark 4.23, such a model is atomic.

Exercise 4.25 *Let M be an L-structure, and let a, b be finite tuples from M. Then $tp_M(a,b)$ is isolated iff $tp_M(a)$ is isolated and $tp_M(b/a)$ is isolated.*

Proposition 4.26 *Let T be countable and complete, and let M be a model of T. Then M is a prime model of T if and only if M is a countable atomic model of T.*

Proof. Assume first M to be prime. As remarked earlier M is countable. Let $p \in S_n(T)$ be nonisolated. By Proposition 4.20, p is omitted in some model N of T. But there is an elementary embedding of M in N. So p is also omitted in M. Thus every complete n-type realized in M must be isolated, and M is atomic.

Conversely, assume M to be countable and atomic. We have to show that M is elementarily embeddable in every model of T. Let N be a model of T. Let us enumerate $M = (a_0, .., a_n, ...)$. We will define inductively an elementary map f of M into N. Let us suppose that we have defined f on $a_0, .., a_{n-1}$ with $f(a_i) = c_i \in N$ and $tp_M(a_0, .., a_{n-1}) = tp_N(c_0, .., c_{n-1})$. Write $b = (a_0, ., a_{n-1})$ and $d = (c_0, .., c_{n-1})$. So $(M, b) \equiv (N, d)$. Let $p_b(x) = tp_M(a_n/b)$. By Exercise 4.25, $p_b(x)$ is isolated (in $S_1(Th(M, b))$). Thus $p_d(x)$ is isolated (in $S_1(Th(N, d))$). By 4.19 (ii), $p_d(x)$ is realized in N by some c_n say. It follows that $tp_M(a_0, ., a_n) = tp_N(c_0, .., c_n)$. Put $f(a_n) = c_n$.

Exercise 4.27 *Suppose T is countable and complete. (i) Show that any atomic model of T is ω-homogeneous.*
(ii) Show that any two prime models of T are isomorphic.

Proposition 4.28 *(T countable, complete.) If $S_n(T)$ is countable then the isolated types are dense in $S_n(T)$.*

Proof. This is purely "topological". If X is a countable, Hausdorff, compact toplogical space with a basis of clopens then the isolated points are dense in X: Suppose for a contradiction that U is an open subset of X containing no isolated points. We may assume U to be clopen. U contains 2 distinct points $a \neq b$. By Hausdorffness there are open disjoint subsets U_0, U_1 of U, one containing a the other containing b. We may assume U_0, U_1 to be clopen. Again each of U_0, U_1 contains at least two distinct points, so we find disjoint nonempty clopen subsets U_{00}, U_{01} of U_0 and U_{10}, U_{11} of U_1. Continuing this way we produce clopen subsets U_η for each $\eta \in 2^{<\omega}$ such that if η_2 prolongs η_1 then $U_{\eta_2} \subseteq U_{\eta_1}$. For each $\rho \in 2^\omega$, put $U_\rho = \cap\{U_{\rho|n} : n < \omega\}$. By compactness each U_ρ is nonempty and for different ρ's they are disjoint. So X has size at least the continuum.

Exercise 4.29 *(i) Let X be a compact Hausdorff topological space with a countable basis of clopens. Show that X is either countable or has cardinality the continuum.*
(ii) Suppose T is countable, complete and that for some n, $S_n(T)$ is uncountable. Prove that T has continuum many countable models (up to isomorphism).

A big question in model theory is Vaught's conjecture: If T is a countable complete theory then the number of countable models of T up to isomorphism is either at most ω or is 2^ω. (A negative solution was recently announced by

Knight.) By virtue of Exercise 4.29(ii), we may assume T to be small (see Remark 4.16). From our study so far we have found some kind of information about the class of countable models of a small theory.

Corollary 4.30 *Let T be a countable, complete, small theory. Then T has a prime model M_0 and a countable saturated model M_1. For any countable model M of T, there is an elementary embedding of M_0 into M and an elementary embedding of M into M_1.*

Proof. We have already dealt with the existence and properties of the countable saturated model. (See Remark 4.16.) By Proposition 4.28, for each n the isolated types are dense in $S_n(T)$. By Proposition 4.24 T has a countable atomic model, which by 4.26 is prime.

Let us consider some of the examples mentioned earlier in the light of the structure and number of countable models. Details are left to you. The theory of dense linear orderings (with no first or last element) has exactly one countable model, which must therefore be both prime and saturated. The theory of independent unary predicates has neither a countable saturated model nor a prime model. The theory of discrete linear orderings (with no first or last element) has a prime model and a countable saturated model (what are they), but has continuum many countable models. The theory of algebraically closed fields of a given characteristic has a prime and countable saturated model. Moreover it has only countably many countable models. The theory of real closed fields has a prime model but no countable saturated model. So it has continuum many countable models.

Lemma 4.31 *Let T be countable and complete. Then T is ω-categorical if and only if for each $n \geq 1$, every type in $S_n(T)$ is isolated.*

Proof. Suppose first that the right hand side is false. So for some n there is $p \in S_n(T)$ which is not isolated. By 4.20, T has a countable model omitting p. But T also a countable model realizing p. These models could not be isomorphic, so T is not ω-categorical.

Conversely, suppose the right hand side holds. Then every model of T is atomic. By 4.26 and 4.27, T is ω-categorical.

From "topological" considerations, we obtain the following characterization of ω-categorical theories.

Corollary 4.32 *Let T be countable and complete. Then the following are equivalent:*
(i) T is ω-categorical,
(ii) For each $n \geq 1$, $S_n(T)$ is finite,
(iii) For each $n \geq 1$ there are only finitely many many L-formulas $\phi(x_1, .., x_n)$ up to equivalence modulo T.

Proposition 4.33 *Suppose T to be countable complete. Then T does not have exactly two countable models (up to isomorphism).*

Proof sketch. Suppose for the sake of a contradiction that T has exactly two countable models. By 4.29 (ii) T is small, so by 4.30 has a prime model M and a countable saturated model M_1. On the other hand by 4.30 there is $p \in S_n(T)$ for some n, which is not isolated. p is realized in M_1 but not in M_0 whereby M_0 and M_1 are not isomorphic. We have to find another countable model. Let $p(\bar{x})$ be the nonisolated type in $S_n(T)$. Adjoin new constants $\bar{c}$ to get a language L' and let T' be $T \cup p(\bar{c})$ (or just $p(\bar{c})$) which is a complete L'-theory). By Lemma 4.14, T' is also small. (Why?) So by 4.30, T' has a prime model. Let us write this prime model as $(M_2, \bar{a})$ where M_2 is a model of T (so an L-structure) and $\bar{a}$ is the interpretation of the new constants $\bar{c}$. Note that M_2 can not be isomorphic to M_0, as M_0 omits p, and M_2 realizes it. We will now show that M_2 is NOT ω-saturated, so is NOT isomorphic to M_1 which will complete the proof.

Now $S_n(T)$ is infinite (Why?), so also $S_n(T')$ is infinite (why?), and so there is a nonisolated type $q(\bar{x})$ in $S_n(T')$. q is omitted in $(M_2, \bar{a})$ as the latter is a prime, so atomic model of T'. As $q(\bar{x}) \in S_n(Th(M_2, \bar{a}))$ is omitted in $(M_2 < \bar{a})$, M_2 could not be ω-saturated.

Example 4.34 *There is a countable complete theory with exactly three countable models, up to isomorphism.*

Explanation. This is the so-called Ehrenfeucht example. Let L consists of a binary relation symbol $<$ and constants c_i for $i < \omega$. T says that $<$ is a dense linear ordering with no first or last element, and that $c_i < c_j$ whenenever $i < j$. A routine back-and-forth argument shows that T is complete with quantifier-elimination. Let M be a countable model, and let $a_i = c_i(M)$. Then exactly one of the following cases occurs:

(i) the a_i are cofinal in M.
(ii) $\{a_i : i < \omega\}$ has a supremum in M.
(iii) $\{a_i : i < \omega\}$ has an upper bound in M but no supremum in M.
Note that each of these three possibilities can occur, and that moreover the isomorphism type of M is determined by which one occurs. Thus T has precisely three countable models. Let us call these M_1, M_2, M_3 according to whether (i), (ii) or (iii) occurs. M_1 has to be the prime model (it elementarily embeds in the other models). Which of M_2, M_3 is saturated? Well consider M_2 and let b be the supremum of the a_i in M_2. Then the set of formulas $\{x < b\} \cup \{x > a_i : i < \omega\}$ is finitely satisfiable in M_2 but not realized in M_2. So M_2 is not saturated. So M_3 is the countable saurated model. Note that all models elementarily embed in M_2 too. So M_2 is weakly saturated but not saturated.

Exercise 4.35 *Let T be a countable complete theory. Assume that every countable model of T is ω-homogeneous. Then either T is ω-categorical, or T has infinitely many countable models up to isomorphism.*

5 ω-stable theories and Morley's theorem

In this section we will develop some machinery and notions required to prove Morley's celebrated theorem. These notions, such as ω-stability and indiscernibles, are of interest in their own right. Roughly speaking T is said to be ω-stable if there are only countably many types over any countable set.

Recall that Morley's Theorem says: If the countable complete theory T is κ-categorical for some uncountable cardinal κ then T is κ-categorical for all uncountable κ.

Our proof will have three steps.
Step 1. Deduce from the κ-categoricity of T (for some some uncountable κ), the ω-stability of T.
Step 2. Use ω-stability (and sometimes) the assmption of categoricity, to find enough saturated models in uncountable cardinals, prime models over arbitrary sets, and indiscernibles in uncountable models.
Step 3. Show using the machinery from Step 2 that if T is κ-categorical in some uncountable cardinality then every uncountable model of T is saturated.

Let us start with Step 1.

Definition 5.1 *The countable complete theory T is said to be ω-stable if for any model M of T, and countable subset A of M, $S_1(A, M)$ is countable (that is, $Th(M, a)_{a \in A}$ has at most countably many complete 1-types).*

Remark 5.2 *As usual, if T is ω-stable then over any countable set there are only countably many complete n-types for all n. Many of the theories we have analyzed earlier are ω-stable: for example the theory of infinite sets in the empty language, the theory of torsion-free divisible abelian groups, the theory of algebraically closed fields,..) On the other hand the theory of dense (or discrete) linear orderings is not ω-stable.*

Definition 5.3 *Let T_0 be a possibly incomplete theory in language L_0. We will say that T_0 has Skolem functions, or is Skolemized, if for each L_0-formula $\phi(x, \bar{y})$ with $l(\bar{y}) = n$ there is a n-ary function symbol f of L_0 such that $T_0 \models \forall \bar{y}(\exists x \phi(x, \bar{y}) \rightarrow \phi(f(\bar{y}), \bar{y}))$.*

Lemma 5.4 *Let T be any theory (in language L). Then there is a language $L_0 \supseteq L$ of cardinality at most $|L| + \omega$ and an L_0 theory T_0 containing T, such that T_0 has Skolem functions.*

Proof. Let M be a model of T. For each $\phi(x, \bar{y}) \in L$ add a new function symbol $f_\phi(\bar{y})$ to get a language L^1. Expand M to an L^1-structure M_1 by defining $f(M^1)(\bar{b})$ to be some c such that $M \models \phi(c, \bar{b})$ if there is such a c, and anything you want otherwise. Iterate this construction to find languages $L \subset L^1 \subset L^2 ...$ and expansions $M^1, M^2, ...$ of M. Let L_0 be the union of the L^i (for $i < \omega$) and M_0 the resulting expansion of M to an L_0-structure. Put $T_0 = Th(M_0)$.

Lemma 5.5 *Suppose that T is Skolemized. Then any substructure of a model of T is an elementary substructure.*

Proof Let M be a substructure of $N \models T$. Suppose $N \models \exists \phi(x, \bar{b})$ where $\phi(x, \bar{y}) \in L$ and $\bar{b}$ is from M. Then $N \models \phi(f(\bar{b}), \bar{b})$ for some function symbol $f \in L$. As M is a substructure of N, $f(\bar{b}) \in M$. So by Tarski-Vaught, M is an elementary substructure of N.

Definition 5.6 *Let M be a structure, A a subset of the universe of M, $(I, <)$ an ordered set and $(b_i : i \in I)$ a subset of the universe of M. We say that $(b_i : i \in I)$ is indiscernible (relative to $(I, <)$) over A in M if for any n and $i_1 < .. < i_n$ and $j_1 < ..j_n$ from I, $tp_M((b_{i_1}, .., b_{i_n})/A) = tp_M((b_{j_1}, .., b_{j_n})/A)$.*

Remark 5.7 *We are mainly interested in the case when I is infinite. We can define likewise when an indexed sequence of k-tuples from a model M is indiscernible over A in M. If $A = \emptyset$ we just say indiscernible. Often I will be an ordinal α with the usual ordering, in which case we will just say that $(b_i : i < \alpha)$ is an indiscernible sequence.*

Lemma 5.8 *Suppose that $(a_i : i < \omega)$ is an indiscernible sequence in a structure M. Let $(I, <)$ be any infinite ordered set. Then there is a structure N containing elements $(b_i : i \in I)$ such that $(b_i : i \in I)$ is indiscernible relative to $(I, <)$ in N and such that for each n and $i_1 < .. < i_n \in I$, $tp_N(b_{i_1}, .., b_{i_n}) = tp_M(a_1, .., a_n)$.*

Proof. Compactness.

We will obtain infinite indiscernible sequences from Ramsey's Theorem. First some notation: if X is a set and $n < \omega$, $X^{[n]}$ denote the set of n-element subsets of X.

Fact 5.9 *Let $n < \omega$. Let X_1, X_2 be disjoint subsets of $\omega^{[n]}$ whose union is $\omega^{[n]}$. Then there is an infinite subset Y of ω, and some i such that $Y^{[n]} \subseteq X_i$.*

Remark 5.10 *By iterating 5.9 we clearly obtain the following strengthening: Let $n_1, .., n_r < \omega$ $(r < \omega)$, and for each $i < r$ let $X_{i,1}, X_{i,2}$ be a partition of $\omega^{[n_i]}$. Then there is an infinite subset Y of ω and for each $i < r$ some $j_r \in \{1, 2\}$ such that $Y^{[n_i]}$ is contained in X_{i,j_i} for $i = 1, .., r$.*

Proposition 5.11 *Let T be a complete theory. For each n let $\Sigma_n(x_1, .., x_n)$ be a (possibly empty) set of L-formulas. Suppose that there is a model M of T and elements $a_i \in M$ for $i < \omega$ such that for each $i_1 < .. < i_n < \omega$, $(a_{i_1}, .., a_{i_n})$ realizes $\Sigma_n(x_1, ., x_n)$ in M. Then there is a model N of T and an indiscernible sequence $(b_i : i < \omega)$ in N such that $(b_1, .., b_n)$ realize $\Sigma(x_1, .., x_n)$ in N.*

Proof. Adjoin new constants $\{c_i : i < \omega\}$ to L to get L'. Let Ω be the following set of L'-sentences: $T \cup \{\phi(c_{i_1}, .., c_{i_n}) \leftrightarrow \phi(c_{j_1}, .., c_{j_n}) : i_1 < .. < i_n < \omega, j_1 < ..j_n < \omega, \phi(x_1, .., x_n) \in L\} \cup \{\Sigma_n(c_{i_1}, ., c_{i_n}) : n < \omega, i_1 < .. < i_n < \omega\}$. It is clearly enough to prove that Ω is consistent. Take a finite subset Ω' of Ω and let $\phi_1(x_1, .., x_{n_1}), .., \phi_r(x_1, .., x_{n_r})$ be the L-formulas appearing in the second part of Ω'. For $i = 1, .., r$, partition $\omega^{[n_i]}$ into two sets $X_{i,1}$ and $X_{i,2}$ as follows: suppose $j_1 < .. < j_{n_i} < \omega$. Put $\{j_1, .., j_{n_i}\}$ into $X_{i,1}$ if $M \models \phi_i(a_{j_1}, .., a_{j_{n_i}})$ and into $X_{i,2}$ otherwise. Let Y be as given by 5.10. Let $Y = \{s_0, s_1, s_2, ...\}$ where $s_0 < s_1 < ....$ Then interpreting c_i as a_{s_i} gives a model of Ω'.

Corollary 5.12 *Let T be a theory with infinite models. Then for any cardinal κ there is a model M of T and a set $\{a_i : i < \kappa\}$ of distinct elements of M such that $(a_i : i < \kappa)$ is an indiscernible sequence in M.*

Proof. By 5.11 and 5.8.

Here is the main conclusion we obtain from the above material on Skolem functions and indiscernibles:

Proposition 5.13 *For any countable theory T and uncountable cardinal κ, there is a model M of T of cardinality κ such that for every countable subset A of M only countably many types in $S_1(A, M)$ are realized in M.*

Proof. By Lemma 5.4 let T' be a countable Skolemization of T in a (countable) language L'. It is clearly enough to prove the Proposition for T'. By Corollary 5.12, let $(b_i : i < \kappa)$ be an indiscernible sequence (of cardinality κ) in a model M of T. By Lemma 5.5 we may assume that M is the substructure of itself generated by $(b_i : i < \kappa)$. Namely every element of M is of the form $t(M)(\bar{b})$ for some term t of L' and some finite tuple $\bar{b}$ of the b_i's. Note that M has cardinality κ.

Now let A be a countable subset of the universe of M. For each $a \in A$, pick a term t_a of L' and a finite tuple $\bar{b}_a$ from $(b_i : i < \kappa)$ such that $a = t_a(\bar{b}_a)$. Let B be the set of b_i appearing in the $\bar{b}_a$ for $a \in A$. So B is countable. Let $I_0 \subset \kappa$ be the set of $i < \kappa$ such that $b_i \in B$. So I_0 is countable. Now for each n, we will define an equivalence relation E_n on n-tuples from κ. Namely, suppose $\alpha_1, .., \alpha_n < \kappa$ and $\beta_1, ..., \beta_n < \kappa$. We say that $E_n((\alpha_1, .., \alpha_n), (\beta_1, .., \beta_n))$ if (a) for each $1 \leq i, j \leq n$ $\alpha_i < \alpha_j$ iff $\beta_i < \beta_j$ and $\alpha_i = \alpha_j$ iff $\beta_i = \beta_j$ and (b) for each $z \in I_0$, and each $i = 1, .., n$, $\alpha_i < z$ iff $\beta_i < z$ and $\alpha_i = z$ iff $\beta_i = z$. As κ is well-ordered it follows that there are at most countably many E_n-classes for each n.

Claim. Suppose that $E_n((\alpha_1, .., \alpha_n), (\beta_1, .., \beta_n))$, and that $s(x_1, .., x_n)$ is an L'-term. Then $tp_M(s(b_{\alpha_1}, .., b_{\alpha_n})/A) = tp_M(s(b_{\beta_1}, ..b_{\beta_n})/A)$.

Proof of claim. This is because every element of A is of the form $t(\bar{b})$ for some $\bar{b}$ from B and because by definition of E_n and indiscernibility of $(b_i : i < \kappa)$, if $\bar{b}$ is a tuple from B then $tp_M(b_{\alpha_1}, .., b_{\alpha_n}, \bar{b}) = tp_M(b_{\beta_1}, .., b_{\beta_n}, \bar{b})$.

There are countably many L'-terms s and countably many E_n-clases for each n. So by the claim, only countably many distinct types over A are realized in M.

Corollary 5.14 *(T countable and complete.) Suppose T is κ-categorical for some uncountable κ. Then T is ω-stable.*

Proof. Suppose for the sake of a contradiction that T is not ω-stable. So there is a model M of T and some countable subset A of M such that $S_1(Th(M, a)_{a \in A})$ is uncountable. So there is an elementary extension N of M realizing all these types. By Lowenheim-Skolem we may choose either an elementary extension N' of N or an elementary substructure N' of N containing A, such that N' has cardinality κ and uncountably many complete 1-types over A are realized in N'. Let N'' be a model of T of cardinality κ given by 5.13. Then N' and N'' are not isomorphic, contradicting κ-categoricity of T.

Step 1 is complete. We now proceed to Step 2, the study of ω-stable theories. We will first show that ω-stability is equivalent to "Morley rank" being defined. We will leave the verification of various things as exercises.

Definition 5.15 *Let T be a complete theory, $n < \omega$, M a model of T and $\phi(\bar{x}, \bar{a})$ a formula with free variables $\bar{x} = (x_1, .., x_n)$ and parameters $\bar{a}$ from M. We first define by induction "$RM_n(\phi(\bar{x}, \bar{a}) \geq \alpha$" where α is an ordinal. On the face of it this depends on M too.*
Anyway
(i) $RM(\phi(\bar{x}, \bar{a})) \geq 0$ if $M \models \exists \bar{x}(\phi(\bar{x}, \bar{a})$.
(ii) For δ a limit ordinal, $RM_n(\phi(\bar{x}, \bar{a})) \geq \delta$ if $RM_n(\phi(\bar{x}, \bar{a})) \geq \alpha$ for all $\alpha < \delta$,
(iii) $RM_n(\phi(\bar{x}, \bar{a})) \geq \alpha + 1$ if there is some elementary extension N of M and there are formulas $\psi_j(\bar{x}, \bar{b}_j)$ for $j < \omega$ where $\bar{b}_j \in N$, such that
(a) $N \models \psi_j(\bar{x}, \bar{b}_j) \to \phi(\bar{x}, \bar{a})$ for all $j < \omega$,
(b) $RM_n(\psi_j(\bar{x}, \bar{b}_j)) \geq \alpha$ for all $j < \omega$, and
(c) for all $i \neq j$, $N \models \neg \exists \bar{x}(\psi_i(\bar{x}, \bar{b}_i) \wedge \psi_j(\bar{x}, \bar{b}_j))$.

Having defined the expression "$RM_n(-) \geq \alpha$" we will say
(iv) $RM_n(\phi(\bar{x},\bar{a})) = \alpha$ iff α is the greatest ordinal such that "$RM_n(\phi(\bar{x},\bar{a})) \geq \alpha$". If $RM_n(\phi(\bar{x},\bar{a})) \geq \alpha$ for all ordinals α, we will say that $RM_n(\phi(\bar{x},\bar{a})) = \infty$.

Exercise 5.16 *Show that $RM_n(\phi(\bar{x},\bar{a}))$ depends only on $tp_M(\bar{a})$.*

Remark 5.17 *(i) We will drop the n in $RM_n(-)$ when it is clear from the context.*
(ii) Let M be an ω-saturated model of T. Identify formulas of L_M in free variables $x_1,..,x_n$ with definable subsets of M^n. Then the crucial clause (iii) in Definition 5.15 can be rephrased as: Let $X \subset M^n$ be definable in M. Then $RM(X) \geq \alpha + 1$ if there are pairwise disjoint definable subsets Y_i of X for $i < \omega$ such that $RM(Y_i) \geq \alpha$ for all $i < \omega$.

Exercise 5.18 *(Work in an ω-saturated model M of the complete theory T.)*
(i) $RM(\phi(\bar{x},\bar{a}) \vee \psi(\bar{x},\bar{b})) = max\{RM(\phi(\bar{x},\bar{a})), RM(\psi(\bar{x},\bar{b}))\}$.
(ii) if $M \models \phi(\bar{x},\bar{a}) \rightarrow \psi(\bar{x},\bar{b})$ then $RM(\phi(\bar{x},\bar{a})) \leq RM(\psi(\bar{x},\bar{b}))$.
(iii) $RM(\phi(\bar{x},\bar{a})) = 0$ if $M \models \exists^{=k}\bar{x}(\phi(\bar{x},\bar{a}))$ for some integer $k \geq 1$.
(iv) Suppose that $RM(\phi(\bar{x},\bar{a})) = \alpha$. Then there is a greatest integer d such that there exist $\psi_i(\bar{x},\bar{b}_i)$ for $i < d$ such that $RM(\psi_i(\bar{x},\bar{b}_i)) = \alpha$ for $i < d$, $M \models \psi_i(\bar{x},\bar{b}_i) \rightarrow \phi(\bar{x},\bar{a})$ for $i < d$, and the $\psi_i(\bar{x},\bar{b}_i)$ are pairwise inconsistent. We call d the Morley degree of $\phi(\bar{x},\bar{a})$, $dM(\phi(\bar{x},\bar{a}))$, and this also depends only on $tp_M(\bar{a})$.

Proposition 5.19 *Let T be countable, complete. Then the following are equivalent.*
(i) T is ω-stable,
(ii) for any $M \models T$ and formula $\phi(\bar{x})$ of L_M, $RM(\phi(\bar{x})) < \infty$.
(iii) For any $\lambda \geq \omega$, T is λ-stable. That is, for any $M \models T$ and subset A of M of cardinality at most λ, $S_1(A,M)$ has cardinality at most λ.

Proof. (i) implies (ii). Let γ be an ordinal, such that for any formula $\phi(\bar{x})$ with parameters from a model M of T, if $RM(\phi(\bar{x})) \geq \gamma$ then $RM(\phi(\bar{x})) = \infty$. ($\gamma$ exists by Exercise 5.16.) Work in an ω-saturated model M of T and work with L_M-formulas. It follows that if $RM(\phi(\bar{x})) = \infty$ then there are $\psi_1(\bar{x}), \psi_2(\bar{x})$ each of which also has $RM = \infty$ and such that

$M \models \neg\exists\bar{x}(\psi_1(\bar{x}) \wedge \psi_2(\bar{x}))$. So we can build a tree of (consistent) formulas $\psi_\eta(\bar{x})$ of formulas of L_M for $\eta \in 2^{<\omega}$, such that if η is an initial segment of τ then $M \models \psi_\tau(\bar{x}) \rightarrow \psi_\eta(\bar{x})$ and such that for each η, $\psi_{\eta 0}(\bar{x})$ and $\psi_{\eta 1}(\bar{x})$ are inconsistent. Let A be the set of parameters from M occuring in the ψ_η. For each $\tau \in 2^\omega$, $\{\psi_{\tau|j}(\bar{x}) : j < \omega\}$ extends to a complete n-type $p_\tau(\bar{x})$ in $S_n(Th(M,a)_{a\in A})$ and for $\tau_1 \neq \tau_2$, $p_{\tau_1} \neq p_{\tau_2}$. So we have 2^ω types over a countable set, contradicting ω-stability.
(ii) implies (iii). Let M be a model of T of cardinality λ. We will show that $S_1(M)(= S_1(Th(M,a)_{a\in M}))$ has cardinality at most λ. For each $p(x) \in S_1(M)$ let $\phi_p(x)$ be a formula in $p(x)$ such that $(RM(\phi(x)), dM(\phi(x)))$ is least possible. (That is first minimize Morley rank then minimize Morley degree).
Claim. Let $p(x) \in S_1(M)$ and let $RM(\phi_p) = \alpha$ and $dM(\phi_p) = d$. Then for any formula $\psi(x)$ of L_M, $\psi(x) \in p(x)$ iff $RM(\phi_p(x) \wedge \psi(x)) = \alpha$ and $dM(\phi_p(x) \wedge \psi(x)) = d$.
Proof of claim. Left implies right is clear. For right implies left; assume the right hand side, and assume for a contradiction that $\psi(x) \notin p$ and thus $\neg\psi(x) \in p(x)$. But then $RM(\phi_p(x) \wedge \neg\psi(x)) = \alpha$ and so $dM(\phi_p(x) \wedge \psi(x)) \geq 1$. But then clearly $dM(\phi_p(x)) \geq d + 1$, a contradiction.

So $p(x) \in S_1(Th(M))$ is "determined by" $\phi_p(x)$ (that is $\phi_p = \phi_q$ implies $p = q$). As there are at most λ many formulas in L_M there are at most
(iii) implies (i) is immediate.

The first application of 5.19 will to be find saturated models.

Lemma 5.20 *Suppose T (countable, complete) is ω-stable. Then for every cardinal κ and regular cardinal $\lambda \leq \kappa$, T has a λ-saturated model of cardinality κ.*

. *Proof.* Let M_0 be a model of T of cardinality κ. By 5.19 (iii), we can build a continuous elementary chain $(M_\alpha : \alpha < \lambda)$ of models of T of cardinality κ such that ALL types in $S_1(M_\alpha)$ are realized in $M_{\alpha+1}$ for all $\alpha < \lambda$. Let M be the union of this chain. So M has cardinality κ and we claim that M is λ-saturated. Let A be a subset of M of cardinality $< \lambda$. As λ is regular $A \subseteq M_\alpha$ for some $\alpha < \lambda$. Any 1-type over A extends to a complete 1-type over M_α which is realized in $M_{\alpha+1}$ so in M.

Corollary 5.21 *Suppose that κ is an uncountable cardinal and that the* *(countable, complete) theory T is κ-categorical. Then T has a saturated model of cardinality κ.*

Proof. By 5.14, T is ω-stable. If κ is regular, then we can apply Lemma 5.20 (taking $\lambda = \kappa$). If κ is NOT regular then in particular κ is a limit cardinal. For each $\lambda < \kappa$, λ^+ is a regular cardinal $< \kappa$, hence by 5.20, T has some model of cardinality κ which is λ^+-saturated. As T has a unique model (say M) of cardinality κ, M is λ^+-saturated for all $\lambda < \kappa$. It follows that M is κ-saturated.

Remark 5.22 *It is actually a theorem that if T is ω-stable then T has a saturated model in every infinite cardinality κ. The proof in the case where κ is singular uses a bit more information about ω-stable theories, such as the rudiments of forking.*

The next fact we need to establish for ω-stable theories is the existence of "constructible" models over any set.

Definition 5.23 *Let M be a structure and A a subset of the universe of M. We will say that M is constructible over A, if we can write the universe of M as $A \cup \{b_\alpha : \alpha < \gamma\}$ for some ordinal γ such that $tp_M(b_\beta / A \cup \{b_\alpha : \alpha < \beta\})$ is isolated, for all $\beta < \gamma$.*

Remark 5.24 *(i) Suppose M is constructible over A. Then for every model $(N, a)_{a \in A}$ of $Th(M, a)_{a \in A}$, M is elementarily embeddable in N over A. That is $(M, a)_{a \in A}$ is a prime model for $Th(M, a)_{a \in A}$.*
(ii) Let M be constructible over A and let $\{b_\alpha : \alpha < \gamma\}$ be as in 5.23. For any given $\beta < \gamma$, $tp(b_\beta / A \cup \{b_\alpha : \alpha < \beta\})$ is isolated by a formula $\phi(x)$ with parameters from $A \cup \{b_\alpha : \alpha < \beta\}$. Now we can write $\phi(x)$ in the form $\phi'(x, \bar{b})$ where $\phi'(x, \bar{y})$ is an L_A-formula, and $\bar{b}$ is a finite tuple from $\{b_\alpha : \alpha < \beta\}$. Note then that $\phi'(x, \bar{b})$ also isolates $tp(b_\beta / A \cup \{\bar{b}\})$. We say that $tp(b_\beta / A \cup \{b_\alpha : \alpha < \beta\})$ is isolated over $A \cup \{\bar{b}\}$.

Lemma 5.25 *Suppose M is constructible over A. Then for each finite tuple $\bar{c}$ from M, $tp_M(\bar{c}/A)$ is isolated. That is, M is atomic over A.*

Proof. Write M as $A \cup \{b_\alpha : \alpha < \gamma\}$ as in Definition 5.23, where we may assume that the $b_\alpha \notin A$ and are all distinct. (Why?) Let $B_\beta = A \cup \{b_\alpha :$

$\alpha < \beta\}$. Let us call a finite sequence $(b_{\beta_1}, .., b_{\beta_n})$ good if, possibly after rearranging the sequence, $\beta_1 < ... < \beta_n$ and for each $i = 1, .., n$, $tp(b_{\beta_i}/B_{\beta_i})$ is isolated over $A \cup \{b_{\beta_1}, .., b_{\beta_{i-1}}\}$. By iterating Exercise 4.25, we see that if $\bar{b}$ is a good sequence, then $tp(\bar{b}/A)$ is isolated.

Claim. Let $\bar{c}$ be a finite tuple from $(b_\alpha : \alpha < \gamma\}$. Then $\bar{c}$ can be extended to a good sequence.
Proof of claim. In fact we will show by induction on $\beta \leq \gamma$, that if $\bar{c}$ is a finite tuple from $B_\beta \setminus A$ then $\bar{c}$ can be extended to a good tuple from $B_\beta \setminus A$. Let $\bar{c}$ be a finite tuple from $B_{\beta+1}$. We may assume that $b_\beta \in \bar{c}$ (otherwi8se use induction hypothesis). Let $\bar{b}$ be such that $tp(b_\beta/B_\beta)$ is isolated over $A \cup \bar{b}$. Let $\bar{c}' = \bar{c} \setminus \{b_\beta\}$ and let $\bar{c}'' = \bar{b}\bar{c}''$. Then $\bar{c}''$ is a finite sequence contained in $B_\beta \setminus A$, so we can apply induction to extend it to a good sequence $\bar{d}$ contained in $B_\beta \setminus A$. Then clearly $(\bar{d}, b_\beta)$ is also a good sequence (contained in $B_{\beta+1}$), and $\bar{d}$ extends $\bar{c}$.

We can now use the claim to prove the lemma. Let $\bar{c}$ be a finite tuple from M. We want to show that $tp(\bar{c}/A)$ is isolated. We may assume that no element of $\bar{c}$ is in A. (Why??) By the claim extend $\bar{c}$ to a good sequence $\bar{c}'$. By a previous remark, $tp(\bar{c}'/A)$ is isolated. Hence so is $tp(\bar{c}/A)$ (by 4.25).

Proposition 5.26 *Let T (countable, complete) be ω-stable. Let $M \models T$ and let $A \subseteq M$. Then there is an elementary substructure N of M which contains A and is constructible over A.*

Proof. If A is already the universe of an elementary substructure of M there is nothing to do. Otherwise, by Tarski-Vaught there is a formula $\phi(x)$ of L_A such that $M \models \exists x\phi(x)$ but $M \models \neg\phi(a)$ for all $a \in A$. Choose such a formula $\phi(x)$ such that $(RM(\phi(x)), dM(\phi(x))) = (\alpha, d)$ is minimized. We claim that $\phi(x)$ isolates a complete type in $S_1(A, M)$. Otherwise there is a formula $\psi(x)$ in L_A such that both $\phi(x) \wedge \psi(x)$ and $\phi(x) \wedge \neg\psi(x)$ are consistent (with $Th(M, a)_{a \in A}$). But then one of $\phi(x) \wedge \psi(x)$, $\phi(x) \wedge \neg\psi(x)$ has lower (RM, dM) than (α, d), a contradiction. Let b_0 realize $\phi(x)$ in M. So $tp(b_0/A)$ is isolated and $b_0 \notin A$. Continue like this to find distinct b_α in M such that $tp(b_\beta/A \cup \{b_\alpha : \alpha < \beta\})$ is isolated. We have to stop some time (as M is a set). At that point we have an elementary substructure of M.

Finally we need to prove the existence of infinite indiscernible sequences in uncountable models of ω-stable theories.

Definition 5.27 *Suppose that T is ω-stable. Let A be a subset of a model M of T. Let $p(\bar{x}) \in S_n(A, M)$. Then $(RM(p), dM(p)) = min\{(RM(\phi(\bar{x})), dM(\phi(\bar{x})) : \phi(\bar{x}) \in p(\bar{x})\}$.*

Let us make explicit something that we used earlier.

Exercise 5.28 *(T ω stable say.) Let M be a model of T and A a subset of M. Let $\phi(x)$ be an L_A-formula with $(RM, dM) = (\alpha, d)$. Then there is at most one type $p(x) \in S_1(A)$ such that $\phi(x) \in p(x)$ and $(RM(p(x)), dM(p(x)) = (\alpha, d)$.*

Proof. Let $p(x) \in S_1(A)$ be such a type. Then note that for any $\psi(x)$ in L_A, $\psi(x) \in p(x)$ if and only if $(RM(\phi(x) \wedge \psi(x)), dM(\phi(x) \wedge \psi(x))) = (\alpha, d)$. So p is uniquely determined.

Lemma 5.29 *Suppose that T is ω-stable. Let M be a model of T and A a subset of the universe of M. Let $\phi(x) \in L_A$ be a formula with $(RM(\phi(x)), dM(\phi(x))) = (\alpha, d)$. Let $(b_i : i < \omega)$ be a sequence of elements of M. Let $p_i(x) = tp_M(b_i/A \cup \{b_j : j < i\})$. Assume that*
(i) $M \models \phi(b_i)$ for all $i < \omega$, and
(ii) $(RM(p_i), dM(p_i)) = (\alpha, d)$ for all $i < \omega$. Then $(b_i : i < \omega)$ is an indiscernible sequence over A.

Proof. We will prove by induction on $n < \omega$ that $tp_M(b_0, .., b_n/A) = tp_M(b_{i_0}, .., b_{i_n}/A)$ whenever $i_0 < ... < i_n$.

Let us first consider the case $n = 0$. Let $i < \omega$. Then $tp(b_i/A)$ has $(RM, dM) \leq (\alpha, d)$ as $\phi(x) \in tp(b_i/A)$. On the other hand $tp(b_i/A) \subseteq p_i(x)$ so by (ii) also has (RM, dM) at most (α, d). So $tp(b_i/A)$ has $(RM, dM) = (\alpha, d)$. The same is true of $tp(b_0/A)$. By (i) and Exercise 5.28, $tp(b_i/A) = tp(b_0/A)$.

Now for the induction step. Fix $n > 0$. Consider $i_0 < .. < i_n < \omega$. Then as above $tp(b_{i_n}/A \cup \{b_{i_0}, .., b_{i_{n-1}}\})$ has $(RM, dM) = (\alpha, d)$, as does p_n. Both these types contain the formula $\phi(x)$. By 5.28, for any L_A-formula $\psi(x, y_0, .., y_{n-1})$, we have
(a) $M \models \psi(b_n, b_0, .., b_{n-1})$ iff $\phi(x) \wedge \psi(x, b_0, .., b_{n-1})$ has $(RM, dM) = (\alpha, d)$.
(b) $M \models \psi(b_{i_n}, b_{i_0}, .., b_{i_{n-1}})$ iff $\phi(x) \wedge \psi(x, b_{i_0}, .., b_{i_{n-1}})$ has $(RM, dM) = (\alpha, d)$.

By induction hypothesis $tp_M(b_0, .., b_{n-1}/A) = tp_M(b_{i_0}, .., b_{i_{n-1}}/A)$. So by 5.16 the right hand sides of (a),(b) are equivalent. So by (a) and (b), $tp_M(b_0, .., b_n/A) = tp_M(b_{i_0}, .., b_{i_n}/A)$, proving the lemma.

Now we are able to obtain the desired consequence of ω-stability.

Proposition 5.30 *Let T be ω-stable. Let M be an model of T of cardinality $\kappa > \omega$, and A a subset of M of cardinality $< \kappa$. Then M contains an infinite indiscernible sequence over A.*

Proof. We may assume A to be infinite. Let $\lambda = |A|$. Note that the formula $x = x$ has $> \lambda$ many realizations in M. Choose a formula $\phi(x)$ in L_M such that $\phi(x)$ has $> \lambda$ many realizations in M and such that $(RM(\phi(x)), dM(\phi(x))) = (\alpha, d)$ say is minimized. (Note $\alpha > 0$.) By adding elements to A we may assume $\phi(x)$ is in L_A. We will construct realizations $b_0, b_1, ...$ of $\phi(x)$ in M such that $tp_M(b_i/A \cup \{b_0, .., b_{i-1}\})$ has $(RM, dM) = (\alpha, d)$.

First we find b_0. Suppose for a contradiction that for every realization c of $\phi(x)$ in M, $(RM(tp(c/A)), dM(tp(c/A))) < (\alpha, d)$. So for each such c this is witnessed by a formula $\psi_c(x)$ in L_A. There are at most λ such formulas. As there are $> \lambda$ possible c', there must be $> \lambda$ many of them with the same $\psi_c(x)$. But then this contradicts the choice of (α, d). So we find b_0.

$b_1, b_2, ...$ are found in precisely the same manner. By 5.29 $(b_i : i < \omega)$ is indiscernible over A.

Exercise 5.31 *Modify the above proof to show that, under the hypothesis of 5.30, M contains an indiscernible sequence (over A) of cardinality κ, assuming κ is regular.*

In any case, Step 2 is complete.

Finally we get to Step 3.

Proposition 5.32 *Suppose that (countable, complete) T is ω-stable, κ is an uncountable cardinal, and every model of T of cardinality κ is saturated. Then every uncountable model of T is saturated.*

Proof. We will show the contrapositive. Assume $\lambda > \omega$ and that M is a non saturated model of T of cardinality λ. So there is a subset A of M of cardinality $< \lambda$ and a type $p(x) \in S_1(A, M)$ which is not realized in M. By Proposition 5.30, let $I = (a_i : i < \omega) \subset M$ be an (infinite) indiscernible sequence over A. Note that that there is no (consistent) formula $\phi(x)$ with parameters from $A \cup I$ such that $M \models \forall x(\phi(x) \to \psi(x))$ for all $\psi(x) \in p(x)$. (For otherwise any realization of $\phi(x)$ in M would realize p.)

That is,
(*) for each consistent formula $\phi(x)$ over $A \cup I$, there is a formula $\psi(x) \in p(x)$ such that $M \models \exists x(\phi(x) \wedge \neg\psi(x))$.

Let A_0 be a countable subset of A. For each consistent $\phi(x)$ over $A_0 \cup I$, pick $\psi_\phi(x) \in p(x)$ given by (*). As $A_0 \cup I$ is countable, there are countably many ψ_ϕ so there is a countable subset A_1 of A which contains A_0 and such that every ψ_ϕ is over A_1. Continue this way to define countable subsets $A_0 \subset A_1 ... \subset A_n ...$ of A. Let A' be the union of the A_i, and let $p'(x) \in S_1(A', M)$ be the restriction of p to A'. So A' is countable, I is indiscernible over A', and
(**) for every consistent formula $\phi(x)$ over $A' \cup I$, there is $\psi(x) \in p'(x)$ such that $M \models \exists x(\phi(x) \wedge \neg\psi(x))$.

By Lemma 5.8, there is a model $(N, a)_{a \in A'}$ of $Th(M, a)_{a \in A'}$ containing a sequence $I' = (b_i : i < \kappa)$ such that I' is indiscernible over A' in N, and such that $tp_N(b_0, .., b_n/A') = tp_M(a_0, ., , , a_n/A')$ for all n. By 5.26, let N' be an elementary substructure of N which contains $A' \cup I'$ and is constructible over $A' \cup I'$. Note that N' is a model of T' of cardinality κ.
Claim. $p'(x)$ is not realized in N'.
Proof. Suppose, for a contradiction that $p'(x)$ is realized by c in N'. By 5.25, $tp_{N'}(c/A' \cup I')$ is isolated, by a formula $\phi(x, b_{i_0}, .., b_{i_n})$ say, where $\phi(x, y_1, .., y_n)$ is in $L_{A'}$, and $i_0 < ... < i_n < \kappa$. So for each $\psi(x) \in p'(x)$, $N' \models \forall x(\phi(x, b_{i_0}, .., b_{i_n}) \rightarrow \psi(x))$. Now $tp_{N'}(b_{i_0}, .., b_{i_n}/A') = tp_M(a_0, .., a_n/A')$. Hence
$M \models \forall x(\phi(x, a_0, .., a_n) \rightarrow \psi(x))$ for all $\psi(x) \in p'(x)$. This contradicts (**), and proves the claim.

So N' is a model of T of cardinality κ which is not saturated (in fact not even ω_1-saturated).

We conclude:

Theorem 5.33 *Suppose T is a countable complete theory which is κ-categorical for some $\kappa > \omega$. Then T is λ-categorical for all $\lambda > \omega$.*

Proof. By 5.14, T is ω-stable. By 5.21, every model of T of cardinality κ is saturated. By Proposition 5.32, for any $\lambda > \omega$ every model of T of cardinality λ is saturated, and thus by 4.5, T has a unique model of cardinality λ.