# Free-cut elimination

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# Intro

- we will prove that the cut-elimination theorem applies in a special form to the case when we use extra non-logical axioms from some set $\mathfrak{G}$
- $\mathfrak{G}$ is assumed to be closed under substitution
- in general, cuts on formulas that have direct ancestors in some non-logical initial sequent cannot be removed, but other cuts can

# Free cuts

## Definition

- a formula in an $LK_{\mathfrak{G}}$-proof is *anchored* if it is a direct descendant of some formula in an initial $\mathfrak{G}$-sequent
- a cut inference on $\varphi$ is *anchored* if $\varphi$ is atomic and both its occurrences in the premises are anchored, or $\varphi$ is not atomic and at least one of its occurrences in the premises is anchored
- a cut inference is *free* if it is not anchored
- $P$ is *free-cut free* if it contains no free cuts

# The procedure

- we use induction on the maximum depth of free cuts in an $LK_{\mathfrak{G}}$-proof $P$ - we have to ensure that anchored cuts do not change into free cuts during the procedure
- hence we have to assume that no cut-formula in $P$ is *only weakly introduced*, i.e. at least one direct ancestor of every cut-formula is in an initial sequent or it is a principal formula of a strong inference

# Why the additional assumption?

- if $A$ is only weakly introduced in the right premise of the following cut inference
$$\frac{\Gamma \to \Delta, A \qquad A, \Gamma \to \Delta}{\Gamma \to \Delta}$$
then, to eliminate this cut, it makes sense to delete all direct ancestors of $A$ in the right sub-proof
- but if a formula $B \in \Gamma$ is anchored in the left premise but not anchored in right premise, the same $B$ becomes only weakly introduced in the resulting proof
- specifically, it becomes unanchored, and if it is later used as a cut-formula, the respective cut inference now becomes free

# The theorem

**Theorem (Free-cut elimination)**

*Let $\mathfrak{G}$ be a set of sequents closed under substitution.*

1. *If $LK_{\mathfrak{G}} \vdash \Gamma \to \Delta$, then there is a free-cut free $LK_{\mathfrak{G}}$-proof of $\Gamma \to \Delta$*
2. *Let $P$ be an $LK_{\mathfrak{G}}$-proof such that no cut-formula is only weakly introduced in $P$. Assume further that every free cut in $P$ has depth less than or equal to $d$. Then there is a free-cut free proof $P^*$ of the same endsequent such that*

$$\|P^*\| \leq 2_{2d+2}^{\|P\|}$$

# The central lemma

**Lemma**

*Let $P$ be an $LK_{\mathfrak{G}}$-proof ending with a free cut of the highest depth $d$ among other free cuts in $P$. Then there is an $LK_{\mathfrak{G}}$-proof $P^*$ of the same endsequent such that every free cut in $P^*$ has depth strictly less than $d$ and with $\|P^*\| < \|P\|^2$.*

*Furthermore, every formula occurring in the endsequent of $P$ which was anchored by an $\mathfrak{G}$-sequent remains anchored in $P^*$, and every formula in the endsequent of $P^*$ which is only weakly introduced was already only weakly introduced in $P$.*

# Proof - cases

Suppose $P$ ends with a topmost maximal-depth free cut

$$\dfrac{\begin{array}{c} Q \\ \vdots \\ \Gamma \to \Delta, A \end{array} \qquad \begin{array}{c} R \\ \vdots \\ A, \Gamma \to \Delta \end{array}}{\Gamma \to \Delta}$$

We distinguish cases according whether $A$ is atomic or not, and if not, what is its outermost connective, $\wedge, \vee$ or $\supset$. For the other connectives, $\neg, \exists$ or $\forall$ the standard argument works.

# Proof - $A$ is atomic

- assume w.l.o.g. that $A$ is not anchored in $R$
- since $A$ is not only weakly introduced, it has a direct ancestor in a logical initial sequent $A \to A$
- replace every sequent $\Pi \to \Lambda$ in $R$ by $\Pi^-, \Gamma \to \Delta, \Lambda$, where $\Pi^-$ is $\Pi$ minus all direct ancestors of $A$
- all inferences remain valid, but the leaves may no longer be initial sequents: those initial sequents containing a direct ancestor of $A$ now become $\Gamma \to \Delta, A$
- since this sequent is provable by $Q$, replace all its occurrences by the sub-proof $Q$

# Proof - $A$ is $B \vee C$

- if the sequent $\Gamma \to \Delta, B \vee C$ is derivable by $Q$, the sequent $\Gamma \to \Delta, B, C$ is also derivable by a non-greater proof $Q^*$ (same argument as in the standard cut elimination)
- form $R'$ from $R$ by replacing every sequent $\Pi \to \Lambda$ in $R$ with $\Pi^-, \Gamma \to \Delta, \Lambda$ by deleting all direct ancestors of $A$ and adding $\Gamma$ and $\Delta$ to the respective cedents
- $R'$ is an invalid proof, every L$\vee$ inference from $R$ with $B \vee C$ principal becomes

$$\frac{B, \Pi^-, \Gamma \to \Delta, \Lambda \qquad C, \Pi^-, \Gamma \to \Delta, \Lambda}{\Pi^-, \Gamma \to \Delta, \Lambda}$$

# Proof - $A$ is $B \vee C$ (cont.)

- but we can combine the valid sub-proofs of the premises of $R'$ with $Q^*[\Pi^-, \Lambda]$ to obtain a valid proof of the sequent $\Pi^-, \Gamma \to \Delta, \Lambda$ by the following transformation

$$\dfrac{\dfrac{\Pi^-, \Gamma \to \Delta, \Lambda, B, C \qquad C, \Pi^-, \Gamma \to \Delta, \Lambda}{\Pi^-, \Gamma \to \Delta, \Lambda, \mathrm{B}} \qquad B, \Pi^-, \Gamma \to \Delta, \Lambda}{\Pi^-, \Gamma \to \Delta, \Lambda}$$

- we thus fix $R'$ and obtain a valid proof $R^*$ of $\Gamma, \Gamma \to \Delta, \Delta$, so the result follows by a series of contraction applications
- this transformation is done at every L$\vee$ inference from $R$ with $C \vee D$ principal, and so we get

$$\|P^*\| \leq \|R\| \times (\|Q\| + 1) < \|P\|^2$$

## Proof - $A$ is $B \to C$

- use a proof $Q^*$ of $B, \Gamma \to \Delta, C$ with $\|Q^*\| \leq \|Q\|$ and form $R'$ from $R$ by adding $\Gamma, \Delta$ to the respective cedents of all sequents in $R$ and deleting all direct ancestors of $A$
- replace every incorrect L$\to$ inference $I$ in $R$ of the form

$$\frac{\Pi^-, \Gamma \to \Delta, \Lambda, B \qquad C, \Pi^-, \Gamma \to \Delta, \Lambda}{\Pi^-, \Gamma \to \Delta, \Lambda}$$

by the combination of $Q^*[\Pi^-, \Lambda]$ and the valid sub-proofs of $I$:

$$\frac{\dfrac{\Pi^-, \Gamma \to \Delta, \Lambda, B \qquad B, \Pi^-, \Gamma \to \Delta, \Lambda, C}{\Pi^-, \Gamma \to \Delta, \Lambda, C} \qquad C, \Pi^-, \Gamma \to \Delta, \Lambda}{\Pi^-, \Gamma \to \Delta, \Lambda}$$

# What was different from the proof for $LK$?

- the overall structure of the algorithm is the same, double induction on the (depth, height) of the *free* cuts in $P$
- but now we do not discard entire sub-proofs when we 'fix' invalid binary inferences during the procedure
- this way anchored formulas in the endsequents remain anchored during the transformations and the same holds for those formulas that were not only weakly introduced

**e.g.**

In the proof for $LK$ in the $\vee$-case Buss forms two intermediate trees $R_1$ and $R_2$ from $R$ whose fixing requires the deletion of entire sub-proofs, while in the proof for $LK_{\mathfrak{S}}$ he forms an intermediate tree $R'$ which is then locally 'repaired' from the top to the bottom in such a way that no sub-proofs are deleted.

# Induction

- we want to formalize induction in the sequent calculus so that we can apply the free-cut elimination theorem to theories where induction is restricted to certain classes of formulas
- we use induction rules instead of induction axioms:

$$\frac{A(b), \Gamma \to \Delta, A(b+1)}{A(0), \Gamma \to \Delta, A(t)}$$

The variable $b$ works as an eigenvariable, $t$ is an arbitrary term.

# Examples of theories with induction

- Robinson's arithmetic $Q$ (basic properties of the successor function, addition and multiplication) together with induction for $\Sigma_n$ formulas form a theory called $I\Sigma_n$
- in the sequent calculus, $Q$ is formalized as additional initial sequents and for $I\Sigma_n$ the induction rule above is restricted to $\Sigma_n$ formulas
- another important theories (to which we will probably get next semester) are fragments of bounded arithmetic $T_2^n$ and $S_2^n$

# Extending some definitions

**Notation**

If $\Phi$ is a set of formulas, $T + \Phi$-IND denotes a theory $T$ together with the above induction rules restricted to $\Phi$

**Definition**

- for a sequent calculus proof $P$ in an arithmetic theory $\mathfrak{G} + \Phi$-IND take the *principal formulas* of an induction inference to be $A(0)$ and $A(t)$
- an occurrence of a formula in $P$ is *anchored* if it is a direct descendent of a formula in an initial sequent from $\mathfrak{G}$ or a direct descendent of a principal formula of an induction inference.
- the notions of an anchored and a free cut are defined as above

# Free-cut elimination for theories with induction

**Theorem (Free-cut elimination for theories with induction)**

*If $T$ is some theory of arithmetic $\mathfrak{G} + \Phi$-IND with $\mathfrak{G}$ and $\Phi$ closed under term substitution and if $\Gamma \rightarrow \Delta$ follows from $T$, then there is a free-cut free $T$-proof of $\Gamma \rightarrow \Delta$. Moreover, the previous upper bounds also apply here.*

**Corollary**

*If $T$ and $\mathfrak{G}$ are as before, $\Phi$ is closed under term substitution and under subformulas, $\mathfrak{G}$-sequents only contain formulas from $\Phi$, $\Gamma \rightarrow \Delta$ is a logical consequence of $T$ and every formula in $\Gamma \rightarrow \Delta$ is in $\Phi$, then there is a $T$-proof $P$ of $\Gamma \rightarrow \Delta$ such that every formula in $P$ is in $\Phi$.*