Δ_0-definable functions

We may want to use in formulas for convenience some function $f(x_1, \dots, x_k)$; for example:

$\text{rem}(x, y) :=$ "the remainder of x modulo y".

But using it we do not want to leave Δ_0-formulas; i.e. we want $\Delta_0 = \Delta_0(f)$.

The following two conditions will suffice:

(i) there is a term of L_{PA}, $t(\bar{x})$, s.t. $\mathbb{N} \models \forall \bar{x},\ f(\bar{x}) \leq t(\bar{x})$,

(ii) there is a Δ_0-fla $\theta_f(\bar{x}, y)$ that defines in $\mathbb{N}$ the graph of f:

$$\mathbb{N} \models \forall \bar{x}, y,\ f(\bar{x}) = y \iff \theta_f(\bar{x}, y).$$

Lemma: Assume f satisfies both (i) + (ii).
Then any $\Delta_0(f)$-formula (i.e. bounded fla in the language $\mathcal{L}_{PA} \cup \{f\}$) is equivalent in $\mathbb{N}$ to a Δ_0-fla.

Prf: The idea is that any statement $\alpha(f(t_1', \dots, t_k'))$ can be replaced by $\exists y \leq t(t_1', \dots, t_k'), [\Theta_f(t_1', \dots, t_k', y) \wedge \alpha(y)]$

In fact, this can be equivalently written using $\forall^{\leq}$:

$\forall y \leq t(t_1', \dots, t_k'), [\Theta_f(t_1', \dots, t_k', y) \to \alpha(y)]$.

Formally one removes occurrences of f one by one, always taking one $f(t_1', \dots, t_k')$ s.t. no term t_i' contains f.

$\square$

Σ_1's

(i) $\mathrm{rem}(x,y) = z \iff (\exists u \leq x;\ x = u \cdot y + z) \wedge z < y$

(ii) coding pairs:

$$\langle x, y \rangle := \frac{(x+y)(x+y+1)}{2} + x$$

satisfies:

$$\langle x, y \rangle \leq (x+y+1)^2$$

$$\langle x, y \rangle = z \iff z + z = (x+y)(x+y+1) + 2x$$

(here $1 := s(0)$).

(iii) Therefore $\langle\ ,\ \rangle$ to code triples and k-tuples, any fixed $k \geq 2$:

$$\langle x, y, z \rangle := \langle x, \langle y, z \rangle \rangle$$

$\vdots$