Recapitulation

By Fagin’s theorem, a class of finite structures is definable in existential second-order logic if, and only if, it is in NP.

It is an open question whether there is similarly a logic for P.

A precise formulation asks for a recursive enumeration of polynomially-clocked Turing machines that are isomorphism-invariant.

This is equivalent to the question of whether there is a problem in P that is complete under first-order reductions.

A logic for P would be intermediate, in expressive power, between first-order logic and second-order logic.
P-complete Problems

If there is any problem that is complete for $P$ with respect to first-order reductions, then there is a logic for $P$.

If $Q$ is such a problem, we form, for each $k$, a quantifier $Q^k$.

The sentence

$$Q^k(\pi_U, \pi_1, \ldots, \pi_s)$$

for a $k$-ary interpretation $\pi = (\pi_U, \pi_1, \ldots, \pi_s)$ is defined to be true on a structure $A$ just in case

$$\pi(A) \in Q.$$

The collection of such sentences is then a logic for $P$. 
Conversely,

**Theorem**
If the polynomial time properties of graphs are recursively indexable, there is a problem complete for $P$ under first-order reductions.

(D. 1995)

**Proof Idea:**
Given a recursive indexing $((M_i, p_i) | i \in \omega)$ of $P$

Encode the following problem into a class of finite structures:

$$\{(i, x) | M_i \text{ accepts } x \text{ in time bounded by } p_i(|x|)\}$$

To ensure that this problem is still in $P$, we need to pad the input to have length $p_i(|x|)$. 
Constructing the Complete Problem

Suppose $M$ is a machine which on input $i \in \omega$ gives a pair $(M_i, p_i)$ as in the definition of recursive indexing. Let $g$ a recursive bound on the running time of $M$.

$Q$ is a class of structures over the signature $(V, E, \preceq, I)$.

$A = (A, V, E, \preceq, I)$ is in $Q$ if, and only if,

1. $\preceq$ is a linear pre-order on $A$;
2. if $a, b \in I$, $a \preceq b$ and $b \preceq a$, i.e. $I$ picks out one equivalence class from the pre-order (say the $i^{th}$);
3. $|A| \geq p_i(|V|)$;
4. the graph $(V, E)$ is accepted by $M_i$; and
5. $g(i) \leq |A|$.
Summary

The following are equivalent:

- \( P \) is recursively indexable.
- There is a logic capturing \( P \) of the form \( \text{FO}(\mathcal{Q}) \), where \( \mathcal{Q} \) is the collection of vectorisations of a single quantifier.
- There is a complete problem in \( P \) under first-order reductions.

Another way of viewing this result is as a dichotomy.

*Either* there is a single problem in \( P \) such that all problems in \( P \) are easy variations of it

*or*, there is no reasonable classification of the problems in \( P \).
Inductive Definitions

Let $\varphi(R, x_1, \ldots, x_k)$ be a first-order formula in the vocabulary $\sigma \cup \{R\}$

Associate an operator $\Phi$ on a given $\sigma$-structure $A$:

$$\Phi(R^A) = \{a \mid (A, R^A, a) \models \varphi(R, x)\}$$

We define the \textit{non-dereasing} sequence of relations on $A$:

$$\Phi^0 = \emptyset$$

$$\Phi^{m+1} = \Phi^m \cup \Phi(\Phi^m)$$

The \textit{inflationary fixed point} of $\Phi$ is the limit of this sequence.

On a structure with $n$ elements, the limit is reached after at most $n^k$ stages.
The logic **IFP** is formed by closing first-order logic under the rule:

If $\varphi$ is a formula of vocabulary $\sigma \cup \{R\}$ then $[\text{ifp}_{R,x} \varphi](t)$ is a formula of vocabulary $\sigma$.

The formula is read as:

the tuple $t$ is in the inflationary fixed point of the operator defined by $\varphi$

**LFP** is the similar logic obtained using *least fixed points* of *monotone* operators defined by *positive* formulas.

**LFP** and **IFP** have the same expressive power ([Gurevich-Shelah 1986; Kreutzer 2004]).
Transitive Closure

The formula

$$\exists p_{T,xy}(x = y \lor \exists z(E(x, z) \land T(z, y)))](u, v)$$

defines the transitive closure of the relation $E$

The expressive power of IFP properly extends that of first-order logic.

On structures which come equipped with a linear order IFP expresses exactly the properties that are in $P$.

(Immerman; Vardi 1982)
Immerman-Vardi Theorem

∃< ∃State_1 · · · State_q ∃Head ∃Tape

< is a linear order ∧

\[ State_1(t + 1) \rightarrow State_i(t) \lor \ldots \]
\[ \land State_2(t + 1) \rightarrow \ldots \] encoding
\[ \land Tape(t + 1, p) \leftrightarrow Head(t, p) \ldots \] transitions
\[ \land Head(t + 1, h + 1) \leftrightarrow \ldots \] of M
\[ \land Head(t + 1, h - 1) \leftrightarrow \ldots \]

∧at time 0 the tape contains a description of A
∧State(max, s) for some accepting s

With a deterministic machine, the relations State, Tape and Head can be define

inductively.
IFP vs. Ptime

The order cannot be built up inductively.

It is an open question whether a canonical string representation of a structure can be constructed in polynomial-time.

If it can, there is a logic for $P$.

If not, then $P \neq NP$.

All $P$ classes of structures can be expressed by a sentence of IFP with $<$, which is invariant under the choice of order. The set of all such sentences is not r.e.

IFP by itself is too weak to express all properties in $P$.

Evenness is not definable in IFP.
Recursive Indexability

Say that a formula $\varphi$ of IFP in the vocabulary $\sigma \cup \{<\}$ is *order-invariant* if, for any $\sigma$-structure $\mathbb{A}$ and any two linear orders $<_1$ and $<_2$ of its universe,

$$\mathbb{A}, <_1 \models \varphi \text{ if, and only if, } \mathbb{A}, <_2 \models \varphi$$

Then, the following are equivalent:

- $P$ is recursively indexable.

- There is an *r.e.* set $S$ of sentences of IFP so that
  - every sentence in $S$ is order-invariant; *and*
  - every order-invariant sentence of IFP has an equivalent sentence in $S$.

Taking $S$ to be the collection of sentences that do not mention $<$ is insufficient.
Finite Variable Logic

We write $L^k$ for the first order formulas using only the variables $x_1, \ldots, x_k$.

$$(A, a) \equiv^k (B, b)$$

denotes that there is no formula $\varphi$ of $L^k$ such that $A \models \varphi[a]$ and $B \not\models \varphi[b]$.

If $\varphi(R, x)$ has $k$ variables all together, then each of the relations in the sequence:

$$\Phi^0 = \emptyset; \quad \Phi^{m+1} = \Phi^m \cup \Phi(\Phi^m)$$

is definable in $L^{2k}$.

Proof by induction, using substitution and renaming of bound variables.
Pebble Game

The $k$-pebble game is played on two structures $A$ and $B$, by two players—\textit{Spoiler} and \textit{Duplicator}—using $k$ pairs of pebbles $\{(a_1, b_1), \ldots, (a_k, b_k)\}$.

\textit{Spoiler} moves by picking a pebble and placing it on an element ($a_i$ on an element of $A$ or $b_i$ on an element of $B$).

\textit{Duplicator} responds by picking the matching pebble and placing it on an element of the other structure.

\textit{Spoiler} wins at any stage if the partial map from $A$ to $B$ defined by the pebble pairs is not a partial isomorphism.

If \textit{Duplicator} has a winning strategy for $q$ moves, then $A$ and $B$ agree on all sentences of $L^k$ of quantifier rank at most $q$.\hfill (Barwise)

$A \equiv^k B$ if, for every $q$, \textit{Duplicator} wins the $q$ round, $k$-pebble game on $A$ and $B$.

Equivalently (on finite structures) \textit{Duplicator} has a strategy to play forever.
Evenness

To show that *Evenness* is not definable in IFP, it suffices to show that:

for every \( k \), there are structures \( A_k \) and \( B_k \) such that \( A_k \) has an even number of elements, \( B_k \) has an odd number of elements and

\[
A \equiv^k B.
\]

It is easily seen that *Duplicator* has a strategy to play forever when one structure is a set containing \( k \) elements (and no other relations) and the other structure has \( k + 1 \) elements.
P-Complete Problems

It is easily seen that IFP can express some P-complete problems such as *Alternating Transitive Closure* (ATC).

\[
[\text{ifp}_{R,x}(x = t \lor (D(x) \land \exists y(E(x,y) \land R(y))) \lor (C(x) \land \forall y(E(x,y) \rightarrow R(y))))](s)
\]

We can conclude that IFP is *not* closed under AC\textsubscript{0}-reductions.

We can also conclude that ATC is not P-complete under FO-reductions.

It can be shown that ATC is complete for IFP under FO-reductions.

There is a P-complete problem under FO-reductions *if, and only if*, there is one under IFP-reductions.
Fixed-point Logic with Counting

Immerman proposed IFP + C—the extension of IFP with a mechanism for counting.

Two sorts of variables:

- $x_1, x_2, \ldots$ range over $|A|$—the domain of the structure;
- $\nu_1, \nu_2, \ldots$ which range over non-negative integers.

If $\varphi(x)$ is a formula with free variable $x$, then $\#x\varphi$ is a term denoting the number of elements of $A$ that satisfy $\varphi$.

We have arithmetic operations ($+, \times$) on number terms.

Quantification over number variables is bounded: $(\exists x < t) \varphi$
Counting Quantifiers

$C^k$ is the logic obtained from first-order logic by allowing:

- allowing *counting quantifiers*: $\exists^i x \varphi$; and
- only the variables $x_1, \ldots, x_k$.

Every formula of $C^k$ is equivalent to a formula of first-order logic, albeit one with more variables.

For every sentence $\varphi$ of IFP + C, there is a $k$ such that if $A \equiv^{C^k} B$, then

$$A \models \varphi \text{ if, and only if, } B \models \varphi.$$
Counting Game

Immerman and Lander (1990) defined a pebble game for $C^k$.

This is again played by Spoiler and Duplicator using $k$ pairs of pebbles $\{(a_1, b_1), \ldots, (a_k, b_k)\}$.

At each move, Spoiler picks a subset of the universe (say $X \subseteq B$)

Duplicator responds with a subset of the other structure (say $Y \subseteq A$) of the same size.

Spoiler then places a $b_i$ pebble on an element of $Y$ and Duplicator must place $a_i$ on an element of $X$.

Spoiler wins at any stage if the partial map from $A$ to $B$ defined by the pebble pairs is not a partial isomorphism

If Duplicator has a winning strategy for $q$ moves, then $A$ and $B$ agree on all sentences of $C^k$ of quantifier rank at most $q$. 
Cai-Fürer-Immerman Graphs

There are polynomial-time decidable properties of graphs that are not definable in IFP + C.  
\[ \text{(Cai, Fürer, Immerman, 1992)} \]

More precisely, we can construct a sequence of pairs of graphs \( G_k, H_k \) \((k \in \omega)\) such that:

- \( G_k \equiv_C^k H_k \) for all \( k \).
- There is a polynomial time decidable class of graphs that includes all \( G_k \) and excludes all \( H_k \).

Still, IFP + C is a natural level of expressiveness within P.
Summary

IFP + C is a logic that extends first-order logic with *inflationary fixed-points* and *counting*.

It forms a natural expressivity class *properly* contained in P.

It captures all of P on many natural classes of graphs.

There are P properties that are not in IFP + C.

*Note:* If there is a P-complete problem under IFP + C-reductions, then there is a logic for P.