Σ_1 - completeness of Q

<u>bounded quantifiers</u>: if $\alpha(\bar{x}, y)$ is a fla and $t(\bar{x})$ a term (without y) then:

$$\exists y \leq t(\bar{x})\, \alpha(\bar{x}, y) \underset{\text{df.}}{\iff} \exists y\, (y \leq t(\bar{x}) \wedge \alpha(\bar{x}, y))$$

$$\forall y \leq t(\bar{x})\, \alpha(\bar{x}, y) \underset{\text{df.}}{\iff} \forall y\, (y \leq t(\bar{x}) \rightarrow \alpha(\bar{x}, y))$$

<u>bounded flas ($= \Delta_0$-f'las)</u> $:\underset{\text{df.}}{=}$

L_{PA}-f'las using only bounded quantifiers.

Δ_0: the class of bounded flas: the smallest class of f'las containing all atomic flas, closed under $\wedge, \vee, \neg$, and under $\exists^{\leq}, \forall^{\leq}$.

<u>Ex</u>: "x is a prime": $\text{Prime}(x) \iff$

$$s(s(0)) \leq x \wedge \forall y, z \leq x, (y \cdot z = x \rightarrow (y = 1 \vee z = 1))$$

Σ_1-flas: $\mathcal{L}_{PA}$-flas of the form

$\alpha(\bar{x}) \rightleftharpoons \exists \bar{y}\, \beta(\bar{x}, \bar{y})$, where

$\beta \in \Delta_0$.

Σ_1: " ~~then~~ there is a prime bigger than x":

$\exists y,\ x<y \wedge \mathrm{Prime}(y)$.

Numerals: particular closed $\mathcal{L}_{PA}$-terms used to denote specific numbers

$s_0 := 0$

$s_1 := s(0)$

$s_{k+1} := s(s_k)$.

I.e.

$s_n = \underbrace{s(s(\cdots\cdots s(0)\cdots)}_{n\text{-times.}}$

Lemma:

(a) For any term $t(x_1, \dots, x_k)$, any $n_1, \dots, n_k \in \mathbb{N}$ and $m = t(n_1, \dots, n_k)$,

$$Q \vdash S_m = t(S_{n_1}, \dots, S_{n_k}).$$

(b) For any $m \in \mathbb{N}$,

$$Q \vdash x \leq S_m \iff (x = 0 \lor x = S_1 \lor \dots \lor x = S_m).$$

Prf:

(a) By induction on the nb. of symbols in t, using ax's of Q about $+$ and $\cdot$ (ax's Q4 – Q7). For example:

$$S_3 + S_2 \underset{Q5}{=} S(S_3 + S_1) \underset{Q5}{=} S(S(S_3 + 0)) \underset{Q4}{=} S(S(S_3)) = S_5.$$

(b) ($\Leftarrow$) is obvious as Q proves $S_n \leq S_m$, if $n \leq m$: use Q9 and Q11.

($\Rightarrow$) By ind. on m.

Case 1: $m = 0$. Then $x \leq S_m$ by Q9.

Case 2: $x \neq 0$. By Q3 $\exists y$ s.t. $S(y) = x$.

So $S(y) \leq S_n$ and by Q11 $y \leq S_{n-1}$.

By ind. assumption $y = 0 \vee \ldots \vee y = S_{n-1}$,

so $x = S_1 \vee \ldots \vee x = S_n$.

Cases 1 & 2 yield statement (3).

$\square$

Theorem (Σ_1-completeness of Q)

For any Σ_1-fla $\varphi(x_1, \ldots, x_k)$ and any $n_1, \ldots, n_k \in \mathbb{N}$: if $\mathbb{N} \models \varphi(n_1, \ldots, n_k)$ then $Q \vdash \varphi(S_{n_1}, \ldots, S_{n_k})$.

Proof: We first prove the statement for Δ_0-flas. By De Morgan rules we may assume that $\neg$ is only in front of atomic flas. Proceed by induction on the complexity of the formula.

Case 1 : $\alpha(\bar{x})$ is a basic ($\pm$ atomic) fla.

(a) $t(\bar{x}) = s(\bar{x})$. If $t(n_1, \dots, n_k) = s(n_1, \dots, n_k) = m$

then by Lemma (a),

$$G \vdash t(s_{n_1}, \dots) = s_m = s(s_{n_1}, \dots).$$

(b) $t(\bar{x}) \neq s(\bar{x})$: if $t(n_1, \dots) = u \neq v = s(n_1, \dots)$

then $G \vdash t(s_{n_1}, \dots) = s_u \wedge s(s_{n_1}, \dots) = s_v$

and using G10 also $G \vdash s_u \neq s_v$.

(c) $t(\bar{x}) \leq s(\bar{x})$ } Analogous cases, using

(d) $t(\bar{x}) \not\leq s(\bar{x})$ } G 9-11.

Case 2 : $\alpha(\bar{x}) = \beta(\bar{x}) \wedge \gamma(\bar{x})$.

$\mathbb{N} \models \alpha(n_1, \dots) \iff \mathbb{N} \models \beta(n_1, \dots)$ and $\mathbb{N} \models \gamma(n_1, \dots)$

$\Downarrow$ $\Downarrow$

$\underbrace{G \vdash \beta(s_{n_1}, \dots) \text{ and } G \vdash \gamma(s_{n_1}, \dots)}$

$\Downarrow$

$$G \vdash \alpha(s_{n_1}, \dots).$$

Case 3 : $\alpha = \beta \vee \gamma$: analogous

Case 4 : $\alpha(\bar{x}) = \exists y \leq t(\bar{x}) \beta(\bar{x}, y)$.

If $m = t(u_1, \ldots, u_k)$, $\mathbb{N} \models \exists y \leq m \beta(u_1, \ldots, y)$

and $\exists y$ is witnessed by w:

$$\mathbb{N} \models w \leq m \wedge \beta(u_1, \ldots, w)$$

Then, by ind. hypothesis:

$$Q \vdash s_w \leq s_m = t(s_{u_1}, \ldots)$$

$$Q \vdash \beta(s_{u_1}, \ldots, s_w)$$

(i.e. (Case 2): and $\exists$-quant) $Q \vdash \alpha(s_{u_1}, \ldots)$.

Case 5 : $\alpha(\bar{x}) = \forall y \leq t(\bar{x}) \beta(\bar{x}, y)$.

Assume $m = t(u_1, \ldots)$, $\mathbb{N} \models \alpha(u_1, \ldots, u_k)$.

We have (Lemma (a) + (b)): that $Q \vdash$:

$$\begin{cases} t(s_{u_1}, \ldots) = s_m \\ y \leq s_m \iff (y = s_0 \vee \ldots \vee y = s_m). \\ \beta(s_{u_1}, \ldots, s_{u_k}, s_a), \text{ for all } a = 0, \ldots, m. \end{cases}$$

This together yields $Q \vdash \alpha(s_{u_1}, \ldots)$.

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It remains to treat the unbounded $\exists$ quantifiers at the beginning of a general Σ_1-fla. But that is completely analogous to the treatment of $\exists^{\leq}$ in Case 4.

$\Box_{\text{thm.}}$

<u>Remark</u> The key to the negative solution of Hilbert's 10^{th} problem is a theorem of Matijasevič-Robinson-Davis-Putnam that any Σ_1-fla is over $\mathbb{N}$ equivalent to a purely existential fla: $\exists \bar{y}\, \alpha(\bar{x}, \bar{y})$, α open.

[Over $\mathbb{Z}$ it has even the form of a Diophantine equation:

$$\exists \bar{y}\ p(\bar{x}, \bar{y}) = 0 \quad .]$$