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Student Logic
Seminar

Language of arithmetic

Logical symbols + "=" + <u>non-logical symbols</u>:

0, S - unary successor f-tion, $+$, $\cdot$, $\leq$ (or $<$)

<u>Numerals</u> n are shorthands for $S^n(0)$.

For weak theories (especially BA) we include unary $\lfloor \frac{1}{2}x \rfloor \left(= \lfloor \frac{x}{2} \rfloor\right)$, unary $|x| \left(= \lceil \log_2(x+1) \rceil\right.$, bit-length$\left.\right)$ and Nelson's binary <u>smash</u> $\#$,

$m \# n = 2^{|m| \cdot |n|}$.

Alternatively to $\#$ one can use unary ω_1, where $\omega_1 = n^{\lfloor \log_2 n \rfloor}$.

$\#$ and ω_1 are equivalent in a sense of growth-rate, i.e.

$\omega_1(n) \approx n \# n$, $m \# n \approx \omega_1(\max\{m, n\})$.

Language of arithmetic cont.

For strong theories it is convinient to enlarge the language by including function symbols for all <u>primitive recursive</u> f-tions.

This can be done by inductively defining the class of p.-r. f-tions as the smallest (under inclusion) class containing 0, S, closed under composition and <u>primitive recursion</u>:

if g-n-arg, h-$n+2$-arg -recursive, then f-$n+1$-arg defined by:

$$f(\bar{x},0)=g(\bar{x})$$

$$f(\bar{x},m+1)=h(\bar{x},m,f(\bar{x},m))$$

is again p.-r. f-tion (defining equations)

A quantifier is <u>bounded</u> if it is of the form $\forall x \leq t$, $\exists x \leq t$, t- does not involve x.

A quantifies is <u>sharply-bounded</u>, if it is of the form $\forall x \leq |t|$, $\exists x \leq |t|$,

Theory is <u>bounded</u> iff axioms are bounded f-las.

Very weak fragments

Theory Q [Tarski, Mostowski, Robinson] has symbols $0, S, +, \cdot$ and axioms:

$$\forall x \; \neg Sx = 0$$

$$\forall x, y \; Sx = Sy \supset x = y$$

$$\forall x \; (x \neq 0) \supset \exists y \; x = Sy$$

$$\forall x \; x + 0 = x$$

$$\forall x, y \; x + Sy = S(x+y)$$

$$\forall x \; x \cdot 0 = 0$$

$$\forall x, y \; x \cdot Sy = x \cdot y + x$$

Q doesn't contain $\leq$, but it can be defined by:

$$x \leq y \leftrightarrow \exists z \; x + z = y.$$

$Q_{\leq}$ is a conservative extension of Q by adding $\leq$ + its defining axiom.

Very weak fragments cont.

Q is very weak.

$Q \nvdash \forall x,y\ x+y=y+x$

and

$Q \nvdash \forall x,y\ x\cdot y=y\cdot x.$

Although, Q is strong enough to prove every true Σ_1-sentence (defined later)

Theory R [Tarski, Mostowski, Robinson]

same language as Q, infinite set of axioms:

$$S^m 0 \neq S^n 0 \quad \forall m \neq n$$

$$S^m 0 + S^n 0 = S^{m+n} 0 \quad \forall m,n$$

$$S^m 0 \cdot S^n 0 = S^{m\cdot n} 0 \quad \forall m,n$$

$$\forall x\ (x \leq S^m 0 \vee S^m 0 \leq x) \quad \forall m$$

$$\forall x\ (x \leq S^m 0 \leftrightarrow x=0 \vee \ldots \vee x = S^m 0) \quad \forall m.$$

where $s \leq t$ abbreviates $\exists z\,(s+z=t)$

It holds, that $Q \vdash R$.

and

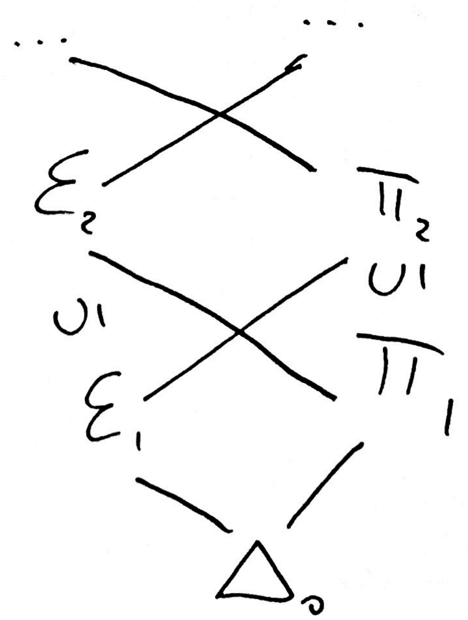
$R \nvdash Q$ (why?)

Strong fragments.

Def: A formula is __bounded__ iff all quantifiers are bounded. The set of bounded f-las is denoted by Δ_0. We then define inductively:

1. $\Sigma_0 = \Pi_0 = \Delta_0$,
2. Σ_{n+1} is (the tautological closure of) the set of f-las of the form $\exists \bar{x} A$, where $A \in \Pi_n$,
3. Π_{n+1} is (the tautological closure of) the set of f-las of the form $\forall \bar{x} A$, where $A \in \Sigma_n$

Classes Σ_n, Π_n form the __arithmetic hierarchy__.



, note that
$\Sigma_i \subseteq \Sigma_{i+1}, \Pi_{i+1}$,
$\Pi_i \subseteq \Sigma_{i+1}, \Pi_{i+1}$.

Strong fragments cont.

Def: Induction axiom scheme for a class Φ is defined as a set of axioms (Φ-IND):

$$A(0) \land \forall x (A(x) \supset A(S(x))) \supset \forall x A(x),$$

for all $A \in \Phi$. Note that A can contain more free variables.

The least number principle (= minimisation) for a class Φ is defined as (Φ-MIN):

$$\exists x A(x) \supset \exists x (A(x) \land \neg \exists y (y < x \land A(y))),$$

for all $A \in \Phi$.

The collection principle (= replacement) for a class Φ is defined as (Φ-REPL):

$$[\forall x \leq t \, \exists y \, A(x,y)] \supset [\exists z \, \forall x \leq t \, \exists y \leq z \, A(x,y)]$$

Def: $PA = Q + \text{ALL-IND}$.

Hierarchy

<u>Def</u>: $I\Sigma_n = Q_{\leq} + \Sigma_n\text{-IND}$.

$I\Pi_n = Q_{\leq} + \Pi_n\text{-IND}$.

of particular importance is $I\Delta_0 = Q_{\leq} + \Delta_0\text{-IND}$.

$L\Sigma_n / L\Pi_n = I\Delta_0 + \Sigma_n\text{-MIN} / \Pi_n\text{-MIN}$

$B\Sigma_n / B\Pi_n = I\Delta_0 + \Sigma_n\text{-REPL} / \Pi_n\text{-REPL}$.

It can be shown [Parsons, Paris, Kirby]:

$$
\begin{array}{c}
I\Sigma_{n+1} \\
\Downarrow \\
B\Sigma_{n+1} \Longleftrightarrow B\Pi_n \\
\Downarrow \\
I\Sigma_n \Longleftrightarrow I\Pi_n \Longleftrightarrow L\Sigma_n \Longleftrightarrow L\Pi_n,
\end{array}
$$

where $\Longrightarrow$ denotes logical implication provably one side.

We'll eventually see all those proofs.

Hierarchy
cont.

There is an alternative definition of the previously-defined hierarchy.

<u>Def.</u> We define Σ_n^+, Π_n^+ inductively:

① $\Sigma_0^+ = \Pi_0^+ = \Delta_0$,

② Σ_{n+1}^+ is the class of formulas obtained from Π_n^+ by prepending them with an arbitrary block of existential quantifiers and <u>bounded universal quantifier</u>:

$$\forall \bar{x} \leq t \, \exists \bar{y} \, \Pi_n^+ \text{ is } \Sigma_n^+$$

③ Π_{n+1}^+ is defined analogously by prepending Σ_n^+ with universal quantifiers and <u>bounded existential quantifiers</u>:

$$\exists \bar{x} \leq t \, \forall \bar{y} \, \Sigma_n^+ \text{ is } \Pi_{n+1}^+ .$$

Hierarchy end.

Let $\forall \bar{x} \leq t\, \exists \bar{y}\, \varphi$ be Σ^+_{n+1}, thus φ is Π^+_n.

Assuming Σ_n-REPL we derive:

$$\underbrace{\exists z\, \underbrace{\forall \bar{x} \leq t\, \underbrace{\exists \bar{y} \leq z\, \varphi}_{\Pi_n^+ \text{ by induction is } \Pi_n}}_{\Pi_n}}_{\Sigma_n}.$$

So, Σ_n-REPL proves $\Sigma_n = \Sigma_n^+$
($= \Pi_n$-REPL)

and since $I\Sigma_n \vdash B\Sigma_n$,

Σ_n is no different from Σ_n^+ over $I\Sigma_n\ (= I\Pi_n)$.

Bootstrapping $I\Delta_0$

(a) addition is commutative: $\forall x,y\ x+y=y+x$

- First, prove $\forall x\ 0+x=x$ by induction:
  - $0+0=0$
  - $0+S(x)=S(0+x)=S(x)$

- second, prove $\forall x,y\ Sx+y=S(x+y)$ by induction on y:
  - $Sx+0=Sx=S(x+0)$
  - $Sx+Sy=S(Sx+y)=SS(x+y)=\ldots$

    $\ldots=S(x+Sy)$

- finally, prove $\forall x,y\ x+y=y+x$ by induction on x:

  - $0+y=y=y+0$

  - $S(x)+y=S(x+y)=S(y+x)=y+S(x)$

(b) addition is associative:

$\forall x,y,z : (x+y)+z = x+(y+z)$

similarly as above (induction w.r.t. x)
- $(0+y)+z=y+z=0+(y+z)$
- $(S(x)+y)+z=S(x+y)+z=S((x+y)+z)=S(x+(y+z))=\ldots$

$=S(x)+(y+z)$

Bootstrapping $I\Delta_0$
cond.

(c) multiplication is commutative $\forall x,y\ xy=yx$

- first, prove $0\cdot x=0$:
  - $0\cdot 0=0$
  - $0\cdot S(x)=0\cdot x+0=0+0=0$
- second, prove $S(x)\cdot y=x\cdot y+y$:
  - $S(x)\cdot 0=0=x\cdot 0+0$
  - $S(x)\cdot S(y)=S(x)\cdot y+S(x)=\dots$

$\dots=x\cdot y+y+S(x)=x\cdot y+S(x)+y=\dots$

$\dots=x\cdot y+S(y)+x=S(x)\cdot y+S(y))$

- Finally, prove $x\cdot y=y\cdot x$
  - $0\cdot y=0=y\cdot 0$
  - $S(x)\cdot y=x\cdot y+y=y\cdot x+y=y\cdot S(x)$

(d) distributive law: $\forall x,y,z:(x+y)\cdot z=x\cdot z+y\cdot z$

- $(0+y)z=yz=\cancel{0}z+yz$
- $(S(x)+y)z=S(x+y)z=(x+y)z+z=\dots$

$\dots=x\cdot z+y\cdot z+z=x\cdot z+z+y\cdot z=S(x)\cdot z+y\cdot z$

(e) multiplication is associative $\forall x,y,z:(x\cdot y)z=x(y\cdot z)$.

similar as above.

Bootrapping $I\Delta_0$
cont.

Similarly as above one shows:

(f) $\forall x,y,z: x+z=y+z \leftrightarrow x=y$ ... cancellation

(g) $\forall x,y: x\leq Sy \supset x\leq y \vee x=Sy$ ... discreteness

(h) $\forall x,y,z: (x\leq y \wedge y\leq z) \supset x\leq z$ .. transitivity

(i) $\forall x,y: x+y=0 \supset (x=0 \wedge y=0)$ ... anti-idempotency

(j) $x\leq x$, $x\leq y \vee y\leq x$, $(x\leq y \wedge y\leq x) \supset x=y$
reflexivity, trichotomy, antisymmetry

(k) $\forall x,y,z: z\neq 0 \wedge x\cdot z=y\cdot z \supset x=y$ ... cancellation

(l)
We define $<$ as a shorthand for $x\leq y \wedge x\neq y$.

Thm: $I\Delta_0 \vdash \Delta_0$-MIN.

Pr.: let $A(x)$ - Δ_0-f-la. The least number principle for $A(x)$ is equivalent to the complete induction on $\neg A(x)$:

$\forall y([\forall z<y\ \neg A(z)] \supset \neg A(y)) \supset \forall x\ \neg A(x)$.

This, in turn, is equivalent to the usual induction on the bounded f-la $\forall y\leq x\ \neg A(y)$, which is provable in $I\Delta_0$

Provably recursive f-tions

__Def__: A predicate symbol $R(\bar{x})$ is said to be __Δ_0-defined__ iff it has a __defining axiom__:

$$R(\bar{x}) \leftrightarrow \varphi(\bar{x}),$$

where $\varphi(\bar{x})$ - Δ_0-f-la with all free variables indicated.

The predicate R is __Δ_1-defined__ by a theory T, iff there are Σ_1-f-las $\varphi(\bar{x})$ and $\psi(\bar{x})$ s.t. R has defining axiom:

$$R(\bar{x}) \leftrightarrow \varphi(\bar{x}),$$

and $$T \vdash \forall \bar{x} (\underbrace{\varphi(\bar{x})}_{\Sigma_1} \leftrightarrow \underbrace{\neg\psi(\bar{x})}_{\Pi_1}).$$

__Def__: Let T be a theory. A function symbol f is __Σ_1-defined__ by T, iff it has a defining axiom:

$$y = f(\vec{x}) \leftrightarrow \varphi(\bar{x}, y),$$

for $\varphi \in \Sigma_1$, and

$$T \vdash \forall \bar{x} \exists! y\, \varphi(\bar{x}, y).$$

Provably recursive
f-tions
cont.

Note that $\forall \bar{x}\, \exists! y\, \varphi(\bar{x}, y) \overset{\text{def.}}{\iff}$

$$\forall \bar{x}\, \exists y\, \varphi(\bar{x}, y) \wedge \forall \bar{x}\, \forall y_1, y_2 \left(\neg\varphi(\bar{x}, y_1) \vee \neg\varphi(\bar{x}, y_2) \vee y_1 = y_2\right),$$

which is a Π_2-sentence.

Σ_1-definable f-tions are called <u>provably recursive</u> or <u>provably total f-tions</u> of the theory T.

This is justified by the following:

Let M be a TM computing some f-tion $M(x) = y$.

Let us code TM computations so that we may form an arithmetic predicate $T_M(x, w, y)$ expressing "w encodes a computation of M on x which outputs y".

It will be shown later that such T can be made into a bounded f-la.

Thus, any p-r. f-tion f is defined by a true Σ_1 sentence: $\forall x\, \exists! y\, \exists w\, T_M(x, w, y)$

Parikh theorem

Conversely, for any true sentence $\forall \bar{x} \exists! y\, \varphi(\bar{x}, y)$, where $\varphi(\bar{x}, y) \in \Sigma_1$ one may create a TM M which given $\bar{x}$ finds y satisfying the sentence above.

For weak theories something even stronger holds:

<u>Thm [Parikh]</u>: Let $A(\bar{x}, y)$ be bounded f-la and T be a bounded theory.
Suppose $T \vdash \forall \bar{x} \exists y\, A(\bar{x}, y)$. Then, there is a term t s.t. $T \vdash \forall \bar{x} \exists y \leq t\, A(\bar{x}, y)$, where t does not depend on ~~A~~ y.

The above can be generalized to a vector of existentially quantified variables.

$I\Delta_0$ is bounded, since one can replace
$x \leq y \leftrightarrow \exists z : x + z = y$ by $x \leq y \leftrightarrow \exists z \leq y : x + z = y$
and the induction axioms can be replaced by:

$\forall z \left[A(0) \wedge \forall x \leq z (A(x) \supset A(Sx)) \supset A(z) \right]$.

Parikh thm.
cond.

<u>Thm</u>: a f-tion symbol $f(\bar{x})$ is Σ_1-defined by $I\Delta_0$ iff it has a defining axiom:

$$y = f(\bar{x}) \Longleftrightarrow \varphi(\bar{x}, y),$$

where $\varphi(\bar{x}, y)$ is Δ_0! and there exists a term $t(\bar{x})$ s.t.

$$I\Delta_0 \vdash \forall \bar{x}\, \exists! y \leq t(\bar{x})\, \varphi(\bar{x}, y)).$$

By $\exists! y \leq t(\bar{x})$ we mean $\exists y \leq t(\bar{x}) \wedge \forall \underset{\substack{\cup \\ y \leq t(\bar{x})}}{z}\, \varphi(\bar{x}, y)$.

A predicate symbol R is Δ_1-defined by $I\Delta_0$ iff it is Δ_0-defined by $I\Delta_0$. Moreover, $I\Delta_0$ proves the equivalence of the above definitions.

<u>Pr.</u> $I\Delta_0 \vdash \forall \bar{x}\, \exists y\, \underbrace{\varphi(\bar{x}, y)}_{\Sigma_1} \Rightarrow I\Delta_0 \vdash \forall \bar{x}\, \exists y\, \exists \bar{u}\, \underbrace{\psi(\bar{x}, \bar{u}, y)}_{\Delta_0}$

$\xRightarrow{\text{Parikh}} I\Delta_0 \vdash \forall \bar{x}\, \exists y \leq t(\bar{x})\, \underbrace{\exists \bar{u} \leq s(\bar{x})\, \psi(\bar{x}, \bar{u}, y)}_{\Delta_0}$.

For the second part, let φ and ψ be defining f-(s). Then, $I\Delta_0 \vdash \forall \bar{x}\, (\exists \bar{z}\, \underbrace{\varphi'(\bar{x}, \bar{z})}_{\Delta_0} \Leftrightarrow \forall \bar{u}\, \underbrace{\psi'(\bar{x}, \bar{u})}_{\Delta_0})$

$\Rightarrow I\Delta_0 \vdash \forall \bar{x}\, (\exists \bar{z}\, \exists \bar{u}\, \varphi'(\bar{x}, \bar{z}) \vee \neg \psi(\bar{x}, \bar{u}))$