LECTURE 6

towards $\boxed{\text{Ajtai's thm}}$

RECALL:

LANG.: $0, 1, \neg, \vee$ (binary)

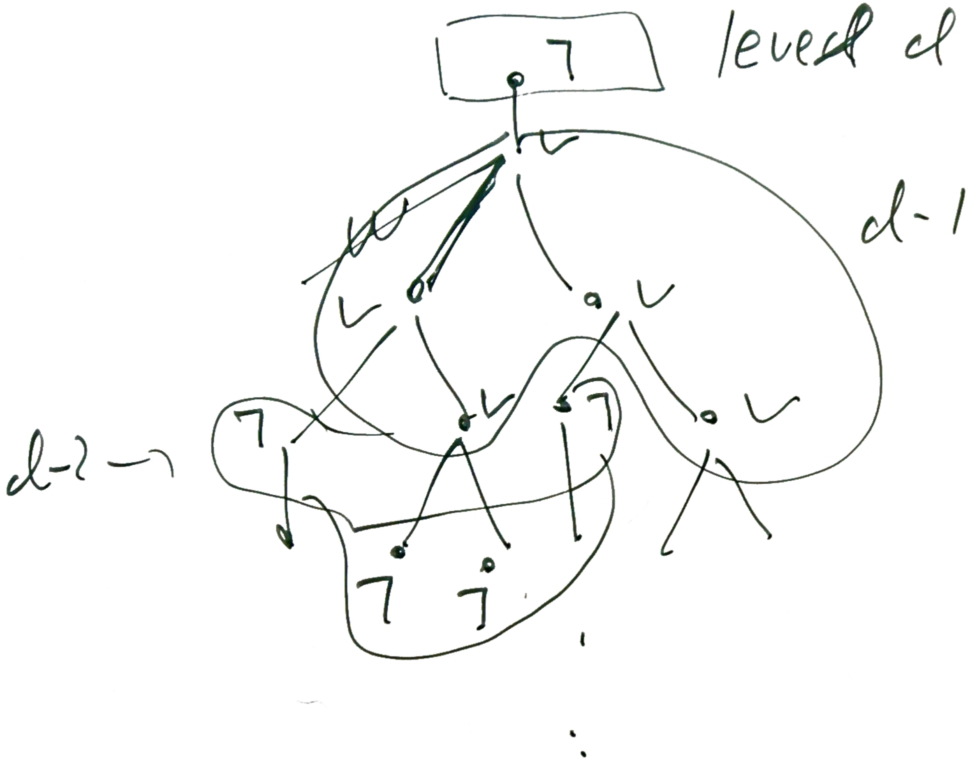
$\wedge$ .... defined

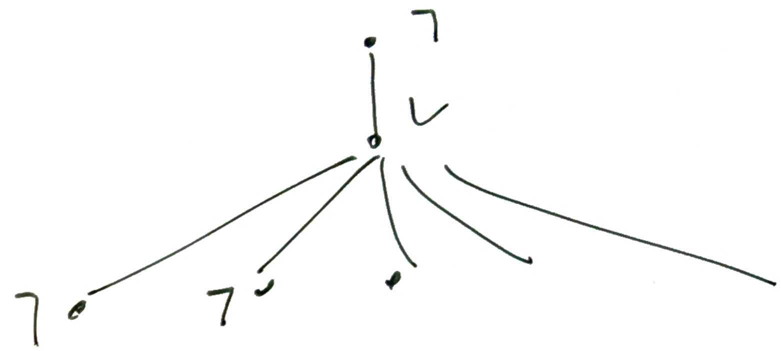
F : A FREGE SYSTEM in

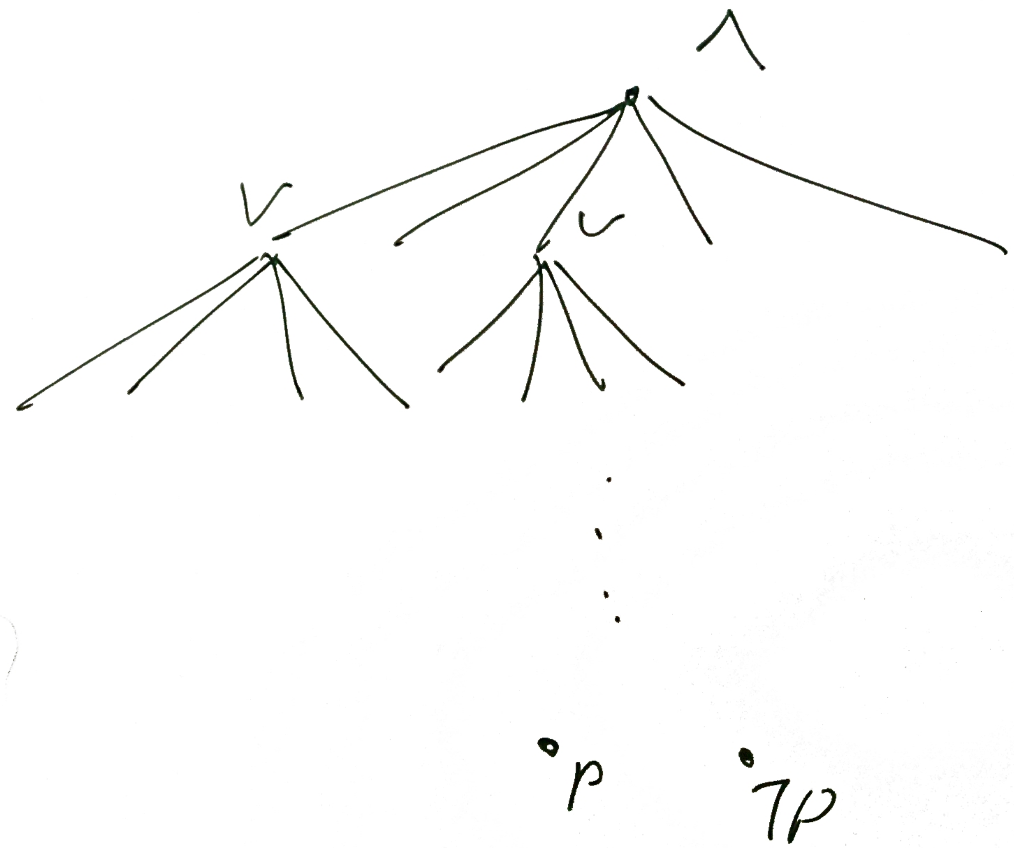
$dp(\varphi)$: the depth of φ

F_d : F using only $\leq d$ FORMULAS

DEPTH


level d
d-1
d-2




p
¬p

FACT 1: $\exists d_0 \geq 0$ F_{d_0} is COMPLETE FOR REFUTATIONS OF UNSAT SETS OF CLAUSES.

FACT 2: PROVING ontoPHP_n (= DNF) is $\underset{p\text{-size}}{\approx}$ REFUTING CLAUSES

$\boxed{\neg \text{onto}PHP_n}$

Tonto PHP_n : p_{ij} , $i \in [n+1]$, $j \in [n]$

- $\bigvee_j p_{ij}$, one for all i
- $\neg p_{i_1 j} \vee \neg p_{i_2 j}$, $i_1 \neq i_2$, j

$\Big\}$ UNSAT

- $\neg p_{i j_1} \vee \neg p_{i j_2}$, i , $j_1 \neq j_2$ FUNCTION
- $\bigvee_i p_{ij}$ INJECTIVITY

$dp \leq 2$

$\boxed{\text{Thm (Ajtai)}}$ - Thm 15.3.1

$\forall d \geq 2\ \forall c \geq 1\ \forall n >> 0$: ANY F_d-REFUTATION OF onto PHP_n MUST HAVE THE SIZE $> n^c$.

In fact, $\forall d \geq 2\ \exists \delta > 0\ \forall n >> 0$:

...... MUST CONTAIN $\geq 2^{n^\delta}$ DIFFERENT SUBFLAS ($\Rightarrow$ SIZE $\geq 2^{n^\delta}$ TOO)

[In fact: K-Pudlák-Woods

Beame-Impagliazzo-Pitassi]

IN LECT. 5 :

- SIZE LOWER B. FOR R^*-REFUT'S

$\approx$ <u>HEIGHT</u> L.B. FOR <u>CLAUSE-DEC. TREES</u>

(USED SPIRA'S L)

$h \geq n$

WORKS FOR <u>k-DNF-DEC. TREES</u> TOO :

$h \geq n/k$

$D_1 \vee \ldots \vee D_t = ?$

term $|D_i| \leq k$

GENERAL STRATEGY:

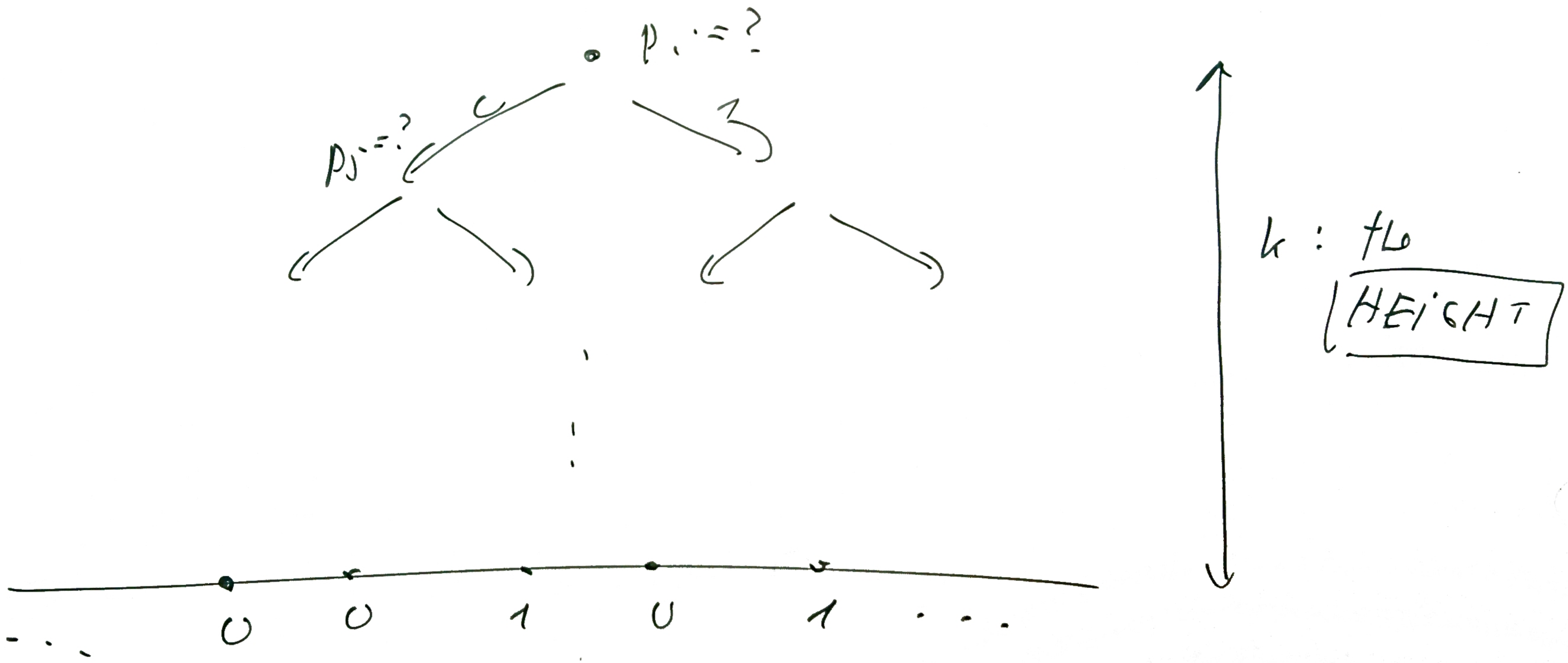
- REDUCE ALL FLAS TO k-DNFs, small k

  [HOW? THIS IS THE TECHNICAL HIGH POINT]

- ARGUE ANALOGOUSLY AS FOR CLAUSE-DEC. TREES

  [THIS IS CONCEPTUAL HIGH POINT]

DEC. TREE DEFINING $\varphi(p_1, \ldots, p_n)$:



$\boxed{k\text{-TREE}}$

T.

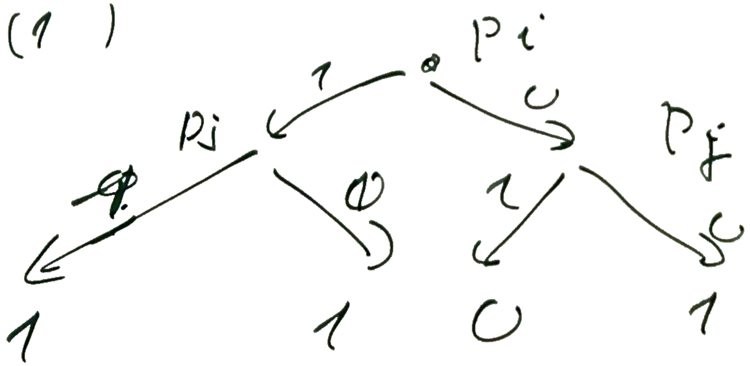
L1.9.5

LEMMA:

(1) IF φ CAN BE DEFINED BY k-TREE $\Rightarrow$ φ and $\neg\varphi$ ARE k-DNFs

(2) IF $\varphi, \neg\varphi$ ARE BOTH ℓ-DNFs THEN φ CAN BE DEFINED BY ℓ^2-TREE.

PRF: (1)

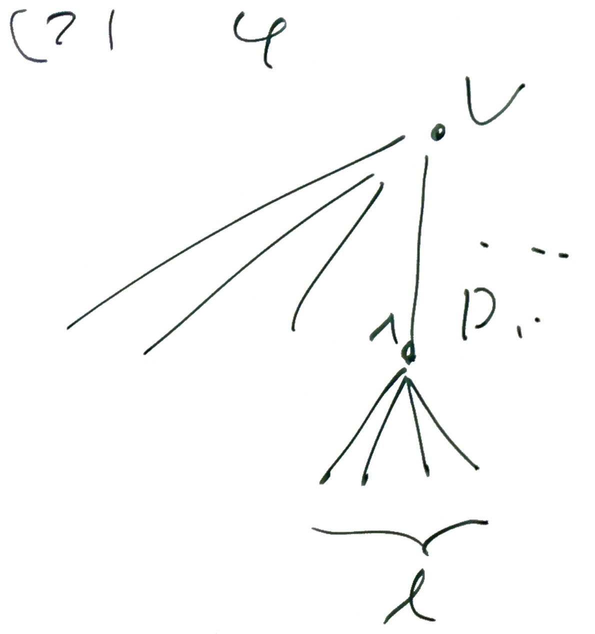


$\varphi \Leftrightarrow$

$\sim (p_i p_j) \vee (p_i \wedge \neg p_j) \vee (p_i \wedge \neg p_j)$

CORRESPOND TO PATHS ENDING WITH 1

w



<u>CLAIM</u>: EACH D_i HAS TO CONFLICT WITH ~~EACH~~ E_j

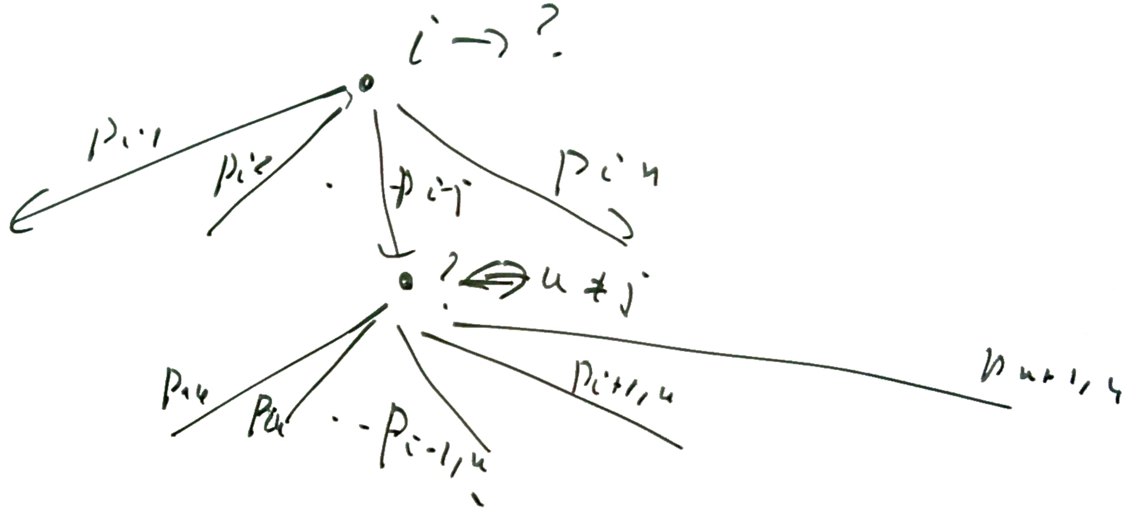
<u>TREE</u>: - ASK FOR ALL ATOM IN SOME D_i (ℓ Qs)

- THIS EITHER KILLS E_j OR MAKES ≥ 1 literal TRUE

- REPEAT ℓ-times

□

PHP - trees



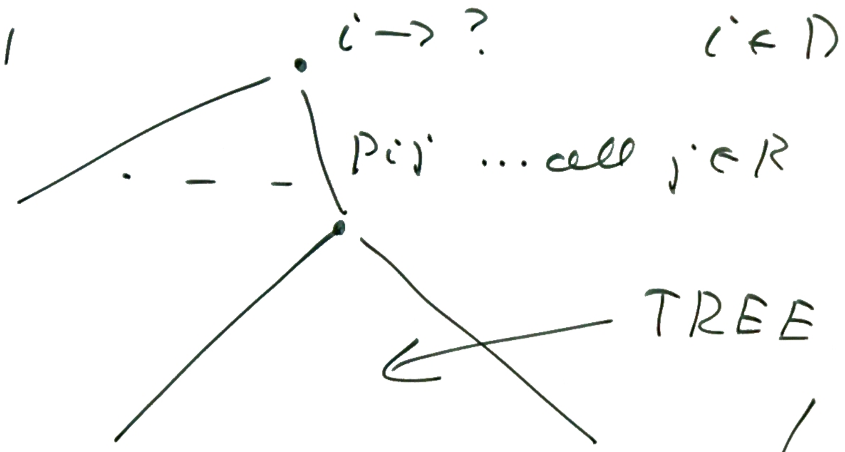
Qs :

$i \to ?$

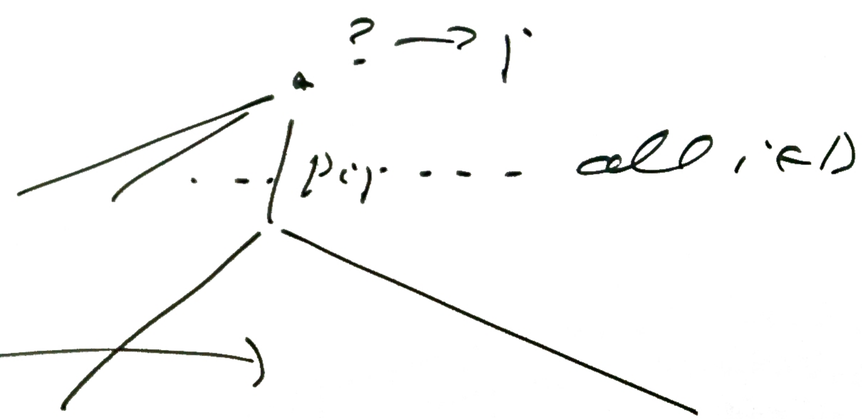
$? \to u$

NO CONFLICTS ON PATHS

DEF. : PHP-TREE OVER $D \subseteq [n+1]$, $R \subseteq [n]$

(i) 

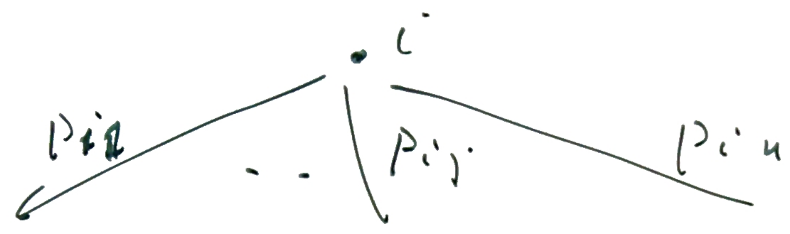
TREE OVER $D \setminus \{i\}$, $R \setminus \{j\}$

(ii) 

(iii) $h(T)$ = HEIGHT = MAX NO. OF EDGES ON A PATH

(iv) k-TREE = PHP-TREE OVER $[n+1]$, $[n]$ of $h(T) \leq k$

Ex:

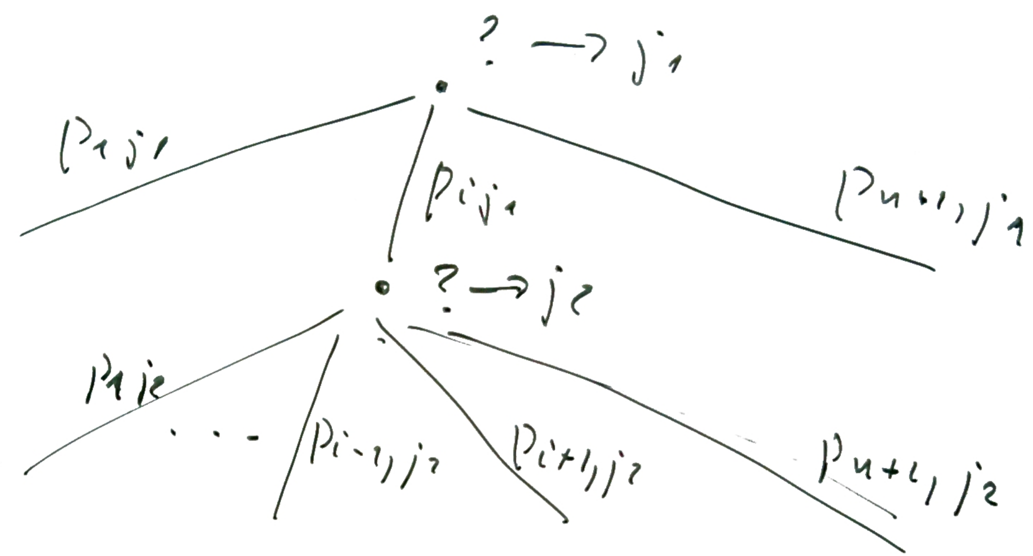


IF THE TREE DESCRIBED "ALL SITUATIONS" $\Longrightarrow$ $\bigvee_j p_{ij}$ IS "TRUE"

↑

WE SHALL MAKE THIS FORMAL

Ex:



CLAUSES $\neg p_{ij_1} \vee \neg p_{ij_2}$ is "TRUE" W.R.T.

THIS TREE

$\mathit{maps} := \text{all } \alpha : \subseteq [n+1] \xrightarrow[\text{1-to-1}]{} [n]$

$\alpha \cup \beta \in \mathit{maps} \overset{=}{\text{def}} \alpha \| \beta$ (α, β COMPATIBLE)

$\alpha \perp \beta \iff \alpha \cup \beta \notin \mathit{maps}$ (α, β CONTRADICTORY)

$\alpha \iff$ CONJUNCTION OF ALL p_{ij}, $(i,j) \in \alpha$

$$H \subseteq \mathit{maps} \iff \bigvee_{\alpha \in H} \alpha$$ ... A DNF

T PHP-tree , $H \subseteq$ maps

$T \rhd H$ (T REFINES H)

$\Updownarrow$

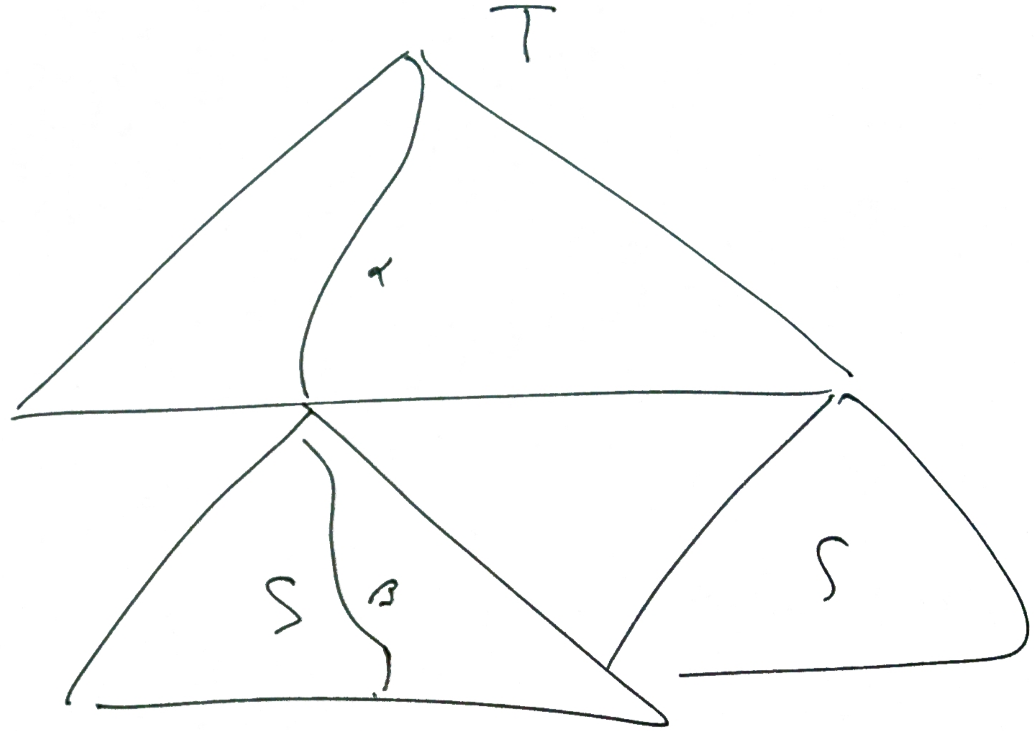
$\forall \alpha \in T : (\exists \beta \in H, \beta \| \alpha) \Rightarrow (\exists \gamma \in H, \gamma \subseteq \alpha)$

α a path in T

$T \Leftrightarrow$ SET OF ALL COMPLETE PATHS IN IT

T, S TREES

$T \times S$: A COMMON REFINEMENT



ONLY : $\alpha \| \beta$

S TREE, $H \subseteq$ Maps

$S(H)$: THE PROJECTION OF H ON S

$:= \{ \alpha \in S \mid \exists \gamma \in H, \gamma \subseteq \alpha \}$.

$H, K, \ldots$: $\subseteq$ Maps

$S, T, \ldots$: PHP-TREES

LEMMA X = LEMMA 15.1.X IN THE BOOK

LEMMA 1: $|\delta| + h(S) \leq n \Rightarrow \exists \rho \in S_i \; \rho \parallel \delta$.

PRF: WALK THROUGH S AND ANSWER CONSISTENTLY WITH δ.

$\Rightarrow$ IMPLIES THAT THIS IS ALWAYS POSSIBLE.

$\square$

LEMMA 2: $h(S) + h(T) \le h$, $H \lhd S \lhd T \Rightarrow H \lhd T$

PRF: $\alpha \in H$, $\cancel{\beta}\delta \in T$; $\alpha \parallel \cancel{\beta}\delta$:

BY L1: $\cancel{\exists \beta \in S;\ \beta \parallel \delta}$ ~~AND AS~~ $S \lhd T$

$\left.\begin{array}{l}\exists \gamma' \in S,\ \gamma' \parallel \delta \\ S \lhd T\end{array}\right\} \Rightarrow \exists \gamma \in S,\ \gamma \le \delta$

NECESSARILY $\alpha \parallel \gamma$

$H \lhd S$

$\Leftarrow$

$\exists \alpha' \in H,\ \alpha' \le \gamma$

$\Rightarrow$ $\alpha' \le \delta$ TOO $\square$

L3 : $h(S) + h(T) \leq n \implies S * T$ is TREE AND
$h(S*T) \leq h(S) + h(T)$

AND $S \lhd S*T,\ T \lhd S*T.$ $\square$

L4 : $h(S) + h(T) \leq n,\ H \lhd S \lhd T \implies$

(i) $T(S(H)) = T(H)$

(ii) $T(S) = T$

(iii) $S(H) = S \iff T(H) = T$ $\square$

L5: (i) $S(\bigcup_i H_i) = \bigcup_i S(H_i)$

(ii) $H_0, H_1 \subseteq T$, $H_0 \cap H_1 = \emptyset$

$\Rightarrow \quad T(H_0) \cap T(H_1) = \emptyset$.

(iii) $S \lhd T$, $h(S) + h(T) \leq n$, $H \subseteq S$

$\Rightarrow T(S \setminus H) = T \setminus T(H)$.

$\square$

KEY NOTION:

- $1 \leq k \leq n$: A PARAMETER (WILL BE $\sim n^{\varepsilon}$)
- Γ : A SET OF FLAS IN ATOMS p_{ij} CLOSED UNDER SUBFLAS

SET-UP $\to$ φ

DEF.: $\boxed{k\text{-EVALUATION OF } \Pi}$

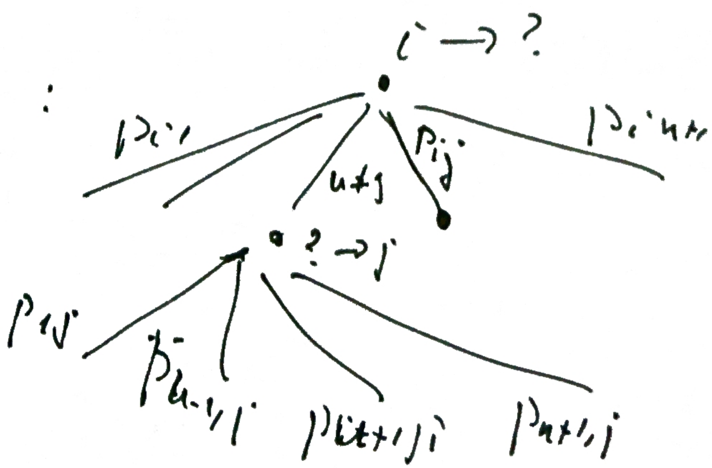
MAP : $(H, S) : \varphi \in \Gamma \longrightarrow H_\varphi \subseteq S_\varphi$.

S_φ a PHP-TREE S.T. : $h(S_\varphi) \leq k$

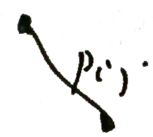
(1) $S_0 = S_1 = \{\emptyset\}$ • root

$H_0 = \emptyset$, $H_1 = S_1$

(2) S_{pij} :



H_{pij}



(3) $S_{\neg\varphi} = S_\varphi$, $H_{\neg\varphi} := S_\varphi \setminus H_\varphi$

(4) $\varphi = \bigcup_i \cdot \psi_i \in \Gamma$

$\bigcup_i \cdot H_{\psi_i} \lhd S_\varphi$ . . . . NOT UNIQUELY SPECIFIED.

$H_\varphi := S_\varphi \left(\bigcup_i \cdot H_{\psi_i} \right)$

$\square$

KEY CONCEPT

DEF: (H, S) 4-eval. of Π

φ is TRUE W.R.T. (H, S)

$\Updownarrow$ df.

$S_\varphi = H_\varphi$

I.E. φ HOLDS IN ALL SITUATIONS "DESCRIBED BY S_φ"

LEMMA 6: $\Gamma \supseteq$ all subflas of all clauses of $\neg$OntoPHP_n (binary $\vee$!)

(H,S) $\bar{k}$-evol. of Γ, $k \leq n-2$

$\Longrightarrow$ ALL CLAUSES OF $\neg$OntoPHP_n ARE TRUE W.R.T. (H,S).

PRF - EX: $\varphi : \bigvee_j p_{ij}$

$H_{p_{ij}} = \{(i,j)\}$ BY DEF.

IF $S_\varphi \rhd \bigcup_j \{(i,j)\} = \{(i,j) \mid j \in [n]\} =: H$

1-tree $H(H) = H$

$\Downarrow$

FOR $T := S_\varphi \times H$ (L4)

$T(H) = T$

$\Downarrow$ L4 again

$S_\varphi = S_\varphi(H) = H_\varphi$. $\square$

LEMMA 7 : FOR ANY FREGE SYSTEM F $\exists c_F \geq 1$

$\forall (H, S)$ k-eval. of Π, $k \leq n/c_F$

Π : contains all subflas of an INSTANCE (∗) OF AN F-RULE.

IF : ALL HYPOTHESES OF (∗) ARE (H,S)–TRUE

THEN: THE CONCLUSION OF (∗) IS (H,S)–TRUE AS WELL.

PRF-IDEA: T: A COMMON REFIN. OF ALL S_φ, φ PLAYS A RULE IN THE INSTANCE

$\Rightarrow$ $\varphi \to T(H_\varphi)$ is a HOMO INTO A BOOL. ALG.

$\Rightarrow$ HAS TO PRESERVE CLASSICAL LOGIC.

PRF-EX :

$$\frac{\varphi \qquad \neg\varphi \vee \psi}{\psi}$$

Γ: all subfms of $\neg\varphi \vee \psi, \varphi, \psi, \ldots$

ASSUME $H_\varphi = S_\varphi$, $H_{\neg\varphi\vee\psi} = S_{\neg\varphi\vee\psi}$

$T := S_\varphi \times S_{\neg\varphi\vee\psi} \times S_\psi$

$\Rightarrow$ $T(H_\varphi) = T$, $T(H_{\neg\varphi\vee\psi}) = T$

$T(H_{\neg\varphi}) = \emptyset$, $T(H_{\neg\varphi\vee\psi}) = T(H_{\neg\varphi}) \cup T(H_\psi)$

$= T \setminus T(H_\varphi) = 0$ $\qquad = \emptyset \cup \underbrace{T(H_\psi)}\ = T$

$\square$

- READ DETAILS IN SEC. 15.1

  ↓

  ELEMENTARY BUT WITHOUT CHECKING THEM IT IS HARD TO UNDERSTAND THAT IT WORKS

- KEY ISSUE: DOES ANY k-EVAL. ACTUALLY EXIST?

[NEXT TIME]