Exercises in Stochastic Processes I

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Chapter 1

Sums of count random variables

1.1 Generating functions

Definition 1.1.1. Let $\{a_n\}_{n\in\mathbb{N}_0}$ be a sequence of real numbers. The function

$$A(s) = \sum_{n=0}^{\infty} a_n s^n,$$

defined for all $s \in \mathbb{R}$ for which the sum converges absolutely, is called a generating function (GF) of the sequence $\{a_n\}_{n\in\mathbb{N}_0}$.

Definition 1.1.2. Let X be a non-negative integer-valued (i.e. count) random variable. Its probability distribution is given by the probabilities $\{p_n\}_{n\in\mathbb{N}_0}$ where $p_n = \mathbb{P}(X = n)$. The generating function of the sequence $\{p_n\}_{n\in\mathbb{N}_0}$ is called a probability generating function (PGF).

The definition of a PGF $P_X(s)$ of a count random variable X can be rewritten as follows:

$$P_X(s) = \mathbb{E}s^X$$

The following theorem summarizes basic facts about probability generating functions.

Theorem 1.1.1. Let X be a count random variable, P_X its probability generating function with the radius of convergence R_X. Then the following holds:

1. $R_X \geq 1$. For all $|s| < R_X$ the derivatives $P_X^{(k)}(s)$ exist and, furthermore, the limit $\lim_{s\to 1-} P_X^{(k)}(s) = P_X^{(k)}(1-)$ also exists.
2. $\mathbb{P}(X = k) = \frac{1}{k!} P_X^{(k)}(0)$ and, in particular, $\mathbb{P}(X = 0) = P_X(0)$.
3. $\mathbb{E}X(X-1)\cdots(X-k+1) = P_X^{(k)}(1-)$, and, in particular, $\mathbb{E}X = P_X'(1-)$.

Clearly from Theorem 1.1.1 it follows that

$$\operatorname{var} X = P''_X(1-) + P'_X(1-) - \left(P'_X(1-)\right)^2 \quad \text{if} \quad \mathbb{E}X < \infty.$$

Exercise 1. Find the generating function of the count random variable X and determine its radius of convergence. Using the generating function find the mean value and variance of X.

1. X has the Bernoulli distribution with parameter $p \in (0,1)$.
2. X has the binomial distribution with parameters $m \in \mathbb{N}_0$ and $p \in (0,1)$.
3. X has the negative binomial distribution with parameters $m \in \mathbb{N}_0$ and $p \in (0,1)$.
4. X has the Poisson distribution with parameter $\lambda > 0$.
5. X has the geometric distribution (on $\mathbb{N}_0$) with parameter $p \in (0,1)$.

Exercise 2. Let $P_X(s)$ be a probability generating function of a count random variable X. Find the probability distribution of X and compute its mean if

1. $P_X(s) = \frac{1}{4-s}$, $|s| < 4$.
2. $P_X(s) = \frac{2}{s^2-5s+6}$, $|s| < 2$.
3. $P_X(s) = \frac{24-9s}{5s^2-30s+40}$, $|s| < 2$.
4. $P_X(s) = \frac{1}{2\max(p,q)}\left(1 - \sqrt{1-4pqs^2}\right)$, $|s| \le 1$ with $p \in (0,1)$ and $q = 1-p$.

Solution to Exercise 2, part (4). The mean can be computed in a standard way by taking the derivative $P'_X(s)$ and considering the limit $\lim_{s\to 1-} P'_X(s)$. We obtain

$$\mathbb{E}X = \begin{cases} \frac{2pq}{\max(p,q)|p-q|}, & p \neq q, \\ \infty, & p = q. \end{cases}$$

Now, in order to obtain the probabilities $\mathbb{P}(X = k)$ recall the Taylor expansion of the function $f(x) = \sqrt{1+x}$ at $x = 0$. We have that

$$\sqrt{1+x} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} x^k = \sum_{k=0}^{\infty} \binom{2k}{k} \frac{(-1)^k}{2^{2k}(1-2k)} x^k.$$

Now, take $x := -4pqs^2$ (notice that such x will never be equal to -1 where the Taylor expansion would not exist since the square root is not smooth at zero). Hence, we obtain

$$P_X(s) = \frac{1}{2\max(p,q)} \left(1 + \sum_{k=0}^{\infty} \binom{2k}{k} \frac{(pq)^k}{2k-1} s^{2k}\right).$$

Comparing the coefficients it follows that

$$\begin{array}{lcll} \mathbb{P}(X=0) & = & 0 & \\ \mathbb{P}(X=2k) & = & \frac{1}{2\max(p,q)}\binom{2k}{k}\frac{(pq)^k}{2k-1}, & k = 1,2,\ldots \\ \mathbb{P}(X=2k-1) & = & 0, & k = 1,2,\ldots \end{array}$$

It should be mentioned that $P(s)$ is not a probability generating function as defined in Definition 1.1.2 - the sequence $\{p_n\}$ here is not a probability distribution on $\mathbb{N}$. △

Exercise 3. Consider a series of Bernoulli trials. Determine the probability that the number of successes in n trials is even.

Solution to Exercise 3. Even though there are many ways how to solve this Exercise, we will make use of the probability generating functions. Notice that the number of successes in n trials can be even in two ways:

1. in the n-th trial, the throw was a success and in $(n-1)$ trials the total number of successes was odd
2. in the n-th trial, the throw was a failure and in $(n-1)$ trials the total number of successes was even

Denote now by a_n the probability that in n-trials there will be an even number of successes. By the law of total probability, we obtain that

$$a_n = p(1-a_{n-1}) + (1-p)a_{n-1}, \quad n = 1, 2, \dots,$$

with $a_0 := 1$ (surely in zero trials there are no successes which is an even number). Now multiplying the whole recurrence equation by s^n and summing from $n = 1$ to ∞ we obtain for $|s| < 1$ that

$$\sum_{n=1}^{\infty} a_n s^n = p \sum_{n=1}^{\infty} +(1-2p) \sum_{n=1}^{\infty} a_{n-1} s^n$$

If we denote by $A(s)$ the GF of the sequence of probabilities (not a distribution!) $\{a_n\}_{n=0}^{\infty}$, we can rewrite this as

$$A(s) - 1 = p \cdot \frac{s}{1-s} + (1-2p)sA(s), \quad |s| < 1,$$

from which we get, using partial fractions, that

$$A(s) = \frac{1-(1-p)s}{(1-s)(1-(1-2p)s)} = \frac{\frac{1}{2}}{1-s} + \frac{\frac{1}{2}}{1-(1-2p)s}, \quad |s| < 1.$$

Going back to the definition of $A(s)$ we can see that

$$\sum_{n=0}^{\infty} a_n s^n = \frac{1}{2} \sum_{n=0}^{\infty} 1 \cdot s^n + \frac{1}{2} \sum_{n=0}^{\infty} (1-2p)^n s^n$$

and comparing the coefficients we get the final formula for a_n

$$a_n = \frac{1}{2}\left(1 + (1-2p)^n\right), \quad n = 0, 1, 2, \dots$$

△

Exercise 4. At a party, the guests play (having had some drinks) a game. Each person takes off one of their socks and puts it into a sack. Then all the socks are shuffled and each person chooses (uniformly randomly) one sock. Assume there is $n \in \mathbb{N}$ guests at the party. What is the probability p_n that no guest gets their own sock?

Solution to Exercise 4. This is a famous and old problem called "The Matching Problem" first solved in 1713 by P. R. de Montmort. Assume that $n \geq 2$. We shall calculate the number of derangements D_n (i.e. permutation of $\{1, 2, \ldots, n\}$ such that no element appears in its original position). Then, of course, the sought probability $p_n = \frac{D_n}{n!}$. Let us number the guests $1, 2, \ldots, n$ and their socks also $1, 2, \ldots, n$. Assume that the guest number 1 chooses sock i (there are $n-1$ ways to make such a choice). The following both possibilities can occur:

1. Person i does not take sock no. 1. But this is the same situation as if we considered only $(n-1)$ guests with $(n-1)$ socks playing the game.
2. Persion i takes sock no. 1. But this is the same situation as if we considered only $(n-2)$ guests with $(n-2)$ socks playing the game.

Hence, the recurrence relation for D_n is

$$D_n = (n-1)\left(D_{n-1} + D_{n-2}\right).$$

Dividing by $n!$ yields a formula for p_n:

$$p_n = \frac{D_n}{n!} = (n-1)\left(\frac{1}{n} \cdot \frac{D_{n-1}}{(n-1)!} + \frac{1}{n(n-1)} \cdot \frac{D_{n-2}}{(n-2)!}\right) = \frac{n-1}{n} p_{n-1} + \frac{1}{n} p_{n-2}, \quad n = 3, 4, \ldots$$

Thus

$$np_n = (n-1)p_{n-1} + p_{n-2}.$$

If we multiply both sides of the recursion formula by s^n and sum for n from 3 to ∞ we obtain

$$\sum_{n=3}^{\infty} np_n s^n = \sum_{n=3}^{\infty} (n-1)p_{n-1} s^n + \sum_{n=3}^{\infty} p_{n-2} s^n.$$

Now, if we denote $P(s) = \sum_{n=1}^{\infty} p_n s^n$ and if we further notice that $p_1 = 0$ and $p_2 = \frac{1}{2}$, we can rewrite this as

$$s(P'(s) - s) = s^2 P'(s) + s^2 P(s)$$

which yields the differential equation

$$P'(s) = \frac{s}{1-s} P(s) + \frac{s}{1-s}$$

with initial condition $P(0) = 0$ (just plug zero into the formula for $P(s)$). Solving this, we obtain

$$P(s) = \frac{e^{-s}}{1-s} - 1.$$

Now, we only need to expand it into a power series to compare coefficients. We have

$$P(s) = \frac{1}{1-s}\left(\sum_{n=0}^{\infty} \frac{(-1)^k}{k!} s^k - 1 + s\right) = \left(\sum_{k=0}^{\infty} s^k\right) \cdot \left(\sum_{n=2}^{\infty} \frac{(-1)^k}{k!} s^k\right) = \sum_{k=2}^{\infty} \left(\sum_{j=2}^{k} \frac{(-1)^j}{j!}\right) s^k.$$

This gives

$$p_1 = 0$$

$$p_n = \sum_{j=2}^{n} \frac{(-1)^j}{j!}, \quad n = 2, 3, \ldots$$

△

1.2 Sums of count random variables

1.2.1 Deterministic sums of count random variables

Theorem 1.2.1. Let X and Y be two independent count random variables with probability generating functions P_X (and radius of convergence R_X) and P_Y (with radius of convergence R_Y), respectively. Denote further $Z = X + Y$ and its probability generating function P_Z. Then

$$P_Z(s) = P_X(s) \cdot P_Y(s), \quad |s| < \min\{R_X, R_Y\}.$$

Exercise 5. Generalize the statement of Theorem 1.2.1 to $n \in \mathbb{N}$ independent count random variables.

Exercise 6. Find the distribution of the sum of $n \in \mathbb{N}$ independent random variables $X_1, \ldots, X_n$, where

1. X_i has the Poisson distribution with parameter $\lambda_i > 0$, $i = 1, \ldots, n$.
2. X_i has the binomial distribution with parameters $p \in (0, 1)$ and $m_i \in \mathbb{N}$.

1.2.2 Random sums of count random variables

Theorem 1.2.2. Let N and $X_1, X_2, \ldots$ be independent and identically distributed count random variables. Set

$$S_0 = 0,$$
$$S_N = X_1 + \ldots X_N.$$

If the common probability generating function of X_i's is P_X and the probability generating function of N is P_N, then the probability generating function of S_N, P_{S_N}, is of the form

$$P_{S_N}(s) = P_N\left(P_X(s)\right).$$

Exercise 7. Prove Theorem 1.2.2 and show that, as a corollary, it holds that

1. $\mathbb{E}S_N = \mathbb{E}N \cdot \mathbb{E}X_1$,
2. $\operatorname{var} S_N = \mathbb{E}N \cdot \operatorname{var} X_1 + \operatorname{var} N \cdot (\mathbb{E}X_1)^2$.

Solution to Exercise 7. Following the definition of S_N and conditioning on the number of summands we obtain

$$\begin{aligned}
P_{S_N}(s) &= \sum_{n=0}^{\infty} \mathbb{P}(S_N = n) s^n \\
&= \sum_{n=0}^{\infty} s^n \sum_{k=0}^{\infty} \mathbb{P}(S_N = n | N = k) \mathbb{P}(N = k) \\
&= \sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} \mathbb{P}(S_N = n | N = k) s^n \right) \mathbb{P}(N = k) \\
&= \sum_{k=0}^{\infty} (P_X(s))^k \mathbb{P}(N = k) \\
&= P_N\left(P_X(s)\right).
\end{aligned}$$

The interchange of the sums is possible due to Fubini-Tonelli's Theorem and since S_k is a deterministic sum of count random variables, we have that

$$P_{S_k}(s) = \sum_{n=0}^{\infty} \mathbb{P}(S_N = n | N = k) s^n = (P_X(s))^k$$

by Theorem 1.2.1. The corollary follows by differentiating the function $P_{S_N}(s)$ and taking its limit as $s \to 1-$. We have

$$\mathbb{E} S_N = \lim_{s \to 1-} P'_{S_N}(s) = \lim_{s \to 1-} P'_N\left(P_X(s)\right) P'_X(s) = P'_N(1-) P'_X(1-) = \mathbb{E}N \cdot \mathbb{E}X_1.$$

Similarly for the variance. $\triangle$

Exercise 8. 1. Let $\{X_i\}_{i \in \mathbb{N}}$ be a sequence of independent, identically distributed random variables with Poisson distribution with parameter $\alpha > 0$. Let N be a random variable, independent of X_i's, with Poisson distribution with parameter $\lambda > 0$. Consider the sum $S_0 = 0, S_N = \sum_{i=1}^{N} X_i$. Find the probability distribution of S_N, its mean and variance.

2. The number of eggs a hen lays, N, has Poisson distribution with parameter $\lambda > 0$. Each egg hatches with probability $p \in (0, 1)$ independently of the other eggs. Find the probability distribution of the number of chickens. How many chickens can a farmer expect?

3. Each day, a random number N of people come to withdraw money from an ATM. Each person withdraws X_i hundred crowns, $i = 1, \ldots, N$. Assume that N has Poisson distribution with parameter $\lambda > 0$ and that X_i's are independent and identically distributed having binomial distribution with parameters $n \in \mathbb{N}$ and $p \in (0, 1)$. Find the probability distribution of the total amount withdrawn each day and compute its mean.

1.3 Galton-Watson branching process

Our aim is to analyse the evolution of a population of identical organisms, each of which lives exactly one time unit and as it dies, it gives birth to a random number of new organisms, thus producing a new generation. Example of such organisms are the males carrying a family name, bacteria, etc.

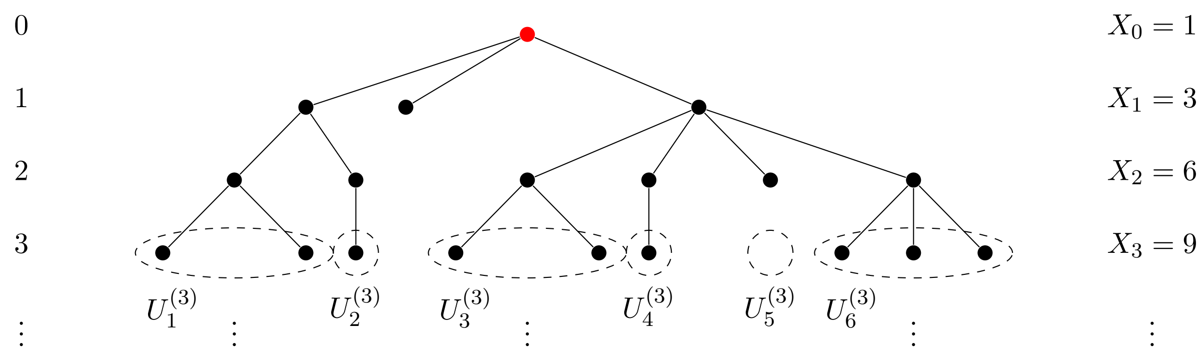
We assume that the family sizes, $U_i^{(k)}$, are independent count random variables which are identically distributed with probabilities

$$\mathbb{P}(U = k) = p_k, \quad k = 0, 1, 2, \ldots$$

and we denote the number of organisms in the n-th generation (i.e. its size) by X_n. Hence, the evolution of our population is described by the random sequence $X = (X_n, n \in \mathbb{N}_0)$. It follows that

$$\begin{aligned} X_0 &= 1, \\ X_n &= \sum_{i=1}^{X_{n-1}} U_i^{(n)} \end{aligned}$$

if $X_{n-1} \neq 0$ and $X_n = 0$ if $X_{n-1} = 0$, $n = 1, 2, \ldots$ A neat way how to visualise one realisation of the Galton-Watson branching process is via random trees as in the following picture.



Since $U_i^{(n)}$ are independent (of each other w.r. to i and n; and of X_{n-1}) using Theorem 1.2.2 we obtain that the probability generating function P_{X_n} can be expressed as

$$P_{X_n}(s) = P_{X_{n-1}}\left(P_U(s)\right) \tag{1.3.1}$$

Using the same argument and Theorem 1.1.1 we also have (cf. Exercise 7) that

$$\mathbb{E}X_n = P'_{X_n}(1) = P'_{X_{n-1}}\left(P_U(1)\right) \cdot P'_U(1) = P'_{X_{n-1}}(1) \cdot P'_U(1) = \mathbb{E}X_{n-1} \cdot \mathbb{E}U$$

and, denoting $\mathbb{E}U =: \mu$ and $\operatorname{var} U =: \sigma^2$, we obtain $\mathbb{E}X_n = \mu^n$. In a similar manner, the formula for $\operatorname{var} X_n$ can be obtained. Indeed, from Exercise 7, part (2), we have that

$$\begin{aligned} \operatorname{var} X_n &= (\mathbb{E}X_1)^2 \operatorname{var} X_{n-1} + \mathbb{E}X_{n-1} \operatorname{var} X_1 \\ &= (\mathbb{E}U)^2 \operatorname{var} X_{n-1} + \mu^{n-1} \operatorname{var} U \\ &= \mu^2 \operatorname{var} X_{n-1} + \mu^{n-1}\sigma^2 \end{aligned}$$

Hence, we obtain a linear difference equation of the first order and recursively it holds that

$$\operatorname{var} X_n = \sigma^2 \mu^{n-1} \left(1 + \mu + \mu^2 + \ldots + \mu^{n-1}\right)$$

with the last term being a partial geometric series. We have arrived at the following result.

Theorem 1.3.1. Let $X = (X_n, n \in \mathbb{N}_0)$ be the Galton-Watson process with the number of children in a family U having the mean μ and variance σ^2. Then

$$\mathbb{E}X_n = \mu^n$$
$$\operatorname{var} X_n = \begin{cases} n\sigma^2, & \mu = 1, \\ \sigma^2\mu^{n-1}\frac{\mu^n-1}{\mu-1}, & \mu \neq 1. \end{cases}$$

Denote $e_n := \mathbb{P}(X_n = 0)$, i.e. the probability that the Galton-Watson branching process X is extinct by the n-th generation. Clearly $\{e_n\}_{n\in\mathbb{N}_0}$ is a bounded and non-decreasing sequence (since if $X_n = 0$ for some $n \in \mathbb{N}_0$ then also $X_{n+1} = 0$) and thus there is a limit $e := \lim_{n\to\infty} e_n$, which is called the probability of ultimate extinction of the process X.

Theorem 1.3.2. The probability of ultimate extinction e is the smallest root of the fixed point equation $P_U(x) = x$ on $[0, 1]$.

Proof. We shall proceed in two steps. Notice, that a solution to the equation $P_U(x) = x$ on the interval $[0, 1]$ always exists - just take $x = 1$.

1. Claim: The number e solves the equation $P_U(x) = x$ on $[0, 1]$.
   First notice, that $e_n = \mathbb{P}(X_n = 0) = P_{X_n}(0)$. From the recursive formula (1.3.1) it follows that

$$\begin{aligned} P_{X_n}(s) &= P_{X_{n-1}}(P_U(s)) \\ &= \dots \\ &= \underbrace{P_U(P_U(\dots(s)\dots))}_{n-\text{times}} \\ &= \dots \\ &= P_U(P_{X_{n-1}}(s)) \end{aligned}$$

   Plugging in $s = 0$ we obtain

$$e_n = P_U(e_{n-1}), \quad n \in \mathbb{N}.$$

   Since P_U is continuous on $[0, 1]$ (it is a power series with radius of convergence $R_U \geq 1$) we can take the limit $n \to \infty$ of both sides, move the limit into the argument of P_U and obtain $e = P_U(e)$.

2. Claim: The number e is the smallest root of $P_U(x) = x$ on $[0, 1]$.
   First notice that the function P_U is non-decreasing on the interval $[0, 1]$. This is because its first derivative is non-negative on $[0, 1]$. Let η be a solution to the fixed point equation

$P_U(x) = x$ on $[0,1]$. We will show that $e \leq \eta$. From (1.3.1) we have that

$$\begin{aligned} e_1 &= P_U(e_0) = P_U(0) \leq P_U(\eta) = \eta, \\ e_2 &= P_U(e_1) \leq P_U(\eta) = \eta, \\ &\vdots \\ e_n &\leq \eta, \quad n \in \mathbb{N}_0. \end{aligned}$$

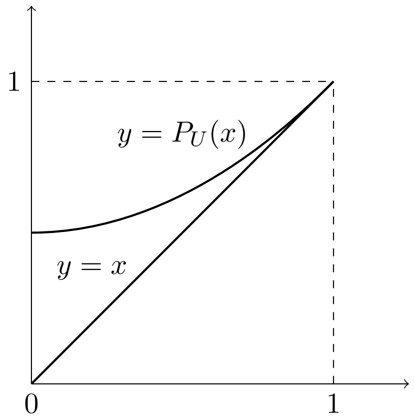
Hence we obtain $e \leq \eta$.

□

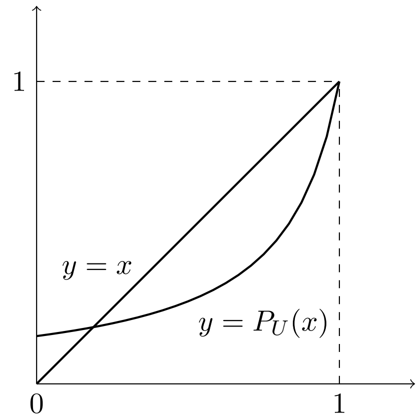
Theorem 1.3.3. Suppose that $p_0 > 0$. Then $e = 1$ if and only if $\mu \leq 1$.

Proof. Let us first comment on the assumption $p_0 > 0$. If $p_0 = 0$, then it would mean that there will always be at least one member of the next generation in one family U, this would mean that $\mathbb{P}(X_n = 0) = 0$ for all $n \in \mathbb{N}_0$ and thus, our process would never extinct (i.e. $e = 0$). Since $p_0 > 0$, then it cannot happen that $p_1 = 1$, which would mean that we always have exactly one member of the next generation and, again, the process X would never extinct.

We already know that P_U is on the interval $[0,1]$ continuous and non-decreasing. Furthermore, P_U is also convex, since $P_U''(s) \geq 0$ on $[0,1]$. Hence, the the equation $P_U(x) = x$ can have only one or two roots in the interval $[0,1]$ as shown in the pictures below.



Case when $P_U'(1-) = \mu \leq 1$.



Case when $P_U'(1-) = \mu > 1$.

By Theorem 1.3.2, we have that e is the smallest non-negative root of the fixed point equation $P_U(x) = x$. Hence, $e = 1$ if and only if $\mu = P_U'(1-) \leq 1$. □

Theorems 1.3.2 and 1.3.3 give us a cookbook how to compute the probability of ultimate extinction of the Galton-Watson branching process. First compute $\mu = \mathbb{E}U$ and if $\mathbb{E}U \leq 1$ we immediately know that $e = 1$. If $\mathbb{E}U > 1$, we have to solve the fixed point equation $x = P_U(x)$.

Exercise 9. Let $X = (X_n, n \in \mathbb{N}_0)$ be the Galton-Watson branching process with U having the distribution $\{p_n\}_{n \in \mathbb{N}_0}$. Find the probability of ultimate extinction e and the expected number of members of the n-the generation.

1. $p_0 = \frac{1}{5}$, $p_1 = \frac{1}{5}$, $p_2 = \frac{3}{5}$ and $p_k = 0$ for $k = 3, 4, \ldots$.
2. $p_0 = \frac{1}{12}$, $p_1 = \frac{5}{12}$, $p_2 = \frac{1}{2}$ and $p_k = 0$ for $k = 3, 4, \ldots$.
3. $p_0 = \frac{1}{10}$, $p_1 = \frac{2}{5}$, $p_2 = \frac{1}{2}$ and $p_k = 0$ for $k = 3, 4, \ldots$.
4. $p_0 = \frac{1}{2}$, $p_1 = 0$, $p_2 = 0$, $p_3 = \frac{1}{2}$ and $p_k = 0$ for $k = 4, 5, \ldots$.
5. $p_k = \left(\frac{1}{2}\right)^{k+1}$, $k = 0, 1, 2, \ldots$.
6. $p_k = pq^k$, $k = 0, 1, 2, \ldots$ with $p \in (0,1)$ and $q = 1 - p$ (i.e. geometric distribution)

Solution to Exercise 9, part (1). First notice, that $\mathbb{E}U = \frac{1}{5} \cdot 0 + \frac{1}{5} \cdot 1 + \frac{3}{5} \cdot 2 = \frac{7}{5}$. Hence $\mathbb{E}X_n = \left(\frac{7}{5}\right)^n$. We further have that $\mathbb{E}U = \mu > 1$ and we thus know by Theorem 1.3.3 that $e < 1$. We thus have to solve the fixed point equation $P_U(s) = s$ as proved in Theorem 1.3.2. In this case,

$$P_U(s) = \frac{1}{5}s^0 + \frac{1}{5}s^1 + \frac{3}{5}s^2, \quad s \in \mathbb{R}.$$

This means that we need to solve a quadratic equation, namely,

$$\frac{1}{5} + \frac{1}{5}s + \frac{3}{5}s^2 = s.$$

There will always be root $s_1 = 1$. The second one is $s_2 = \frac{1}{3}$. Hence, we obtain $e = \frac{1}{3}$. △

Exercise 10. Suppose that the family sizes U have geometric distribution on $\{0, 1, 2, \ldots\}$ with parameter $p \in (0,1)$. Find the distribution of X_n, i.e. the probabilities $\mathbb{P}(X_n = j)$ for $j = 0, 1, 2, \ldots$.

Solution to Exercise 10. The starting point is to determine the probability generating function P_{X_n}. From the proof of Theorem 1.3.2, we know that

$$P_{X_n}(s) = \underbrace{P_U(P_U(\ldots(s)\ldots))}_{n-\text{times}}$$

We know that the PGF of the geometric distribution on $\mathbb{N}_0$ is given by

$$P_U(s) = \frac{p}{1-qs} = \frac{1}{1+\mu-\mu s}, \quad |s| < \frac{1}{q},$$

where μ is the mean $\mu = \mathbb{E}U = \frac{p}{q}$. Writing down the first few iterates of P_U, we can guess the pattern

$$\begin{aligned}
P_{X_1}(s) &= \frac{1}{1+\mu-\mu s} \\
P_{X_2}(s) &= \frac{1}{1+\mu-\mu\frac{1}{1+\mu-\mu s}} = \frac{(1+\mu)-\mu s}{(1+\mu+\mu^2)-s(\mu+\mu^2)} \\
&\vdots \\
P_{X_n}(s) &= \frac{\sum_{k=0}^{n-1}\mu_k - \left(\sum_{k=1}^{n-1}\mu^k\right)s}{\sum_{k=0}^{n}\mu_k - \left(\sum_{k=1}^{n}\mu^k\right)s}
\end{aligned}$$

which can be proved by induction on n. Furthermore, we can simplify the formula as

$$P_{X_n}(s) = \begin{cases} \frac{(\mu^{n-1}-1)-(\mu^{n-2}-1)\mu s}{(\mu^n-1)-(\mu^{n-1}-1)\mu s} & \mu \neq 1 \\ \frac{n-(n-1)s}{n+1-ns} & \mu = 1 \end{cases}$$

<u>The case $\mu = 1$:</u> Now we only need to expand the formula above into a power series and compare coefficients. This is easily done by dividing the nominator by the denominator to obtain

$$\begin{aligned} P_{X_n}(s) &= \frac{n-(n-1)s}{(n+1)-ns} \\ &= 1 - \frac{1}{n} + \frac{\frac{1}{n}}{(n+1)-ns} \\ &= 1 - \frac{1}{n} + \frac{1}{n(n+1)} \cdot \frac{1}{1-\frac{n}{n+1}s} \\ &= 1 - \frac{1}{n} + \frac{1}{n(n+1)} \cdot \sum_{k=0}^{\infty} \left(\frac{n}{n-1}\right)^k s^k \end{aligned}$$

Which gives

$$\begin{aligned} \mathbb{P}(X_n = 0) &= 1 - \frac{1}{n+1} \\ \mathbb{P}(X_n = k) &= \frac{1}{n(n+1)} \left(\frac{n}{n-1}\right)^k, \quad k = 1, 2, \ldots \end{aligned}$$

<u>The case $\mu \neq 1$:</u> Similarly as in the previous case, we have to write $P_{X_n}(s)$ as a power series and compare coefficients. We have

$$\begin{aligned} P_{X_n}(s) &= \frac{(\mu^{n-1}-1)-(\mu^{n-2}-1)\mu s}{(\mu^n-1)-(\mu^{n-1}-1)\mu s} \\ &= \frac{\mu^{n-2}-1}{\mu^{n-1}-1} + \frac{(\mu^{n-1}-1)-(\mu^n-1)\frac{\mu^{n-2}-1}{\mu^{n-1}-1}}{(\mu^n-1)-(\mu^{n-1}-1)\mu s} \\ &= \frac{\mu^{n-2}-1}{\mu^{n-1}-1} + \left(\frac{\mu^{n-1}-1}{\mu^n-1} - \frac{\mu^{n-2}-1}{\mu^{n-1}-1}\right) \cdot \frac{1}{1-\frac{\mu^{n-1}-1}{\mu^n-1}\mu s} \\ &= \frac{\mu^{n-2}-1}{\mu^{n-1}-1} + \left(\frac{\mu^{n-1}-1}{\mu^n-1} - \frac{\mu^{n-2}-1}{\mu^{n-1}-1}\right) \sum_{k=0}^{\infty} \left(\frac{\mu^{n-1}-1}{\mu^n-1}\mu\right)^k s^k \end{aligned}$$

which gives

$$\begin{aligned} \mathbb{P}(X_n = 0) &= \frac{\mu^{n-1}-1}{\mu^n-1} \\ \mathbb{P}(X_n = k) &= \frac{\mu^{n-2}-1}{\mu^{n-1}-1} + \left(\frac{\mu^{n-1}-1}{\mu^n-1} - \frac{\mu^{n-2}-1}{\mu^{n-1}-1}\right)\left(\frac{\mu^{n-1}-1}{\mu^n-1}\mu\right)^k \end{aligned}$$

△

1.4 Answers to exercises

Answer to Exercise 1. 1. $P_X(s) = 1 - p + ps$ for $s \in \mathbb{R}$, $\mathbb{E}X = p$, $\operatorname{var} X = p(1-p)$.

2. $P_X(s) = (1 - p + ps)^m$ for $s \in \mathbb{R}$, $\mathbb{E}X = mp$, $\operatorname{var} X = mp(1-p)$.
3. $P_X(s) = \left(\frac{ps}{1-(1-p)s}\right)^m$, for $|s| < \frac{1}{1-p}$, $\mathbb{E}X = \frac{pm}{1-p}$, $\operatorname{var} X = \frac{pm}{(1-p)^2}$.
4. $P_X(s) = e^{\lambda(s-1)}$ for $s \in \mathbb{R}$, $\mathbb{E}X = \lambda$, $\operatorname{var} X = \lambda$.
5. $P_X(s) = \frac{p}{1-(1-p)s}$ for $|s| < \frac{p}{1-p}$, $\mathbb{E}X = \frac{1-p}{p}$ and $\operatorname{var} X = \frac{1-p}{p^2}$.

Answer to Exercise 2. 1. $\mathbb{E}X = \frac{1}{3}$ and $\mathbb{P}(X = k) = \frac{3}{4} \cdot \left(\frac{1}{4}\right)^k$, $k \in \mathbb{N}_0$.

2. $\mathbb{E}X = 3$ and $\mathbb{P}(X = k) = \frac{1}{2^k} - \frac{2}{3} \cdot \frac{1}{3^k}$, $k \in \mathbb{N}_0$.
3. $\mathbb{E}X = \frac{11}{15}$ and $\mathbb{P}(X = k) = \frac{3}{10} \cdot \frac{1}{2^k}\left(1 + \frac{1}{2^k}\right)$, $k \in \mathbb{N}_0$.
4. $$\mathbb{E}X = \begin{cases} \frac{2pq}{\max(p,q)|p-q|}, & p \neq q, \\ \infty, & p = q. \end{cases}$$

   and
   $$\begin{aligned} \mathbb{P}(X = 0) &= 0 \\ \mathbb{P}(X = 2k) &= \tfrac{1}{2\max(p,q)}\tbinom{2k}{k}\tfrac{(pq)^k}{2k-1}, \qquad k = 1, 2, \dots \\ \mathbb{P}(X = 2k-1) &= 0, \qquad k = 1, 2, \dots \end{aligned}$$

Answer to Exercise 3. Let a_n be the probability that in n trials, the number of successes is even. Then
$$a_n = \frac{1}{2}\left(1 + (1-2p)^n\right), \quad n = 0, 1, 2, \dots$$

Answer to Exercise 4.

$$\begin{aligned} p_1 &= 0 \\ p_n &= \sum_{j=2}^{n} \frac{(-1)^j}{j!}, \quad n = 2, 3, \dots \end{aligned}$$

Answer to Exercise 5. Let $X_1, \dots, X_n$ be independent count random variables with probability generating functions $P_{X_1}, \dots, P_{X_n}$ and the corresponding radii of convergence $R_1, \dots, R_n$. Denote $Z = \sum_{k=1}^n X_i$ and its probability generating function P_Z. Then

$$P_Z(s) = \prod_{i=1}^{n} P_{X_i}(s), \quad |s| < \min_{i=1,\dots,n} R_i.$$

Answer to Exercise 6. The sum $Z = \sum_{i=1}^n X_n$

1. has the Poisson distribution with parameter $\sum_{i=1}^n \lambda_i$.
2. has the binomial distribution with parameter p and $\sum_{i=1}^n m_i$.

Answer to Exercise 7. Both claims are true.

Answer to Exercise 8.

1. $P_{S_N}(s) = e^{\lambda(e^{\alpha(s-1)}-1)}$ for $s \in \mathbb{R}$, $\mathbb{E}S_N = \lambda\alpha$, $\operatorname{var} S_N = \lambda\alpha(1+\alpha)$.
2. The number of chicks has the Poisson distribution with parameter λp. The farmer can expect λp chicks.
3. Let S_N be the total amount withdrawn each day. Then $P_{S_N}(s) = e^{\lambda[(1-p+ps)^n-1]}$ for $s \in \mathbb{R}$, $\mathbb{E}S_N = \lambda np$.

Answer to Exercise 9.

1. $e = \frac{1}{3}$, $\mathbb{E}X_n = \left(\frac{7}{5}\right)^n$.
2. $e = \frac{1}{6}$, $\mathbb{E}X_n = \left(\frac{17}{12}\right)^n$.
3. $e = \frac{1}{5}$, $\mathbb{E}X_n = \left(\frac{7}{5}\right)^n$.
4. $e = \frac{\sqrt{5}-1}{2}$, $\mathbb{E}X_n = \left(\frac{3}{2}\right)^n$.
5. $e = 1$, $\mathbb{E}X_n = 1$.
6. If $p < \frac{1}{2}$ then $e = \frac{1}{\mu} = \frac{p}{1-p}$. If $p \geq \frac{1}{2}$, then $e = 1$. $\mathbb{E}X_n = \left(\frac{1-p}{p}\right)^n$.

Answer to Exercise 10. If $\mu = 1$, then

$$\mathbb{P}(X_n = 0) = 1 - \frac{1}{n+1}$$
$$\mathbb{P}(X_n = k) = \frac{1}{n(n+1)}\left(\frac{n}{n-1}\right)^k, \quad k = 1, 2, \ldots$$

If $\mu \neq 1$, then

$$\mathbb{P}(X_n = 0) = \frac{\mu^{n-1}-1}{\mu^n - 1}$$
$$\mathbb{P}(X_n = k) = \frac{\mu^{n-2}-1}{\mu^{n-1}-1} + \left(\frac{\mu^{n-1}-1}{\mu^n-1} - \frac{\mu^{n-2}-1}{\mu^{n-1}-1}\right)\left(\frac{\mu^{n-1}-1}{\mu^n-1}\mu\right)^k$$

Chapter 2

Discrete time Markov Chains

2.1 Markov property and time homogeneity

Definition 2.1.1. A $\mathbb{Z}$-valued random sequence $X = (X_n, n \in \mathbb{N})$ is called a discrete time Markov chain with a state space S if

1. $S = \{i \in \mathbb{Z} : \text{ there exists } n \in \mathbb{N}_0 \text{ such that } \mathbb{P}(X_n = i) > 0\}$,
2. and it holds that

$$\mathbb{P}(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0) = \mathbb{P}(X_{n+1} = j | X_n = i)$$

for all times $n \in \mathbb{N}_0$ and all states $i, j, i_{n-1}, \dots, i_0 \in S$ such that $\mathbb{P}(X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0) > 0$.

Roughly speaking, the first condition says that the state space contains all the possible values of the stochastic process X but no other values. The second condition, called the Markov property, says that given a current state the past and the future of X are independent. For a discrete time Markov chain X, the probability

$$p_{ij}(n, n+1) := \mathbb{P}(X_{n+1} = j | X_n = i)$$

is called the transition probability from state i at time n to the state j at time $n+1$. With these, we can create a stochastic matrix[1] $\boldsymbol{P(n, n+1)} = (p_{ij}(n, n+1))_{i,j \in S}$ which is called the transition probability matrix of X at time n to time $n+1$. Similarly, we can define (for $k \in \mathbb{N}_0$)

$$p_{ij}(n, n+k) := \mathbb{P}(X_{n+k} = j | X_n = i)$$

and a stochastic matrix $\boldsymbol{P(n, n+k)} = (p_{ij}(n, n+k))_{i,j \in S}$. Although it is possible to define matrix multiplication for stochastic matrices of the type $\mathbb{Z} \times \mathbb{Z}$ rigorously, whenever we will multiply infinite matrices, we will assume that $S \subset \mathbb{N}_0$ (thus excluding the case $S = \mathbb{Z}$).

Definition 2.1.2. If there is a stochastic matrix $\boldsymbol{P}$ such that $\boldsymbol{P(n, n+1)} = \boldsymbol{P}$ for all

[1] A stochastic matrix is a matrix $\boldsymbol{P} = (p_{ij})_{i,j \in S}$ such that each row is a probability distribution on S.

$n \in \mathbb{N}_0$, then we say that X is a (time) homogeneous discrete time Markov chain.

For a homogeneous discrete time Markov chain X it is thus possible to define a transition matrix after $k \in \mathbb{N}_0$ steps by $\boldsymbol{P(k)} := \boldsymbol{P(n, n+k)}$. It holds that

$$\boldsymbol{P(k)} = \boldsymbol{P}^k$$

and, as a corollary, we obtain the famous Chapman-Kolmogorov equation

$$\boldsymbol{P(m+n)} = \boldsymbol{P(m)} \cdot \boldsymbol{P(n)}, \quad m, n \in \mathbb{N}_0.$$

If we are interested in the probability distribution of X_n, denoted by $\boldsymbol{p(n)}$, we have that

$$\boldsymbol{p(n)}^T = \boldsymbol{p(0)}^T \boldsymbol{P}^n, \quad n \in \mathbb{N}_0.$$

where $\boldsymbol{p(0)}$ denote the initial distribution of X, i.e. the vector which contains the probabilities $\mathbb{P}(X_0 = k)$ for $k \in S$.

Exercise 11. Let X be the Galton-Watson branching process with $X_0 = 1$ and $p_0 > 0$.

1. Show that X is a homogeneous discrete time Markov chain with a state space $S = \mathbb{N}_0$.
2. Find its transition matrix $\boldsymbol{P}$.
3. Compute the distribution of X_2 in the case when the distribution of U is $p_0 = \frac{1}{5}$, $p_1 = \frac{1}{5}$, $p_2 = \frac{3}{5}$, $p_k = 0$ for all $k = 3, 4, \dots$

Solution to Exercise 11. Let X be the Galton-Watson branching process for which $X_0 = 1$ and $p_0 > 0$.

1. We will proceed in three steps: first we find the effective states S, then we show the Markov property (these two are the key requirement for X to be a Markov chain) and then we will show homogeneity.
State space S: Since $X_0 = 1$, we immediately have that $1 \in S$. Since $p_0 > 0$, then it can happen (with probability p_0) that $X_1 = 0$. Hence also $0 \in S$ and we know that $\{0, 1\} \subset S$. The case when $\{0, 1\} = S$ is only possible if U takes either only the value 0 or takes values in the set $\{0, 1\}$. If there is $k \in \mathbb{N} \setminus \{1\}$ such that $p_k > 0$, then we have that $S = \mathbb{N}_0$.
Markov property: Now that we have the set of effective states S, take some $n \in \mathbb{N}_0$ and $i, j, i_{n-1}, \dots, i_1 \in S$ such that $\mathbb{P}(X_n = i, X_{n-1} = i_{n-1}, \dots, X_1 = i_1, X_0 = 1) > 0$ and consider
$$\mathbb{P}(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \dots, X_1 = i_1, X_0 = 1).$$
If $i = 0$, this probability is 1 if $j = 0$ and 0 if $j \neq 0$. This is, however, the same as $\mathbb{P}(X_{n+1} = j | X_n = i)$. If $i \neq 0$, we have that
$$\begin{aligned}\mathbb{P}(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \dots, X_1 = i_1, X_0 = 1) = \\ &= \mathbb{P}\left(\sum_{k=1}^{X_n} U_k^{(n)} = j | X_n = i, \dots, X_0 = 1\right) \\ &= \mathbb{P}\left(\sum_{k=1}^{X_n} U_k^{(n)} = j | X_n = i\right)\end{aligned}$$

since the sum depends only on X_n and $U_i^{(n)}$ are independent of $X_{n-1}, \ldots, X_0$. Hence, the Markov property (2) of Definition 2.1.1 also holds and we can infer, that X is a discrete time Markov chain.
Homogeneity: We have that

$$p_{ij}(n, n+1) = \mathbb{P}\left(\sum_{k=1}^{X_n} U_k^{(n)} = j | X_n = i\right) = \mathbb{P}\left(Z := \sum_{k=1}^{i} U_k^{(n)} = j\right)$$

We need to show, that this number does not depend on n. Notice, that Z represents a sum of a deterministic number of independent, identically distributed count random variables $U_k^{(n)}$. Hence, we can use Theorem 1.2.1. The PGF of the sum is $P_Z(s) = P_U(s)^i$ and the sought probability is

$$p_{ij}(n, n+1) = \mathbb{P}\left(\sum_{k=1}^{X_n} U_k^{(n)} = j | X_n = i\right) = \mathbb{P}\left(S := \sum_{k=1}^{i} U_k^{(n)} = j\right) = [P_U(s)^i]_j$$

where $[P(s)]_k$ denotes the coefficient at s^k. This, however, does not depend on n.

2. The transition matrix is $\boldsymbol{P} := (p_{ij})_{i,j \in S}$ with $p_{ij} := [P_U(s)^i]_j$.
3. Consider now the particular case when U has the distribution $p_0 = \frac{1}{5}$, $p_1 = \frac{1}{5}$, $p_2 = \frac{3}{5}$ and $p_k = 0$ for all $k = 3, 4, \ldots$ Then we have

$$P_U(s) = \frac{1}{5} + \frac{1}{5}s + \frac{3}{5}s^2$$

and

$$p_{ij} = [P_U(s)^i]_j = \left[\left(\frac{1}{5} + \frac{1}{5}s + \frac{3}{5}s^2\right)^i\right]_j.$$

In particular,

$$\begin{aligned}
P_U(s)^0 &= 1 \\
P_U(s)^1 &= \frac{1}{5} + \frac{1}{5}s + \frac{3}{5}s^2 \\
P_U(s)^2 &= \frac{1}{25} + \frac{2}{25}s + \frac{7}{25}s^2 + \frac{6}{25}s^3 + \frac{9}{25}s^4 \\
P_U(s)^3 &= \frac{1}{125} + \frac{3}{125}s + \frac{12}{125}s^2 + \frac{19}{125}s^3 + \frac{36}{125}s^4 + \frac{27}{125}s^5 + \frac{27}{125}s^6 \\
&\vdots
\end{aligned}$$

and

$$\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{5} & \frac{1}{5} & \frac{3}{5} & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{25} & \frac{2}{25} & \frac{7}{25} & \frac{6}{25} & \frac{9}{25} & 0 & 0 & 0 & \cdots \\ \frac{1}{125} & \frac{3}{125} & \frac{12}{125} & \frac{19}{125} & \frac{36}{125} & \frac{27}{125} & \frac{27}{125} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Since we know that the initial distribution is given by $\boldsymbol{p(0)}^T = (0, 1, 0, \ldots)$ (since $X_0 = 1$), we have that $\boldsymbol{p(2)}^T = \boldsymbol{p(0)}^T \boldsymbol{P}^2$ which will give

$$\boldsymbol{p(2)}^T = \left(\frac{33}{125}, \frac{11}{125}, \frac{36}{125}, \frac{18}{125}, \frac{27}{125}, 0, \ldots\right).$$

△

Exercise 12 (Symmetric random walk)**.** Consider $Y_1, Y_2, \ldots$ independent identically distributed random variables such that $\mathbb{P}(Y_1 = -1) = \mathbb{P}(Y_1 = +1) = \frac{1}{2}$. Define $X_0 = 0$ and $X_n = \sum_{i=1}^{n} Y_i$ for $n \in \mathbb{N}$. Decide whether $X = (X_n, n \in \mathbb{N}_0)$ is a homogeneous discrete time Markov chain or not.

Hint: Notice that we can write $X_n = \sum_{i=1}^{n-1} Y_i + Y_n = X_{n-1} + Y_n$.

Exercise 13 (Running maximum)**.** Consider $Y_1, Y_2, \ldots$ independent identically distributed integer-valued random variables and define $X_n = \max\{Y_1, \ldots, Y_n\}$ for $n \in \mathbb{N}$. Decide whether $X = (X_n, n \in \mathbb{N}_0)$ is a homogeneous discrete time Markov chain or not.

Hint: Can you find a formula for X_n which uses only X_{n-1} and Y_n?

Exercise 14 (Recursive character of Markov chains)**.** Consider $Y_1, Y_2, \ldots$ independent identically distributed integer-valued random variables. Let X_0 be another S-valued random variable, $S \subset \mathbb{Z}$, which is independent of the sequence $\{Y_i\}$ and consider a measurable function $f : S \times \mathbb{Z} \to S$. Define

$$X_{n+1} = f(X_n, Y_{n+1}), \quad n \in \mathbb{N}_0.$$

Show that $X = (X_n, n \in \mathbb{N}_0)$ is a homogeneous discrete time Markov chain.

If X is a homogeneous discrete time Markov chain, then it holds that

$$\begin{aligned}\mathbb{P}(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) &= \\ &= \mathbb{P}(X_{n+1} = j | X_n = i, X_{n-1} = j_{n-1}, \ldots, X_0 = j_0)\end{aligned}$$

for all $n \in \mathbb{N}_0$ and all possible trajectories $(i, i_{n-1}, \ldots, i_0)$ and $(i, j_{n-1}, \ldots, j_0)$ of $(X_n, \ldots, X_0)$. Thus, to show that a stochastic process with discrete time is not a Markov chain, it suffices to find $n \in \mathbb{N}$ and different possible trajectories of length n such that the equality between the above conditional probabilities does not hold. Notice that i and j are the same for both considered trajectories.

Exercise 15. Let $Y_0, Y_1, \ldots$ be independent, identically distributed random variables with a discrete uniform distribution on $\{-1, 0, 1\}$. Set $X_n := Y_n + Y_{n+1}$ for $n \in \mathbb{N}_0$. Decide whether $X = (X_n, n \in \mathbb{N}_0)$ is a homogeneous discrete time Markov chain or not.

Solution to Exercise 15. By definition, we have that

$$\begin{aligned} X_0 &= Y_0 + Y_1 \\ X_1 &= Y_1 + Y_2 \\ X_2 &= Y_2 + Y_3 \end{aligned}$$

We will show that the values $(2, 0, 2)$ and $(-2, 0, 2)$ of a random vector (X_0, X_1, X_2) have different probabilities. Observe that

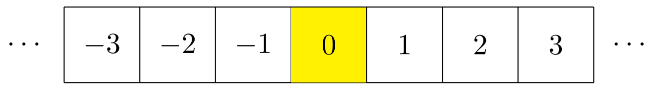
$$\begin{aligned} (X_0, X_1, X_2) = (-2, 0, 2) &\leftrightarrow (Y_0, Y_1, Y_2, Y_3) = (-1, -1, 1, 1) \\ (X_0, X_1, X_2) = (2, 0, 2) &\leftrightarrow (Y_0, Y_1, Y_2, Y_3) = \text{ not possible} \end{aligned}$$

Hence, for the corresponding probabilities, we obtain

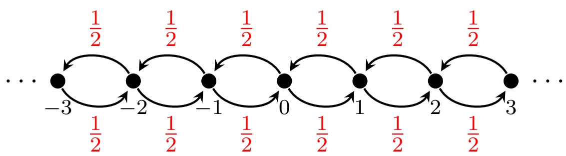
$$\mathbb{P}(X_2 = 2|X_1 = 0, X_0 = 2) = \frac{\mathbb{P}(X_0 = 2, X_1 = 0, X_2 = 2)}{\mathbb{P}(X_0 = 2, X_1 = 0)} = 0$$
$$\mathbb{P}(X_2 = 2|X_1 = 0, X_0 = -2) \frac{\mathbb{P}(Y_0 = -1, Y_1 = -1, Y_2 = 1, Y_3 = 1)}{\mathbb{P}(Y_0 = -1, Y_1 = -1, Y_2 = 1)} = \frac{1}{3}.$$

△

Imagine we do the following experiment. We put a board game figurine on a board which looks like this:

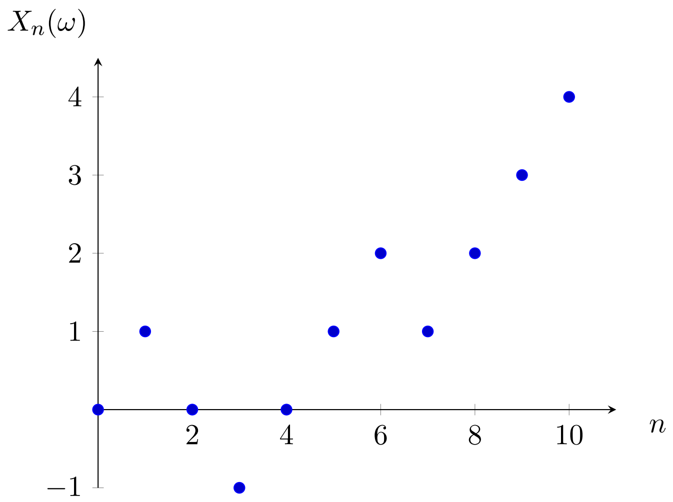


Then, we will flip a coin. If heads, the figurine steps to the right and if tails, it moves to the left. Then we flip a coin again and so on. This can be modelled by the symmetric random walk X from Exercise 12. The board can be visualized as an infinite graph whose vertices are labelled by integers (i.e. corresponding to the state space $S = \mathbb{Z}$). If $\boldsymbol{P}$ is the transition matrix of X, then if $p_{ij} > 0$, we will draw an arrow from i to j. These arrows will be the edges. So we obtain the so-called transition diagram:

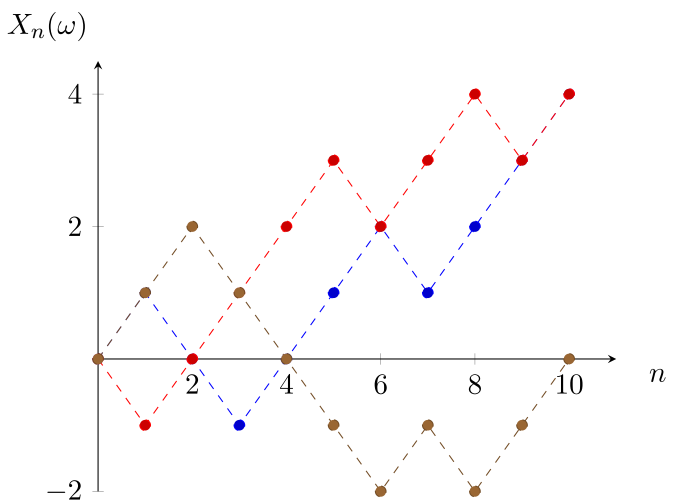


The transition diagram will be very useful when analysing the irreducibility of the chain X which will be needed for classification of states in the next section.

A different way, helpful for intuition, is to draw the so-called sample path of X. For this reason, flip a coin ten times, move the figurine on the board and note down the cell in which it ended. The aim is to draw an honest graph of the function. Put the numbers of cells on the y axis and the number of flips on the x axis. Since we start on the yellow cell at the 0th flip, we put a black dot at $(0, 0)$. Now flip a coin and put the next dot at either $(1, 1)$ or $(1, -1)$ depending on whether we got heads or tails. Flip a coin again and so on. We may obtain the following:



Repeating the experiment several times, we may obtain different graphs. For example, repeating it three times, we could obtain the following:



The dots of same colors are called a sample path of X. To be more precise, $X : \mathbb{N}_0 \times \Omega \to S$ is a random process. Fix $\omega \in \Omega$. Then the function (of one variable) $X(\cdot, \omega) : \mathbb{N}_0 \to S$ is called the trajectory or sample path of X. Unfortunately, having one trajectory of your process does not really tell you much about it since even two sample paths can be dramatically different. Thus, when analysing a stochastic process, you typically want to say what all its trajectories have in common.

2.2 Classification of states based on $\boldsymbol{P}^n$

Throughout this section we only consider homogeneous discrete time Markov chains. For convenience, we will use the symbol

$$\mathbb{P}_i(\cdot) := \mathbb{P}(\cdot | X_0 = i), \quad \mathbb{E}_i(\cdot) := \int_\Omega (\cdot) \mathrm{d}\mathbb{P}_i$$

Definition 2.2.1. Let $X = (X_n, n \in \mathbb{N}_0)$ be a homogeneous discrete time Markov chain with a state space S.

1. A transient state is a state $i \in S$ such that if X starts from i, then it might not ever return to it (i.e. $\mathbb{P}_i(\forall n \in \mathbb{N}_0 : X_n \neq i) > 0$).

2. A recurrent state is a state $i \in S$ such that if X starts from i, then it will visit it again almost surely (i.e. $\mathbb{P}_i(\exists n \in \mathbb{N}_0 : X_n = i) = 1$.)

   (a) Let
   $$\tau_i(1) := \inf\{n \in \mathbb{N} : X_n = i\}.$$
   the first return time to the state i. A recurrent state i is called null recurrent, if the mean value of the first return time to i is infinite, i.e. if $\mathbb{E}_i\tau_i(1) = \infty$.

   (b) A recurrent state $i \in S$ is called positive recurrent if $\mathbb{E}_i\tau_i(1) < \infty$.

3. Denote by $p_{ij}^{(n)}$ the elements of a stochastic matrix $\boldsymbol{P(n)}$ (i.e. the transition probabilities from a state $i \in S$ to a state $j \in S$ in n steps). If a state $i \in S$ is such that $D_i := \{n \in \mathbb{N}_0 : p_{ii}^{(n)} > 0\} \neq \emptyset$, we can define $d_i := GCD(D_i)$ and if $d_i > 1$, we say that i is periodic with a period d_i and non-periodic if $d_i = 1$ (here GCD denotes the greatest common divisor).

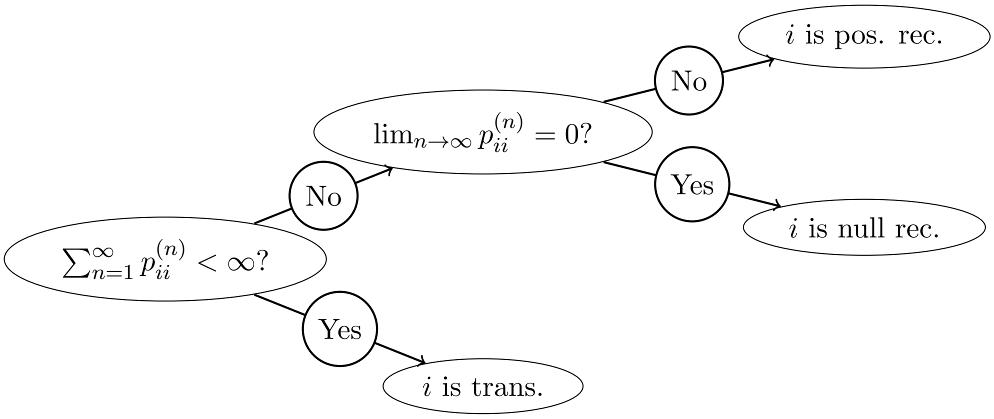
Recall that the transition matrix $\boldsymbol{P(n)}$ can be obtained as the n-th power of the transition matrix $\boldsymbol{P}$ of the chain X since we only deal with homogeneous discrete time Markov chains.

Theorem 2.2.1. Denote the elements of the transition matrix of a homogeneous Markov chain X in n steps $\boldsymbol{P(n)}$ by $p_{ij}^{(n)}$.

1. A state $i \in S$ is recurrent if and only if $\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$.

2. A recurrent state $i \in S$ is null-recurrent if and only if $\lim_{n\to\infty} p_{ii}^{(n)} = 0$.

A little care is needed in applying Theorem 2.2.1. Clearly all states must be either recurrent or transient. Hence, part (1) says that a state $i \in S$ is transient if and only if $\sum_{n=1}^{\infty} p_{ii}^{(n)} < \infty$. However, this means that $p_{ii}^{(n)} \to 0$ as $n \to \infty$ for transient states as well as for null-recurrent states. Hence, showing that $\lim_{n\to\infty} p_{ii}^{(n)} = 0$ yields that the state i is either transient or null-recurrent. Theorem 2.2.1 provides the following cookbook for classification of states based on the limiting behaviour of $p_{ii}^{(n)}$.



It is clear, that for a successful application of these criteria, we need to be able to know the behaviour of $p_{ii}^{(n)}$ when n tends to infinity. We will have to find an explicit formula for $p_{ii}^{(n)}$. This can be done in various ways, either using probabilistic reasoning or computing the matrix $\boldsymbol{P}^n$ and taking its diagonal elements.

Exercise 16. Let $Y_1, Y_2, \dots$ be independent, identically distributed random variables with a discrete uniform distribution on $\{-1, 1\}$. Define $X_0 := 0$ and $X_n := \sum_{i=1}^{n} Y_i$.This is the symmetric random walk $X = (X_n, n \in \mathbb{N}_0)$ considered in Exercise 12. Find the probabilities $p_{ii}^{(n)}$ and classify the states of X.

Solution to Exercise 16. Transient or recurrent states? Our aim is to use the cookbook above. For this, we need to know the limiting behaviour of $p_{ii}^{(n)}$ as $n \to \infty$. Suppose we make an even number $n = 2k$ of steps. Then in order to return to the original position (say i) we have to make k steps to the right and k steps to the left. On the other hand, if we make an odd number $n = 2k + 1$ of steps, then we will never be able to get back to the original position. Hence,

$$p_{ii}^{(n)} = \begin{cases} \binom{2k}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^k, & n = 2k \\ 0, & n = 2k + 1 \end{cases}$$

Now, we need to decide if $\sum_{n=1}^{\infty} p_{ii}^{(n)} < \infty$ or not. In order to do so, notice that, using the Stirling's formula ($m! \approx \sqrt{2\pi m} m^m e^{-m}$), we have

$$\binom{2k}{k} \frac{1}{2^{2k}} = \frac{(2k)!}{(k!)^2} \cdot \frac{1}{2^{2k}} \approx \frac{1}{\sqrt{\pi k}}, \quad k \to \infty,$$

and, using the comparison test, we have that the sum $\sum_{n=1}^{\infty} p_{ii}^{(n)}$ converges if and only if the sum $\sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi n}}$ converges. This is, however, not true and hence we can infer that

$$\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty.$$

This means that the state i is recurrent and we need to say if it is null recurrent of positive recurrent. But this is easy since $\frac{1}{\sqrt{\pi k}}$ goes to 0 (which corresponds to $n = 2k$) and also 0 goes to 0 trivially (which corresponds to $n = 2k + 1$). Hence, the state i is null recurrent. At the beginning, i was arbitrary and our reasoning is valid for any $i \in \mathbb{N}_0$. Hence, all the states of

the symmetric random walk are null recurrent.
Periodicity: For every i, we have that for all k, $p_{ii}^{(2k+1)} = 0$ and $p_{ii}^{2k} > 0$. Hence, the set

$$\{n \in \mathbb{N}_0 : p_{ii}^{(n)} > 0\} = \{2k, k \in \mathbb{N}_0\} = \{0, 2, 4, \ldots\}.$$

Its greatest common divisor is 2 and hence, $d_i = 2$ for every i. Altogether, the symmetric random walk has 2-periodic, null recurrent states. △

Exercise 17. Let $Y_1, Y_2, \ldots$ be independent, identically distributed random variables with a discrete uniform distribution on $\{-1, 0, 1\}$. Define $X_n := \max\{Y_1, \ldots, Y_n\}$. Then $X = (X_n, n \in \mathbb{N})$ is a homogeneous discrete time Markov chain (see Exercise 13). Find the probabilities $p_{ii}^{(n)}$ and classify the states of X.

Hint: Use similar probabilistic reasoning as in Exercise 12 to find the matrix $\boldsymbol{P}^n$. What must the whole trajectory look like, if we start at -1, make n steps, and finish at -1 again?

Exercise 18. Consider a series of Bernoulli trails with the probability of success $p \in (0, 1)$. Denote X_n the length of a series of successes in the n-th trial (if the n-th trial is not a success then we set $X_n = 0$). Show that $X = (X_n, n \in \mathbb{N}_0)$ is a homogeneous Markov chain, find the transition matrix $\boldsymbol{P}$ and $\boldsymbol{P}^n$ and classify its states.

If we want to find the whole matrix $\boldsymbol{P}^n$ we have more options. One is, of course, the Jordan decomposition. Using the Jordan decomposition we obtain matrices $\boldsymbol{S}$ and $\boldsymbol{J}$ such that $\boldsymbol{P} = \boldsymbol{S}\boldsymbol{J}\boldsymbol{S}^{-1}$. Then

$$\boldsymbol{P}^n = \underbrace{(\boldsymbol{S}\boldsymbol{J}\boldsymbol{S}^{-1}) \cdot (\boldsymbol{S}\boldsymbol{J}\boldsymbol{S}^{-1}) \cdot \ldots \cdot (\boldsymbol{S}\boldsymbol{J}\boldsymbol{S}^{-1})}_{n-\times} = \boldsymbol{S}\boldsymbol{J}^n\boldsymbol{S}^{-1}$$

and finding the matrix $\boldsymbol{J}^n$ is easy due to its form. Another way how to find $\boldsymbol{P}^n$ is via the Perron's formula as stated in the following theorem.

Theorem 2.2.2 (Perron's formula for matrix powers)**.** Let $\boldsymbol{A}$ be an $n \times n$ matrix with eigenvalues $\lambda_1, \cdots, \lambda_k$ with multiplicities $m_1, \cdots, m_k$. Then it holds that

$$\boldsymbol{A}^n = \sum_{j=1}^{k} \frac{1}{(m_j - 1)!} \frac{d^{m_j - 1}}{d\lambda^{m_j - 1}} \left[\frac{\lambda^n \mathbf{Adj}\,(\boldsymbol{\lambda I} - \boldsymbol{A})}{\psi_j(\lambda)} \right]_{\lambda = \lambda_j}$$

where

$$\psi_j(\lambda) = \frac{\det(\boldsymbol{\lambda I} - \boldsymbol{A})}{(\lambda - \lambda_j)^{m_j}}.$$

Exercise 19. Consider a homogeneous discrete time Markov chain X with the state space $S = \{0, 1\}$ and the transition matrix

$$\boldsymbol{P} = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix},$$

where $a, b \in (0, 1)$. Find $\mathbf{P}^n$ and classify the states of X.

Solution to Exercise 19. We will use the Jordan decomposition to find the matrix $\boldsymbol{P}^n$, then we will take its diagonal elements and use the cookbook above to classify the states 0 and 1. The key ingredients for Jordan decomposition are the eigenvalues and eigenvectors. The eigenvalues are the roots of the characteristic polynomial $\psi(\lambda)$. We thus have to solve

$$\psi(\lambda) = \det(\lambda \boldsymbol{I} - \boldsymbol{P}) = \begin{pmatrix} \lambda - 1 + a & -a \\ -b & \lambda - 1 + b \end{pmatrix} = \lambda^2 - (2 - (a+b))\lambda + 1 - (a+b) = 0.$$

The corresponding eigenvectors $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ are found as solutions to $\boldsymbol{P}\boldsymbol{v_i} = \lambda_i \boldsymbol{v_i}$, $i = 1, 2$. We arrive at

$$\begin{array}{lll} \lambda_1 = 1 & \dots & \boldsymbol{v_1}^T = (1, 1) \\ \lambda_2 = 1 - (a+b) & \dots & \boldsymbol{v_2}^T = \left(-\frac{a}{b}, 1\right) \end{array}$$

The Jordan form is then

$$\begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -\frac{a}{b} \\ 1 & 1 \end{pmatrix}}_{=:\boldsymbol{S}} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1-(a+b) \end{pmatrix}}_{=:\boldsymbol{J}} \underbrace{\begin{pmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ -\frac{b}{a+b} & \frac{b}{a+b} \end{pmatrix}}_{=\boldsymbol{S}^{-1}}.$$

Now it is easy to find the power $\boldsymbol{J}^n$. We have that

$$\boldsymbol{J}^n = \begin{pmatrix} 1 & 0 \\ 0 & (1-(a+b))^n \end{pmatrix}$$

so that

$$\begin{aligned} \boldsymbol{P}^n &= \boldsymbol{S}\boldsymbol{J}^n\boldsymbol{S}^{-1} \\ &= \frac{1}{a+b} \begin{pmatrix} 1 & -\frac{a}{b} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (1-(a+b))^n \end{pmatrix} \begin{pmatrix} b & a \\ -b & b \end{pmatrix} \\ &= \frac{1}{a+b} \begin{pmatrix} b + a(1-(a+b))^n & a - a(1-(a+b))^n \\ b - b(1-(a+b))^n & a + b(1-(a+b))^n \end{pmatrix}. \end{aligned}$$

Clearly,

$$p_{ii}^{(n)} = \begin{cases} b + a(1-(a+b))^n, & i = 0, \\ a + b(1-(a+b))^n, & i = 1, \end{cases} \quad \underset{n\to\infty}{\longrightarrow} \quad \begin{cases} b, & i = 0, \\ a, & i = 1, \end{cases} \quad \neq 0.$$

Hence, the sum $\sum_{n=1}^{\infty} p_{ii}^{(n)}$ cannot converge and because of this, the Markov chain under consideration has positive recurrent states. Since also $p_{ii} > 0$ for both $i = 0, 1$, the states are also non-periodic. △

Exercise 20 (Random walk on a triangle)**.** Consider a homogeneous discrete time Markov chain with the state space $S = \{0, 1, 2\}$ and the transition matrix

$$\boldsymbol{P} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix},$$

Find $\mathbf{P}^n$ and classify the states of X.

Solution to Exercise 20. In order to successfully apply the Perron's formula we need the following ingredients: the eigenvalues $\lambda_1,\ldots,\lambda_k$ of $\boldsymbol{P}$, their multiplicities $m_1,\ldots,m_k$, the corresponding polynomials $\psi_j(\lambda)$ and the adjoint matrix of $\lambda\boldsymbol{I}-\boldsymbol{P}$. The we plug everything into the formula to obtain $\boldsymbol{P}^n$.

Ingredients:

As in the previous exercise, we have to find the roots of characteristic polynomial

$$\psi(\lambda)=\det\left(\lambda\boldsymbol{I}-\boldsymbol{P}\right)=\begin{vmatrix}\lambda & -\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \lambda & -\frac{1}{2}\\ -\frac{1}{2} & -\frac{1}{2} & \lambda\end{vmatrix}=\lambda^3-\frac{3}{4}\lambda-\frac{1}{4}=(\lambda-1)\left(\lambda+\frac{1}{2}\right)^2=0$$

which yields

$$\begin{array}{lllll}\lambda_1=1 & \ldots & m_1=1 & \ldots & \psi_1(\lambda)=\left(\lambda+\frac{1}{2}\right)^2\\ \lambda_2=-\frac{1}{2} & \ldots & m_2=2 & \ldots & \psi_2(\lambda)=(\lambda-1)\end{array}$$

The adjoint matrix can be obtained as follows. Given a square $n\times n$ matrix $\boldsymbol{A}$, the adjoint

$$\mathbf{Adj}\,\boldsymbol{A}=\begin{pmatrix}+\det M_{11} & -\det M_{12} & \cdots & (-1)^{n+1}\det M_{1n}\\ -\det M_{21} & +\det M_{22} & \cdots & (-1)^{n+2}\det M_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ (-1)^{n+1}\det M_{n1} & (-1)^{n+1}\det M_{n1} & \cdots & (-1)^{n+n}\det M_{n1}\end{pmatrix}^T$$

where M_{ij} is the (i,j) minor (i.e. the determinant of the $(n-1)\times(n-1)$ matrix which is obtained by deleting row i and column j of $\boldsymbol{A}$). We obtain

$$\mathbf{Adj}\,(\boldsymbol{\lambda I}-\boldsymbol{P})=\mathbf{Adj}\begin{pmatrix}\lambda & -\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \lambda & -\frac{1}{2}\\ -\frac{1}{2} & -\frac{1}{2} & \lambda\end{pmatrix}=\begin{pmatrix}\lambda^2-\frac{1}{2} & \frac{1}{2}\lambda+\frac{1}{4} & \frac{1}{2}\lambda+\frac{1}{4}\\ \frac{1}{2}\lambda+\frac{1}{4} & \lambda^2-\frac{1}{4} & \frac{1}{2}\lambda+\frac{1}{4}\\ \frac{1}{2}\lambda+\frac{1}{4} & \frac{1}{2}\lambda+\frac{1}{4} & \lambda^2-\frac{1}{4}\end{pmatrix}.$$

Plugging everything into the formula, we obtain

$$\begin{aligned}\boldsymbol{P}^n&=\frac{1}{(1-1)!}\frac{\mathrm{d}^0}{\mathrm{d}\lambda^0}\left(\frac{\lambda^n\mathbf{Adj}\,(\boldsymbol{\lambda I}-\boldsymbol{P})}{\left(\lambda+\frac{1}{2}\right)^2}\right)\Bigg|_{\lambda=1}+\frac{1}{(2-1)!}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\lambda^n\mathbf{Adj}\,(\boldsymbol{\lambda I}-\boldsymbol{P})}{\lambda-1}\right)\Bigg|_{\lambda=-\frac{1}{2}}\\ &=\frac{4}{9}\cdot\frac{3}{4}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}+\frac{\lambda^n}{\lambda-1}\begin{pmatrix}2\lambda & \frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & 2\lambda & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2} & 2\lambda\end{pmatrix}\Bigg|_{\lambda=-\frac{1}{2}}\\ &\quad+\frac{n\lambda^{n-1}(\lambda-1)-\lambda^n}{(\lambda-1)^2}\begin{pmatrix}\lambda^2-\frac{1}{2} & \frac{1}{2}\lambda+\frac{1}{4} & \frac{1}{2}\lambda+\frac{1}{4}\\ \frac{1}{2}\lambda+\frac{1}{4} & \lambda^2-\frac{1}{4} & \frac{1}{2}\lambda+\frac{1}{4}\\ \frac{1}{2}\lambda+\frac{1}{4} & \frac{1}{2}\lambda+\frac{1}{4} & \lambda^2-\frac{1}{4}\end{pmatrix}\Bigg|_{\lambda=-\frac{1}{2}}\\ &=\begin{pmatrix}\frac{1}{3}+\frac{2}{3}\left(-\frac{1}{2}\right)^n & \frac{1}{3}-\frac{1}{3}\left(-\frac{1}{2}\right)^n & \frac{1}{3}-\frac{1}{3}\left(-\frac{1}{2}\right)^n\\ \frac{1}{3}-\frac{1}{3}\left(-\frac{1}{2}\right)^n & \frac{1}{3}+\frac{2}{3}\left(-\frac{1}{2}\right)^n & \frac{1}{3}-\frac{1}{3}\left(-\frac{1}{2}\right)^n\\ \frac{1}{3}-\frac{1}{3}\left(-\frac{1}{2}\right)^n & \frac{1}{3}-\frac{1}{3}\left(-\frac{1}{2}\right)^n & \frac{1}{3}+\frac{2}{3}\left(-\frac{1}{2}\right)^n\end{pmatrix}.\end{aligned}$$

Clearly for all $i\in S$ we have that

$$p_{ii}^{(n)}=\frac{1}{3}+\frac{2}{3}\left(-\frac{1}{2}\right)^n$$

and hence, the sum

$$\sum_{n=1}^{\infty} p_{ii}^{(n)} = \sum_{n=1}^{\infty} \frac{1}{3} + \frac{2}{3} \sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^n = \infty$$

which implies that all the states are recurrent and, moreover, $p_{ii}^{(n)} \to \frac{1}{3} \neq 0$ which implies that they are positive recurrent. Further,

$$\{n \in \mathbb{N}_0 : p_{ii}^{(n)} > 0\} = \mathbb{N}$$

and hence, the states are all non-periodic since the greatest common divisor of all natural numbers is 1. △

Looking back:

1. When computing the characteristic polynomial of $\boldsymbol{P}$, one has to solve a polynomial equation of higher order. It helps to know that when $\boldsymbol{P}$ is a stochastic matrix, one of its eigenvalues is always 1 (hence, we can divide the polynomial $\psi(\lambda)$ by $\lambda - 1$ to reduce its order and find the other eigenvalues quickly).
2. Do not forget to take the transpose of the matrix when computing the adjoint $\mathbf{Adj}(\boldsymbol{\lambda I} - \boldsymbol{P})$.
3. Compare the periodicity of the symmetric random walk on $\mathbb{Z}$ with the periodicity of the symmetric random walk on a triangle.

2.3 Classification of states based on accessibility and state space

Although Theorem 2.2.1 gives a characterisation of recurrence of a state in terms of the power of a matrix $\boldsymbol{P}$ (more precisely in terms of its diagonal entries), it may be very tedious to apply in practice since one has to check the behaviour of $p_{jj}^{(n)}$ for every $j \in S$. As it turns out, however, the situation can be made easier.

Definition 2.3.1. Let $X = (X_n, n \in \mathbb{N}_0)$ be a homogeneous Markov chain with a state space S and let $i, j \in S$ be two of its states. We say, that j is accessible from i (and write $i \to j$) if there is $m \in \mathbb{N}_0$ such that $p_{ij}^{(m)} > 0$. If the states i, j are mutually accessible, then we say that they communicate (and write $i \leftrightarrow j$). X is called irreducible if all its states communicate.

Theorem 2.3.1. Let $X = (X_n, n \in \mathbb{N}_0)$ be a homogeneous Markov chain with a state space S. The following claims hold:

1. If two states communicate, they are of the same type.
2. If a state $j \in S$ is accessible from a recurrent state $i \in S$, then j is also recurrent and i and j communicate.

Some remarks are in place.

- When we say that two states are of the same type, we mean that both states are either null-recurrent, positive recurrent or transient and if one is periodic, the other is also periodic with the same period.
- Theorem 2.3.1 holds for both finite and countable state space S. Hence, if the Markov chain is irreducible (all its states communicate), then it suffices to classify only one state - the rest will be of the same type.
- If for a state $i \in S$ there is $j \in S$ such that $i \to j \not\to i$, then i must be transient.
- Clearly, the relation $i \leftrightarrow j$ is an equivalence relation on the state space and thus, we may factorize S into equivalence classes. If i is a recurrent state, then $[i]$ simply consists of all the states which are accessible from i.

2.3.1 Finite chains

In the case when the state space S is finite, the situation becomes very easy.

Theorem 2.3.2. Let X be an irreducible homogeneous Markov chain with a finite state space. Then all its states are positive recurrent.

Exercise 21. Suppose we have an unlimited source of balls and k drawers. In every step we pick one drawer at random (uniformly chosen) and we put one ball into this drawer. Let X_n be a number of drawers with a ball at the time n. Show that $X = (X_n, n \in \mathbb{N})$ is a homogeneous Markov chain, find its transition probability matrix and classify its states.

Exercise 22 (Gambler's ruin, B. Pascal (1656))**.** Suppose a gambler goes to a casino to repeatedly play a game which ends by either the gambler losing the round or the gambler winning the round. Suppose further that our gambler has initial fortune $\$\,a$ and the croupier has $\$\,(s-a)$ with $\$\,s$ being the total amount of \$ in the game. Each turn they play and the winner gets one dollar from his opponent. The game ends when either side has no money. Suppose that the probability that our gambler wins is $0 < p < 1$. Model the evolution of capital of the gambler by a homogeneous Markov chain (i.e. say what is X_n and prove that it is a homogeneous Markov chain), find its transition matrix and classify its states.

Solution to Exercise 22. Forget for a second, that we are given an exercise to solve and imagine yourself watching two your friends play at dice. Naturally, you ask yourself: Who is going to win? How long will they play? And so on. In order to answer these questions, you have to build a model of the situation. There are various ways how to do that. For instance, you can assume that the whole universe follows Newton's laws. These laws are deterministic in nature - if you know the initial state of the universe and if you have enough computational power, you could, in theory, get exact answers to these questions.

<u>Analysis:</u> This is not an option for us. Instead, we will use a probabilistic model. But what kind of model should we use? First notice, that dice are played in turns and that each time a specific amount of money travels from one player to his opponent. Further, there is a finite amount, say $\$\,s$, in the game so that it is enough to observe how much money one of the

player has and the fortune of the other can be simply deduced from this. The observed player starts with a certain fortune, say $\$\,a$.

Building a mathematical model: Denote by X_n the fortune of the observed player after the n-th round. So far, X_n is just a symbol. We have to give a precise mathematical meaning to this symbol and it is very natural to assume that X_n is a random variable for each round n. The game ends when either side has no money, this means that all X_n's can take values in the set $S := \{0, 1, \dots, s\}$ with $X_0 = a$. Now, the i-th round can be naturally modelled by another random variable Y_i taking only two values - the observed player won or lost. This corresponds to the player either getting $\$\,1$ or losing it. Hence, each Y_i can take either the value -1 or $+1$ and we assume that the result of every round does not depend on the results of all the previous rounds (e.g. the player does not learn to throw the dice better, the die does not become damaged over time, etc.) - this allows us to assume that Y_i's are independent and identically distributed. Obviously, the X_n's and Y_i's are connected - X_n is the sum of X_0 and all Y_i for $i = 1, \dots, n$ unless X_n is either s or 0 in which case it does not change anymore. The formal model follows:
Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. For each $i \in \mathbb{N}$, let

$$Y_i : (\Omega, \mathscr{F}, \mathbb{P}) \to (\{-1, 1\}, \{\emptyset, \{-1\}, \{1\}, \{-1, 1\}\})$$

be a random variable and assume that $Y_1, Y_2, \dots$ are all mutually independent with the same probability distribution on $\{-1, 1\}$ given by

$$\mathbb{P}(Y_i = -1) = 1 - p, \quad \mathbb{P}(Y_i = 1) = p.$$

Define $X_0 := a$ and

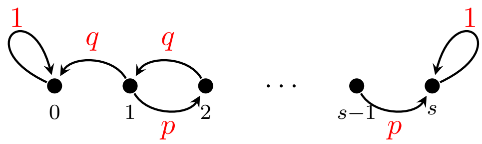
$$X_n := X_0 + (X_{n-1} + Y_n)\mathbf{1}_{[X_{n-1} \notin \{0,s\}]} + X_{n-1}\mathbf{1}_{[X_{n-1} \in \{0,s\}]}, \quad n \in \mathbb{N}.$$

Then each X_n is a random variable taking the values in $S = \{0, 1, \dots, s\}$ (show this!) and, using Exercise 14, we can see that $X := (X_n, n \in \mathbb{N}_0)$ is a homogeneous discrete-time Markov chain with the state space S. Looking closely, we see that it is a random walk on the following graph

with absorption states at 0 and s. The transition matrix of X is built in the following way. Clearly, once we step into the state 0 or s, we will never leave it. Hence $p_{00} = p_{ss} = 1$ and $p_{0i} = p_{si} = 0$ for all $i \in \{1, \dots, s-1\}$. Furthermore, from the state j, $j \in \{1, \dots, s-1\}$, we can always reach only state $j-1$ (with probability $q = 1-p$) and $j+1$ (with probability p), i.e. $p_{j,j-1} = q$ and $p_{j,j+1} = p$. We arrive at

$$\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & q & 0 & p & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & q & 0 & p \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{pmatrix}$$

The transition diagram representing the Markov chain X is the following:



Classification of states: In order to classify the states, we make use of Theorem 2.3.1 and Theorem 2.3.2. We first notice that the chain is reducible. In particular, the state space can be written as

$$S = \{0\} \cup \{s\} \cup \{1, \ldots, s-1\}.$$

All the states in the set $\{1, \ldots, s-1\}$ communicate and thus, they are of the same type. It suffices to analyse only one of them, say the state 1. Obviously,

$$\{n \in \mathbb{N}_0 : p_{11}^{(n)} > 0\} = \{2k, k \in \mathbb{N}_0\}$$

and hence, the state 1 is 2-periodic. Furthermore, $1 \to 0 \not\to 1$ and hence, 1 is transient. This means that all the states $\{1, \ldots, s-1\}$ are 2-periodic and transient. Further, since $p_{00} > 0$ and $p_{ss} > 0$, both 0 and s are non-periodic. Since $p_{00} = \mathbb{P}_i(X_1 = 0) = 1$, we immediately have that 0 is a recurrent state and since $\tau_0(1) = 1, \mathbb{P}$-a.s., we also have that $\mathbb{E}_0\tau_0(1) = 1 < \infty$ which implies that 0 is positive recurrent. Alternatively, one can argue that we can define a (rather trivial) sub-chain of X which only has one state (i.e. 0) and which is therefore irreducible with a finite state space. Then we may appeal to Theorem 2.3.2 to infer that 0 is positive recurrent. The same holds for the state s. △

Exercise 23 (Heat transmission model, D. Bernoulli (1796))**.** The temperatures of two isolated bodies are represented by two containers with a number of balls in each. Altogether, we have $2l$ balls, numbered by $1, \ldots, 2l$. The process is described as follows: in each turn, pick a number from $\{1, \ldots, 2l\}$ uniformly at random and move the ball to the other container. Model the temperature in the first container by a homogeneous Markov chain. Then find its transition matrix and classify its states.

Exercise 24 (Blending of two incompressible fluids, T. and P. Ehrenfest (1907))**.** Suppose we have two containers, each containing l balls (these are the molecule of our fluid). Altogether, we have l black balls and l white balls. The process of blending is described as follows: each turn, pick two balls, each from a different container, and switch them. This way the number of balls in each urn is constant in time. Model the process of blending by a homogeneous Markov chain. Then find its transition matrix and classify its states.

2.3.2 Infinite chains

When the state space is infinite the situation is a bit more difficult.

Definition 2.3.2 (Stationary distribution)**.** Let X be a homogeneous Markov chain with a state space S and transition matrix $\boldsymbol{P}$. We say that a probability distribution $\boldsymbol{\pi} = (\pi_i)_{i\in S}$ is a stationary distribution of X if

$$\boldsymbol{\pi}^T = \boldsymbol{\pi}^T \boldsymbol{P}. \tag{2.3.1}$$

Clearly, a stationary distribution must solve the equation (2.3.1) but at the same time, it has to be a probability distribution on the state space. This may be true only if

$$\sum_{i \in S} \pi_i = 1. \tag{2.3.2}$$

Typically, when solving the system (2.3.1), one fixes an arbitrary π_0, finds a general solution in terms of π_0 and then tries to choose π_0 in such a way that (2.3.2) holds.

Theorem 2.3.3. Let X be an irreducible homogeneous Markov chain with a state space S (finite or infinite). Then all its states are positive recurrent if and only if X admits a stationary distribution.

Obviously, if the stationary distribution does not exist, then we are left with the question to determine if the states are all null-recurrent or transient. This can be decided using the following result.

Theorem 2.3.4. Let X be an irreducible homogeneous Markov chain with a state space $S = \{0, 1, 2, \dots\}$. Then all its states are recurrent if and only if the system

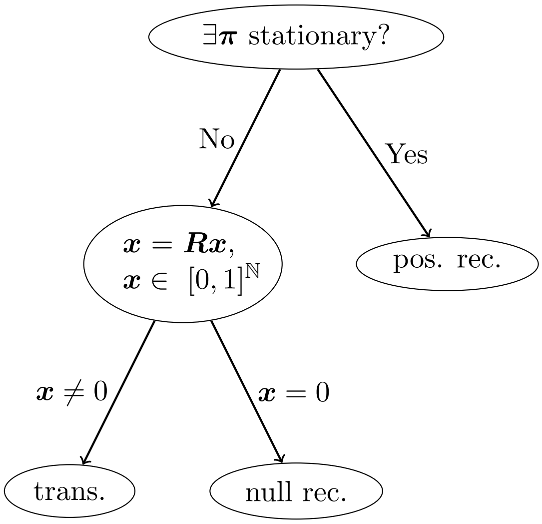
$$x_i = \sum_{j=1}^{\infty} p_{ij} x_j, \quad i = 1, 2, \dots, \tag{2.3.3}$$

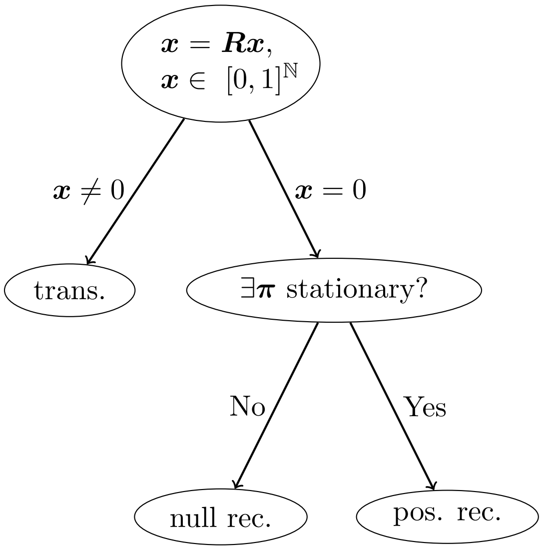
has only a trivial solution in $[0, 1]$, i.e. $x_j = 0$ for all $j \in \mathbb{N}$.

If we adopt the notation $\boldsymbol{x} = (x_i)_{i \in \mathbb{N}}$ and $\boldsymbol{R} = (p_{ij})_{i,j \in \mathbb{N}}$ (notice that i, j are not zero - we simply forget the first column and row), the system (2.3.3) can be neatly written as

$$\boldsymbol{x} = \boldsymbol{R}\boldsymbol{x}$$

which is to be solved for $\boldsymbol{x} \in [0, 1]^{\mathbb{N}}$. Theorems 2.3.3 and 2.3.4 together suggest two possible courses of action which can be taken. We either first try to find the stationary distribution and if that fails (i.e. the states can be either null-recurrent or transient), we can look on the reduced system (2.3.3) (which only tells us if all the states are recurrent or transient) OR we can start with solving the reduced system (2.3.3) to see if the states are transient or recurrent and if we show that they are recurrent we may move on and look for a stationary distribution to see if the states are null or positive recurrent. This reasoning is summarized in the following cookbook.





So, basically, the choice of which one to follow is yours. That being said, you are usually asked to find the stationary distribution anyway so it is efficient to start with the first one.

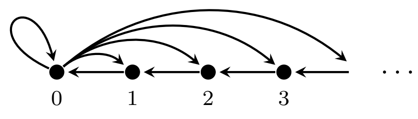
Exercise 25. Classify the states of a homogeneous Markov chain $X = (X_n, n \in \mathbb{N}_0)$ with the state space $S = \mathbb{N}_0$ given by the transition matrix

$$\boldsymbol{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2^2} & \frac{1}{2^3} & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Solution to Exercise 25. We will follow the first guideline. First we should notice that the state space is infinite. When we look at $\boldsymbol{P}$ we see that $p_{0i} > 0$ for all $i \in \mathbb{N}_0$ and also $p_{i,i-1} > 0$ for $i \in \mathbb{N}$. Hence,

$$\begin{aligned} 0 \to i, &\quad i \in \mathbb{N}_0, \\ i \to i-1, &\quad i \in \mathbb{N}, \end{aligned}$$

and thus $i \leftrightarrow j$ for all $i, j \in S$. For a better visualisation we can draw a transition diagram of X as follows:



This means that all the states of X are of the same type (Theorem 2.3.1). At this point, we are already able to determine the periodicity of all states. Since $p_{00} > 0$, the state 0 is non-periodic and thus all the other states are non-periodic. In order to say more, we employ our cookbook according to which we have to decide if X admits a stationary distribution. We have to solve the equation $\boldsymbol{\pi}^T = \boldsymbol{\pi}^T \boldsymbol{P}$ for an unknown stochastic vector $\boldsymbol{\pi}^T = (\pi_0, \pi_1, \pi_2, \ldots)$.

The system reads as

$$
\begin{aligned}
\pi_0 &= \frac{1}{2}\pi_0 + \pi_1 \\
\pi_1 &= \frac{1}{2^2}\pi_0 + \pi_2 \\
\pi_2 &= \frac{1}{2^3}\pi_0 + \pi_3 \\
&\vdots \\
\pi_k &= \frac{1}{2^{k+1}}\pi_0 + \pi_{k+1}, \quad k = 0, 1, 2, \ldots
\end{aligned} \tag{2.3.4}
$$

which translates as (for a fixed π_0 whose value will be determined later)

$$
\begin{aligned}
\pi_1 &= \pi_0 - \frac{1}{2}\pi_0 = \left(1 - \frac{1}{2}\right)\pi_0 \\
\pi_2 &= \pi_1 - \frac{1}{2^2}\pi_0 = \left(1 - \frac{1}{2} - \frac{1}{2^2}\right)\pi_0 \\
&\vdots \\
\pi_{k+1} &= \left(1 - \frac{1}{2} - \frac{1}{2^2} - \ldots - \frac{1}{2^{k+1}}\right)\pi_0, \quad k = 0, 1, 2, \ldots
\end{aligned} \tag{2.3.5}
$$

The formula (2.3.5) is easily proved from (2.3.4) by induction (do not forget this step!). Solving for a general $k \in \mathbb{N}_0$ we have

$$
\pi_{k+1} = \left(1 - \sum_{j=1}^{k+1} \frac{1}{2^j}\right)\pi_0 = \left(1 - \frac{1}{2} \cdot \frac{1 - \frac{1}{2^{k+1}}}{1 - \frac{1}{2}}\right)\pi_0 = \frac{1}{2^{k+1}}\pi_0, \quad k = 0, 1, 2, \ldots
$$

So far, we know that every vector $\boldsymbol{\pi}^T = (\pi_0, \frac{1}{2}\pi_0, \frac{1}{2^2}\pi_0, \ldots)$ solves the equation $\boldsymbol{\pi}^T = \boldsymbol{\pi}^T \boldsymbol{P}$. Now we further require that the vector $\boldsymbol{\pi}$ is also a probability distribution, i.e. all its elements are non-negative and they sum up to 1.

$$
1 \stackrel{!}{=} \sum_{k=0}^{\infty} \pi_k = \pi_0 + \sum_{k=1}^{\infty} \frac{1}{2^k}\pi_0 = 2\pi_0.
$$

This means that the only π_0 for which $\boldsymbol{\pi}$ is a probability distribution on $\mathbb{N}_0$ is $\pi_0 = \frac{1}{2}$. Hence, we arrive at

$$
\pi_k = \frac{1}{2^{k+1}}, \quad k = 0, 1, 2, \ldots
$$

which is the sought stationary distribution of the Markov chain X. Hence, by our cookbook, we know that the chain has only positive recurrent states. △

Exercise 26. Classify the states of a homogeneous Markov chain $X = (X_n, n \in \mathbb{N}_0)$ with the state space $S = \mathbb{N}_0$ given by the transition matrix

$$
\boldsymbol{P} = \begin{pmatrix}
\frac{1}{2\cdot 1} & \frac{1}{3\cdot 2} & \frac{1}{4\cdot 3} & \cdots \\
1 & 0 & 0 & \cdots \\
0 & 1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
$$

Solution to Exercise 26. Clearly the chain X has infinitely many states and in the same way as in the previous exercise, all its states communicate and are non-periodic. Now we try to find the stationary distribution. The equation $\boldsymbol{\pi}^T = \boldsymbol{\pi}^T \boldsymbol{P}$ reads as

$$\begin{aligned}
\pi_0 &= \frac{1}{2\cdot 1}\pi_0 + \pi_1\\
\pi_1 &= \frac{1}{3\cdot 2}\pi_0 + \pi_2\\
&\vdots\\
\pi_k &= \frac{1}{(k+1)(k+2)}\pi_0 + \pi_{k+1}
\end{aligned}$$

which translates as

$$\begin{aligned}
\pi_1 &= \frac{1}{2}\pi_0\\
\pi_2 &= \frac{1}{3}\pi_1\\
\pi_3 &= \frac{1}{4}\pi_2\\
&\vdots
\end{aligned}$$

from which we can guess that

$$\pi_k = \frac{1}{k+1}\pi_0, \quad k = 1, 2, \ldots \tag{2.3.6}$$

This has to be proven by induction as follows. When $k = 1$, we immediately obtain that $\pi_1 = \frac{1}{2}\pi_0$ which is correct. Now we assume that the formula (2.3.6) holds for some $k \in \mathbb{N}$ and we wish to show that it also holds for $k+1$. We compute

$$\pi_{k+1} = \pi_k - \frac{1}{(k+1)(k+2)}\pi_0 \overset{\text{ind.}}{=} \frac{1}{k+1}\pi_0 - \frac{1}{(k+1)(k+2)}\pi_0 = \frac{1}{k+2}\pi_0$$

and hence, the formula (2.3.6) is valid. This means that each vector $\boldsymbol{\pi}^T = (\pi_0, \frac{1}{2}\pi_0, \frac{1}{3}\pi_0, \ldots)$ is a good candidate for a stationary distribution. Now we have to choose π_0 in such a way that we have $\sum_{k=0}^{\infty} \pi_k = 1$. This is, however, impossible as the following computation shows.

$$1 \overset{!}{=} \sum_{k=0}^{\infty} \pi_k = \pi_0 + \sum_{k=1}^{\infty} \frac{1}{k+1}\pi_0 = \infty.$$

We thus see, that X does not admit a stationary distribution. Hence, the states cannot be positive recurrent and we need to decide whether they are null recurrent or transient. We define

$$\boldsymbol{R} := \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \ddots \\ 0 & 1 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

the $\mathbb{N} \times \mathbb{N}$ matrix which is obtained by crossing out the first row and column of $\boldsymbol{P}$. Now we need to solve $\boldsymbol{x} = \boldsymbol{R}\boldsymbol{x}$ for an unknown vector $\boldsymbol{x}^T = (x_1, x_2, \ldots)$. We get

$$\begin{aligned} x_1 &= 0 \\ x_2 &= x_1 \\ &\vdots \\ x_{k+1} &= x_k, \quad k = 1, 2, \ldots \end{aligned}$$

which implies that $0 = x_1 = x_2 = \ldots$ and thus, this equation has only a trivial solution (in particular on $[0, 1]$). This means that all the states are recurrent and since they cannot be positive recurrent, they must be null-recurrent. $\triangle$

Looking back: The equation $\boldsymbol{x} = \boldsymbol{R}\boldsymbol{x}$ always has the trivial solution. The question is whether this is the only solution which lives in the interval $[0, 1]$ (more precisely, $\boldsymbol{x} \in [0, 1]^{\mathbb{N}}$ or, $x_j \in [0, 1]$ for all $j \in \mathbb{N}$). In this exercise, the interval $[0, 1]$ was not important. However, sometimes the interval $[0, 1]$ becomes important - it is in the situation when there is another solution to $\boldsymbol{x} = \boldsymbol{R}\boldsymbol{x}$ such that $x_j \to c$ as $j \to \infty$ where $c \in (1, \infty]$ (see hint to Exercise 28).

Exercise 27. Classify the states of a homogeneous Markov chain $X = (X_n, n \in \mathbb{N}_0)$ given by the transition matrix

$$\boldsymbol{P} = \begin{pmatrix} p_0 & p_1 & p_2 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $0 < p_i < 1$ for all $i \in \mathbb{N}_0$ such that $\sum_{i=0}^{\infty} p_i = 1$.

Hint: Convince yourself that

$$\sum_{k=1}^{\infty} \left(1 - \sum_{j=0}^{k-1} p_j \right) = \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} p_j = \sum_{k=1}^{\infty} k p_k.$$

Exercise 28. Classify the states of a homogeneous Markov chain $X = (X_n, n \in \mathbb{N}_0)$ given by the transition matrix

$$\boldsymbol{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 & \cdots \\ \frac{1}{4} & 0 & 0 & \frac{3}{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Hint: Show that there is no stationary distribution and that the reduced system $\boldsymbol{x} = \boldsymbol{R}\boldsymbol{x}$ is solved by $x_k = \frac{k+1}{2} x_1$ for a fixed x_1. Now, if $x_1 \neq 0$, then $x_j \to \infty$ as $j \to \infty$ and hence, there must be an index j^* such that $x_{j^*} > 1$. Hence, one cannot choose $x_1 \neq 0$ in such a way that all x_k's are in $[0, 1]$ and we are left only with $x_1 = 0$ for which all x_k's belong to $[0, 1]$.

Exercise 29. Classify the states of a homogeneous Markov chain $X = (X_n, n \in \mathbb{N}_0)$ given by the transition matrix

$$\boldsymbol{P} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 & \cdots \\ \frac{2}{9} & \frac{2}{3} & \frac{1}{9} & 0 & 0 & \ddots \\ \frac{2}{27} & 0 & \frac{8}{9} & \frac{1}{27} & 0 & \ddots \\ \frac{2}{81} & 0 & 0 & \frac{26}{27} & \frac{1}{81} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Exercise 30. Classify the states of a homogeneous Markov chain $X = (X_n, n \in \mathbb{N}_0)$ given by the transition matrix

$$\mathbf{P} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 & \cdots \\ \frac{3}{16} & \frac{3}{4} & \frac{1}{16} & 0 & 0 & \ddots \\ \frac{3}{64} & 0 & \frac{15}{16} & \frac{1}{64} & 0 & \ddots \\ \frac{3}{256} & 0 & 0 & \frac{63}{64} & \frac{1}{256} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Exercise 31. Classify the states of a homogeneous Markov chain $X = (X_n, n \in \mathbb{N}_0)$ given by the transition matrix

$$\boldsymbol{P} = \begin{pmatrix} p_0 & p_1 & p_2 & \cdots \\ 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $0 < p_i < 1$ for all $i \in \mathbb{N}_0$ and $\sum_{i=0}^{\infty} p_i = 1$.

Exercise 32. Classify the states of a homogeneous Markov chain $X = (X_n, n \in \mathbb{N}_0)$ given by the transition matrix

$$\boldsymbol{P} = \begin{pmatrix} q_0 & p_0 & 0 & 0 & \cdots \\ q_1 & 0 & p_1 & 0 & \cdots \\ q_2 & 0 & 0 & p_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $0 < p_i < 1$ for all $i \in \mathbb{N}_0$ and $q_i = 1 - p_i$.

Exercise 33. Classify the states of a homogeneous Markov chain $X = (X_n, n \in \mathbb{N}_0)$ given by the transition matrix

$$\boldsymbol{P} = \begin{pmatrix} q_0 & 0 & p_0 & 0 & 0 & \cdots \\ q_1 & 0 & 0 & p_1 & 0 & \cdots \\ q_2 & 0 & 0 & 0 & p_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $0 < p_i < 1$ for all $i \in \mathbb{N}_0$ and $q_i = 1 - p_i$.

Hint: In this exercise, one could think that the state 1 (i.e. the one corresponding to the column of zeroes) is not reachable and should not belong to S. This depends on the initial distribution $\boldsymbol{p(0)}^T = (p(0)_0, p(0)_1, p(0)_2, \ldots)$. If $p(0)_1 = 0$, then we may cross out the state 1, relabel the states and this exercise becomes Exercise 32. Assume then that $p(0)_1 \neq 0$.

Exercise 34. Suppose a snail climbs an infinitely high tree. Each hour, our snail moves one centimetre up with probability 1/3 and with probability 2/3 it moves one centimetre down (it does not really want to fight gravity). If the snail reaches ground level, it moves one centimetre up in the next hour. Formulate the model for the height above ground in which our snail is so that you obtain a homogeneous Markov chain, find its transition matrix and classify its states.

Exercise 35. Modify Exercise 34 in such a way that the probability of going up is $p \in (0, 1)$ and the probability of going down is $q := 1 - p$. Classify the states of the resulting Markov chain in terms of p.

2.4 Absorption probabilities

Recall the Gampler's ruin problem (Exercise 22). One may ask, what is the probability (which will clearly depend on the initial wealth a) that the whole game will eventually result in the ruin of the gambler. In general, what is the probability, that if a Markov chain starts at a transient state i, the first visited recurrent state is a given j. These are called absorption probabilities.

Let X be a Markov chain with a state space S and denote by T the set of transient states and by C the set of recurrent states ($S = C \cup T$). Define

$$\tau := \min\{n \in \mathbb{N} : X_n \notin T\},$$

the time of leaving T and denote by X_τ the first visited recurrent state. Assume that $\mathbb{P}_i(\tau = \infty) = 0$ for all $i \in T$. Then we can define the *absorption probabilities*

$$u_{ij} := \mathbb{P}_i(X_\tau = j), \quad i \in T, j \in C.$$

Theorem 2.4.1. It holds that

$$u_{ij} = p_{ij} + \sum_{k \in T} p_{ik} u_{kj}, \quad i \in T, j \in C. \tag{2.4.1}$$

Theorem 2.4.1 can be easily applied in practice. Notice that if we relabel all the states in such a way that all the recurrent states have smaller index than all the transient states, we can write the transition matrix in the following way:

$$\boldsymbol{P} = \begin{array}{c} \\ C \\ T \end{array} \!\!\begin{array}{c} \begin{array}{cc} C & T \end{array} \\ \begin{pmatrix} \boldsymbol{P_C} & \boldsymbol{0} \\ \boldsymbol{Q} & \boldsymbol{R} \end{pmatrix} \end{array}.$$

Here $\boldsymbol{P_C} = (p_{ij})_{i,j \in C}$, $\boldsymbol{Q} = (p_{ij})_{i \in T, j \in C}$ and $\boldsymbol{R} = (p_{ij})_{i,j \in T}$. The formula (2.4.1) can then be written as

$$\boldsymbol{U} = \boldsymbol{Q} + \boldsymbol{RU}$$

which has one solution in $[0,1]$ if and only if $\mathbb{P}_i(\tau = \infty) = 0$ for every $i \in T$ (i.e. the chain leaves the set of transient states almost surely). The solution is, in general, written as a matrix geometric series

$$\hat{\boldsymbol{U}} = \sum_{n=0}^{\infty} \boldsymbol{R}^n \boldsymbol{Q}$$

and, in the case when $|S| < \infty$ it takes the form

$$\hat{\boldsymbol{U}} = (\boldsymbol{I}_T - \boldsymbol{R})^{-1} \boldsymbol{Q}. \tag{2.4.2}$$

Exercise 36. Classify the states of a Markov chain given by the transition probability matrix

$$\boldsymbol{P} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

and, if applicable, compute the matrix $\boldsymbol{U}$ of absorption probabilities into the set of recurrent states.

Solution to Exercise 36. Let us denote the Markov chain under consideration by X. Clearly, the state space of X is $S = \{0,1,2,3\}$ and we have that $1 \to 2 \to 3 \to 1$ and $0 \to 1 \not\to 0$. Hence, by Theorem 2.3.1, we have that the states $1,2,3$ are positive recurrent (they define a finite irreducible sub-chain) and 0 is a transient state. All the states are non-periodic ($p_{00} > 0$ and $p_{22} > 0$).
Let us further compute the absorption probabilities $\boldsymbol{U}$. The cookbook is as follows: first rearrange the states in such a way that all the recurrent states are in the upper left corner of $\boldsymbol{P}$, then compute $\hat{\boldsymbol{U}}$ from the formula (2.4.2). We obtain

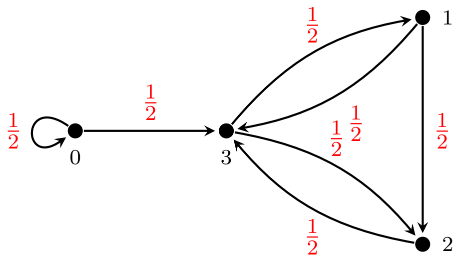
$$\boldsymbol{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 0 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 0 \end{array} \\ \left(\begin{array}{ccc|c} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \hline 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{array}\right) \end{array}$$

Now we compute $\hat{\boldsymbol{U}}$ from formula (2.4.2):

$$\hat{\boldsymbol{U}} = (\boldsymbol{I}_1 - \boldsymbol{R})^{-1} \boldsymbol{Q} = \left(1 - \frac{1}{2}\right)^{-1} \begin{pmatrix} 0 & 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} u_{01} & u_{02} & u_{03} \end{pmatrix}$$

The interpretation of the computed $\hat{\boldsymbol{U}}$ is that if X starts from 0, it jumps on spot for an unspecified amount of time but once it jumps to a different state then 0, it will be the state 3 almost surely. △

Looking back: Being given a chain in Exercise 36 with this particular $\boldsymbol{P}$, we can immediately answer the question which of the recurrent states will be the first one X visits after leaving the set of transient states. Since $p_{03} > 0$ and $p_{01} = p_{02} = 0$, it must be that the first state which is visited after leaving 0 is 3. This becomes even clearer once we draw the transition diagram of X:



Exercise 37. Let $p \in [0,1]$. Classify the states of the Markov chain given by the transition matrix

$$\boldsymbol{P} = \begin{pmatrix} p & 0 & 1-p & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & 1-p & p \\ 0 & p & 0 & 1-p \end{pmatrix}$$

and, if applicable, compute the matrix $\boldsymbol{U}$ of absorption probabilities into the set of recurrent states.

Exercise 38. Classify the states of the Markov chain given by the transition matrix

$$\boldsymbol{P} = \begin{pmatrix} \frac{1}{3} & \frac{1}{6} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} \end{pmatrix}$$

and, if applicable, compute the matrix $\boldsymbol{U}$ of absorption probabilities into the set of recurrent states.

Exercise 39. Classify the states of the Markov chain given by the transition matrix

$$\boldsymbol{P} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{6} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{6} & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

and, if applicable, compute the matrix $\boldsymbol{U}$ of absorption probabilities into the set of recurrent states.

Exercise 40. Classify the states of the Markov chain given by the transition matrix

$$\boldsymbol{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

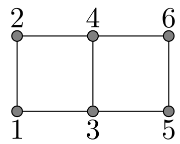
and, if applicable, compute the matrix $\boldsymbol{U}$ of absorption probabilities into the set of recurrent states.

Exercise 41. Classify the states of the Markov chain given by the transition matrix

$$\boldsymbol{P} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{8} & 0 & \frac{7}{8} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and, if applicable, compute the matrix $\boldsymbol{U}$ of absorption probabilities into the set of recurrent states.

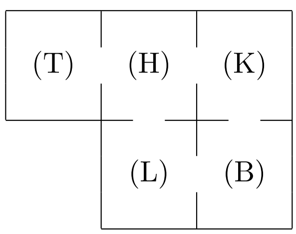
Exercise 42. A baby echidnea was born in the zoo. In order not to hurt itself, it is now only allowed to move in the following maze:



At each step, our baby echindea decides (uniformly randomly) upon one direction (west, east, north or south) and then it goes in that direction until it reaches a wall. If there is a wall in the chosen direction, our baby echidnea is confused and stays on the spot. Denote by X_n its position at step $n \in \mathbb{N}_0$. Classify the states of the Markov chain $X = (X_n, n \in \mathbb{N}_0)$ and compute the matrix of absorption probabilities $\boldsymbol{U}$ into the set of recurrent states.

Exercise 43. Consider an urn and five balls. At each step, we shall add or remove the balls to/from the urn according to the following scheme. If the urn is empty, we will add all the five balls into it. If it is not empty, we remove either four or two balls or none at all, every case with probability 1/3. Should we remove more balls than the urn currently contains, we remove all the remaining balls. Denote by X_n the number of balls in the urn at the n-th step. Classify the states of the Markov chain $X = (X_n, n \in \mathbb{N}_0)$ and compute the matrix of absorption probabilities $\boldsymbol{U}$ into the set of recurrent states.

Exercise 44. There is a mouse in the house! The mouse moves in such a way that in each room, it chooses one of the adjacent rooms (each with the same probability) and runs there (the movements occur at times $n = 1, 2, \ldots$). This is what our flat looks like:

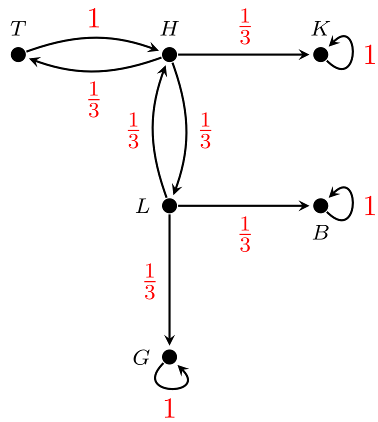


The rooms are the hall (H), kitchen (K), toilet (T), bedroom (B) and the living room (L). We can set two traps - one is mischievously installed in the bedroom and the other one in the kitchen since we really should not have a mouse there. As soon as the mouse enters a room with a trap, it is caught and it will never ever run into another room. Denote by X_n the position of the mouse at time n. Classify the states of the Markov chain $X = (X_n, n \in \mathbb{N}_0)$ and compute the matrix of absorption probabilities $\boldsymbol{U}$.

Exercise 45. Modify the previous Exercise. Suppose now that we open the door from our living room (L) to the garden (G). Once the mouse leaves the flat and enters the garden, it will never come inside again. If the mouse starts in the toilet, what is the probability that it will escape the flat before it is caught in a trap?

Solution to Exercise 45. Building the formal model (i.e. defining X_n and proving that it is a homogeneous discrete time Markov chain) is simple and could be done in a similar way as for any random walk with absorbing states (see e.g. Exercise 22). We will focus on the task at hand: finding the probabilities u_{ij}. First, we shall find the transition probabilities $\boldsymbol{P}$ and the transition diagram describing X. We have that

$$\boldsymbol{P} = \begin{array}{c} \\ K \\ B \\ G \\ T \\ H \\ L \end{array} \begin{array}{c} \begin{array}{cccccc} K & B & G & T & H & L \end{array} \\ \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \end{array}\right) \end{array}$$



Now we have two options. Either to compute the whole matrix $\hat{\boldsymbol{U}}$ or just to compute u_{TG}.
Computing $\hat{\boldsymbol{U}}$: This can be done as in the previous exercises using the formula (2.4.2). We have

$$\hat{\boldsymbol{U}} = (\boldsymbol{I}_3 - \boldsymbol{R})^{-1}\boldsymbol{Q} = \begin{pmatrix} 1 & -1 & 0 \\ -\frac{1}{3} & 1 & -\frac{1}{3} \\ 0 & -\frac{1}{3} & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \end{pmatrix}.$$

Of course, one can make many mistakes in computing the inverse of a matrix 3×3. You should always check that all the rows of your final matrix $\hat{\boldsymbol{U}}$ are probability distributions (i.e. that they sum up to 1).
Computing only u_{TG}: We can also appeal to the formula (2.4.1) and write down only those equations which interest us. We are interested in u_{LG} so let us rewrite (2.4.2) for $i = L$ and $j = G$. We obtain

$$u_{TG} = p_{TG} + p_{TT}u_{TG} + p_{TH}u_{HG} + p_{TL}u_{LG} = u_{HG}$$

Hence, we also need an equation for u_{HG}. We again apply formula (2.4.2) and look into $\boldsymbol{P}$ to obtain

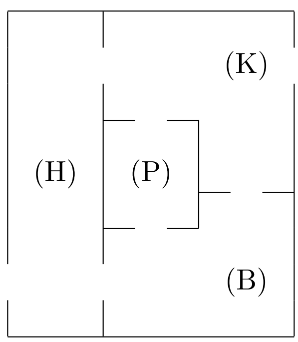
$$u_{HG} = p_{HG} + p_{HT}u_{TG} + p_{HH}u_{HG} + p_{HL}u_{LG} = \frac{1}{3}u_{TG} + \frac{1}{3}u_{LG}$$

which means that we also need an equation for u_{LG}, namely,

$$u_{LG} = p_{LG} + p_{LT}u_{TG} + p_{LH}u_{HG} + p_{LL}u_{LG} = \frac{1}{3} + \frac{1}{3}u_{HG}.$$

Now you can choose your favourite method of solving this system of equation to obtain the solution $u_{TG} = \frac{1}{5}$. $\triangle$

Exercise 46. There is a mouse in the house again! This time, this is what our flat looks like:



The rooms are kitchen (K), hall (H), bedroom (B) and pantry (P). The mouse moves in such a way that in every room it chooses one of the doors (each with the same probability) and runs through those to a different room (or out of the flat). The movements occur at times $n = 1, 2, \ldots$. We have set one trap in the pantry. If the mouse enters a room with a trap, it is caught and it will never run again. If the mouse leaves the house, it forgets the way back and lives happily ever after (and it will never come back to our flat). Denote by X_n the position of the mouse at time n. Classify the states of the Markov chain $X = (X_n, n \in \mathbb{N}_0)$. What is the probability that if the mouse starts in the bedroom, it will escape the flat before it is caught in the trap?

Exercise 47. Recall the gambler's ruin problem (Exercise 22) and assume that the gambler starts with the initial wealth \$$a$ and that the total wealth in the game is \$$s$. Compute the probability that the game will eventually result in the ruin of the gambler.

Solution to Exercise 47. Recall that if X is the Markov chain modelling the gambler's wealth, then its state space is $S = \{0\} \cup \{s\} \cup \{1, 2, \ldots, s-1\}$ where 0 and s are absorbing states and $1, 2, \ldots, s-1$ are transient states. Let us assume that $a \in \{1, 2, \ldots, s-1\}$ (otherwise the game would be somewhat quick). Our task is to compute the probabilities of absorption from the transient state a to the absorbing state 0, i.e. we will compute u_{a0}. Rearranging the states, we obtain

$$\begin{array}{cc} & \begin{array}{cccccccc} 0 & s & 1 & 2 & 3 & \dots & s-2 & s-1 \end{array} \\ \boldsymbol{P} = \begin{array}{c} 0 \\ s \\ 1 \\ 2 \\ 3 \\ \vdots \\ s-2 \\ s-1 \end{array} & \left(\begin{array}{cc|cccccc} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \ddots & 0 & 0 \\ \hline q & 0 & 0 & p & 0 & \ddots & 0 & 0 \\ 0 & 0 & q & 0 & p & \ddots & 0 & 0 \\ 0 & 0 & 0 & q & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 & p \\ 0 & p & 0 & 0 & 0 & \cdots & q & 0 \end{array}\right) \end{array}.$$

With these ingredients in hand, we shall make use of the formula

$$\boldsymbol{U} = \boldsymbol{Q} + \boldsymbol{R}\boldsymbol{U}.$$

Now, we see that

$$\begin{aligned} u_1 - q - pu_2 &= 0 \\ u_i - qu_{i-1} - pu_{i+1} &= 0, \quad i = 2, 3, \dots, s-2 \\ u_{s-1} - qu_{s-2} &= 0 \end{aligned}$$

Let us write $u_i := u_{i0}$ for $i = 1, 2, \dots, s-1$ and define $u_0 := 1$ and $u_s := 0$. We obtain the following difference equation with boundary conditions

$$\begin{aligned} u_i - qu_{i-1} - pu_{i+1} &= 0, \quad i = 1, 2, \dots, s-1 \\ u_0 &= 1 \\ u_s &= 0 \end{aligned}$$

Its characteristic polynomial is

$$\chi(\lambda) = -p\lambda^2 + \lambda - q$$

and its roots are $\lambda_1 = 1$ and $\lambda_2 = q/p$.
The case $p \neq q$: In this case, we have two different roots λ_1 and λ_2, the fundamental solution to the difference equation is then $\{1, q/p\}$ and the general solution to the homogeneous equation is

$$u_i = A \cdot 1^i + B \cdot \left(\frac{q}{p}\right)^i, \quad i = 1, 2, \dots, s-1.$$

where A and B are constants which will be determined from the boundary conditions. Namely, we have that

$$\begin{aligned} 1 &= A + B \\ 0 &= A + B\left(\frac{q}{p}\right)^s \end{aligned}$$

which gives

$$A = -\frac{\left(\frac{q}{p}\right)^s}{1-\left(\frac{q}{p}\right)^s}, \qquad B = \frac{1}{1-\left(\frac{q}{p}\right)^s}.$$

Finally, we arrive at

$$u_i = \frac{\left(\frac{q}{p}\right)^i - \left(\frac{q}{p}\right)^s}{1-\left(\frac{q}{p}\right)^s}, \quad i = 0, 1, \ldots, s.$$

The case $p = q$: In this case, $\chi(\lambda)$ only has one root and the general solution is given by

$$u_i = A + Bi, \quad i = 1, 2, \ldots, s-1,$$

where

$$\begin{aligned} 1 &= A \\ 0 &= A + Bs. \end{aligned}$$

Hence,

$$u_i = 1 - \frac{i}{s}, \quad i = 0, 1, \ldots, s.$$

△

2.5 Steady-state and long-term behaviour

Theorem 2.5.1 (Strong law of large numbers for Markov chains)**.** Let X be an irreducible homogeneous discrete time Markov chain with positive recurrent states S. Let $\boldsymbol{\pi} = (\pi_j)_{j \in S}$ be its stationary distribution. Then

$$\frac{1}{n}\sum_{k=0}^{n} \mathbf{1}_{[X_k = j]} \quad \underset{n\to\infty}{\longrightarrow} \quad \pi_j, \quad j \in S$$

Exercise 48. Consider the mouse moving in our flat from Exercise 44. Assume that there are no traps and the doors to the garden are closed. We wish to analyse the behavioural patterns of our mouse. The mouse moves in the same way as before - in each room, it chooses (uniformly randomly) one of the adjacent rooms and moves there. What is the long-run proportion of time the mouse spends in each of the rooms?

2.6 Answers to exercises

Answer to Exercise 11. The Galton-Watson branching process is a homogeneous discrete time Markov chain with a the transition matrix

$$\boldsymbol{P} = (p_{ij})_{i,j=0}^{\infty} = [P_U(s)^i]_j$$

where $[R(s)]_k$ denotes the coefficient of a polynomial R at s^k. For the particular case of U, the distribution of X_2 is

$$\boldsymbol{p(s)}^T = \left(\frac{33}{125}, \frac{11}{125}, \frac{36}{125}, \frac{18}{125}, \frac{27}{125}, 0, \dots\right).$$

Answer to Exercise 12. The symmetric random walk is a homogeneous discrete time Markov chain.

Answer to Exercise 13. The running maximum is a homogeneous discrete time Markov chain.

Answer to Exercise 14. The claim holds. What is more, the converse is also true as shown for example in [**?**], Proposition 7.6, p. 122.

Answer to Exercise 15. The process X is not a Markov chain. Consider trajectories $(-2, 0, 2)$ and $(2, 0, 2)$.

Answer to Exercise 16. The symmetric random walk on $\mathbb{Z}$ has 2-periodic, null-recurrent states.

Answer to Exercise 17. We have that $p^{(n)}_{-1,-1} = \left(\frac{1}{3}\right)^n$, $p^{(n)}_{0,0} = \left(\frac{2}{3}\right)^n$ and $p^{(n)}_{1,1} = 1$. Hence, $-1, 0$ are non-periodic, transient states and 1 is a non-periodic, positive recurrent state (the so-called absorbing state).

Answer to Exercise 18. The transition matrix is

$$\boldsymbol{P} = \begin{pmatrix} q & p & 0 & 0 & \cdots \\ q & 0 & p & 0 & \cdots \\ q & 0 & 0 & p & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \quad \boldsymbol{P}^n = \begin{pmatrix} q & qp & qp^2 & \cdots & qp^{n-1} & p^n & 0 & 0 & \cdots \\ q & qp & qp^2 & \cdots & qp^{n-1} & 0 & p^n & 0 & \cdots \\ q & qp & qp^2 & \cdots & qp^{n-1} & 0 & 0 & p^n & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

All the states are non-periodic, positive recurrent.

Answer to Exercise 19. We have that

$$\boldsymbol{P}^n = \frac{1}{a+b} \begin{pmatrix} b + a(1-(a+b))^n & a - a(1-(a+b))^n \\ b - b(1-(a+b))^n & a + b(1-(a+b))^n \end{pmatrix}$$

and all the states are positive recurrent and non-periodic.

Answer to Exercise 20. We have that

$$\boldsymbol{P}^n = \begin{pmatrix} \frac{1}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^n & \frac{1}{3} - \frac{1}{3}\left(-\frac{1}{2}\right)^n & \frac{1}{3} - \frac{1}{3}\left(-\frac{1}{2}\right)^n \\ \frac{1}{3} - \frac{1}{3}\left(-\frac{1}{2}\right)^n & \frac{1}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^n & \frac{1}{3} - \frac{1}{3}\left(-\frac{1}{2}\right)^n \\ \frac{1}{3} - \frac{1}{3}\left(-\frac{1}{2}\right)^n & \frac{1}{3} - \frac{1}{3}\left(-\frac{1}{2}\right)^n & \frac{1}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^n \end{pmatrix}$$

and all the states are positive recurrent and non-periodic.

Answer to Exercise 21. X is a homogeneous discrete-time Markov chain with the state space $S = \{0, 1, \ldots, k\}$ and the transition matrix

$$\boldsymbol{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{k} & 1 - \frac{1}{k} & 0 & \ddots & 0 & 0 \\ 0 & 0 & \frac{2}{k} & 1 - \frac{2}{k} & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & \cdots & \frac{k-1}{k} & 1 - \frac{k-1}{k} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

The states $0, \ldots, k-1$ are transient and non-periodic, the state k is absorbing (i.e. positive recurrent, non-periodic).

Answer to Exercise 22. X is a homogeneous discrete-time Markov chain with the state space $S = \{0, 1, \ldots, s\}$ and the transition matrix

$$\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & q & 0 & p & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & q & 0 & p \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{pmatrix}$$

The states $1, \ldots, s-1$ are transient and 2-periodic, the states 0 and s are absorbing.

Answer to Exercise 23. Denote X_n the number of balls in the first container at time n. Then $X = (X_n, n \in \mathbb{N}_0)$ is a homogeneous discrete-time Markov chain whose state space is $S = \{0, 1, \ldots, 2l\}$ and transition matrix

$$\boldsymbol{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{2l} & 0 & 1 - \frac{1}{2l} & 0 & \cdots & 0 & 0 \\ 0 & \frac{2}{2l} & 0 & 1 - \frac{2}{2l} & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2l}{2l} & 0 \end{pmatrix}$$

(i.e. $p_{01} = p_{2l-1,2l} = 1$ and $p_{k,k-1} = \frac{k}{2l}, p_{k,k+1} = 1 - \frac{k}{2l}$ for $k = 1, \ldots, 2l-1$ and $p_{ij} = 0$ otherwise). All the states are positive recurrent and 2-periodic.

Answer to Exercise 24. Denote by X_n the number of white balls in the first container. Then $X = (X_n, n \in \mathbb{N}_0)$ is a homogeneous discrete-time Markov chain whose state space is $S = \{0, 1, \ldots, l\}$ and transition matrix

$$\boldsymbol{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \left(\frac{1}{l}\right)^2 & \frac{2(l-1)}{l^2} & \left(\frac{l-1}{l}\right)^2 & 0 & \cdots & 0 & 0 \\ 0 & \left(\frac{2}{l}\right)^2 & \frac{4(l-2)}{l^2} & \left(\frac{l-2}{l}\right)^2 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \left(\frac{l}{l}\right)^2 & 0 \end{pmatrix}$$

(i.e. $p_{0,1} = p_{l-1,l} = 1$ and $p_{k,k-1} = \left(\frac{k}{l}\right)^2, p_{k,k} = \frac{2k(l-k)}{l^2}, p_{k,k+1} = \left(\frac{l-k}{l}\right)^2$ for $k = 1, \dots, l-1$ and $p_{ij} = 0$ otherwise). All the states are positive recurrent and non-periodic.

Answer to Exercise 25. There is a stationary distribution $\boldsymbol{\pi}^T = (p_k)_{k=0}^{\infty}$ where $p_k = \frac{1}{2^{k+1}}$ for $k \in \mathbb{N}_0$. All the states are non-periodic and positive recurrent.

Answer to Exercise 26. There is no stationary distribution. All the states are non-periodic and null-recurrent.

Answer to Exercise 27. The stationary distribution exists if and only if the sum $\sum_{k=1}^{\infty} kp_k < \infty$. If this is so, the states are all positive recurrent and non-periodic. If $\sum_{k=1}^{\infty} kp_k = \infty$, then there is no stationary distribution and all the states are null-recurrent and non-periodic.

Answer to Exercise 28. There is no stationary distribution. All the states are non-periodic and null-recurrent.

Answer to Exercise 29. There is no stationary distribution. All the states are non-periodic and null-recurrent.

Answer to Exercise 30. There is no stationary distribution. All the states are non-periodic and null-recurrent.

Answer to Exercise 31. There is a stationary distribution $\boldsymbol{\pi}^T = (p_k)_{k=0}^{\infty}$ where $p_0 = \frac{1}{2-p_0}$ and $p_k = \frac{p_k}{1-p_0}$ for $k \in \mathbb{N}$. All the states are non-periodic and positive recurrent.

Answer to Exercise 32. The stationary distribution exists if and only if $\sum_{k=1}^{\infty} \prod_{j=0}^{k-1} p_j < \infty$. If this is so, all the states are positive recurrent and non-periodic. If $\sum_{k=1}^{\infty} \prod_{j=0}^{k-1} p_j = \infty$, then all the states are null-recurrent and non-periodic.

Answer to Exercise 33. All the states are non-periodic. The state 1 is transient. If $\sum_{k=1}^{\infty} \prod_{j=0}^{k-1} p_j < \infty$, the states $0, 2, 3, 4, \dots$ are positive recurrent. If $\sum_{k=1}^{\infty} \prod_{j=0}^{k-1} p_j = \infty$, then the state $0, 2, 3, 4, \dots$ are null-recurrent.

Answer to Exercise 34. Let $Y = (Y_i, i \in \mathbb{N})$ be a sequence of independent, identically distributed random variables with values in $\{-1, 1\}$ defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let their common distribution be

$$\mathbb{P}(Y_1 = -1) = \frac{2}{3}, \quad \mathbb{P}(Y_1 = 1) = \frac{1}{3}.$$

Define $X_0 := 0$ and

$$X_{n+1} := \mathbf{1}_{[X_n = 0]} + \mathbf{1}_{[X_n \neq 0]}(X_n + Y_{n+1}), \quad n = 0, 1, 2, \dots$$

X_n represents the height in which the snail is at time n. Then $X = (X_n, n \in \mathbb{N})$ is a homogeneous discrete time Markov chain with the state space $S = \mathbb{N}_0$. Its transition matrix is

$$\boldsymbol{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 & \cdots \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 & \ddots \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

X admits a stationary distribution $\pi_0 = \frac{1}{4}$ and $\pi_k = \frac{3}{4}\left(\frac{1}{2}\right)^k$ for $k = 1, 2, \ldots$ All its states are 2-periodic and positive recurrent.

Answer to Exercise 35. Regardless of the value of the parameter p, X defined similarly as in Exercise 34 has 2-periodic states. If $p \leq q$, then all the states are null-recurrent. If $p > q$, then all the states are transient.

Answer to Exercise 36. If the states are labeled $S = \{0, 1, 2, 3\}$, then all the states are non-periodic, $1, 2, 3$ are positive recurrent and 0 is transient. The absorption probabilities are given by

$$\hat{\boldsymbol{U}} = (0, 0, 1).$$

Answer to Exercise 37. If $p \in (0, 1)$, then the chain is irreducible and all the states are non-periodic, positive recurrent. If $p = 1$, then the chain is not irreducible, the state space can be written as $S = \{0\} \cup \{1\} \cup \{2, 3\}$ where 0 and 1 are absorbing states and $2, 3$ are non-periodic and positive recurrent. If $p = 0$, then the chain is not irreducible, the state space can be written as $S = \{0, 1\} \cup \{2\} \cup \{3\}$ where $0, 1$ are non-periodic transient states, 2 and 3 are absorbing states and the absorption probabilities are given by

$$\hat{\boldsymbol{U}} = \begin{pmatrix} u_{02} & u_{03} \\ u_{12} & u_{13} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

Answer to Exercise 38. The chain is not irreducible, its state space can be written as $S = \{0, 3\} \cup \{1, 2, 4\}$ where $0, 3$ are non-periodic transient states and $1, 2, 4$ are non-periodic positive recurrent states. The absorption probabilities are

$$\hat{\boldsymbol{U}} = \begin{pmatrix} u_{01} & u_{02} & u_{04} \\ u_{31} & u_{32} & u_{34} \end{pmatrix} = \begin{pmatrix} \frac{2}{7} & \frac{2}{7} & \frac{3}{7} \\ \frac{2}{21} & \frac{16}{21} & \frac{3}{21} \end{pmatrix}.$$

Answer to Exercise 39. The chain is not irreducible, its state space can be written as $S = \{1, 3\} \cup \{0, 2, 4\}$ where $1, 3$ are 2-periodic, transient and $0, 2, 4$ are non-periodic, positive recurrent. The absorption probabilities are given by

$$\hat{\boldsymbol{U}} = \begin{pmatrix} u_{10} & u_{12} & u_{14} \\ u_{30} & u_{32} & u_{34} \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 4 & 3 & 2 \\ 2 & 3 & 4 \end{pmatrix}.$$

Answer to Exercise 40. The chain is not irreducible, its state space can be written as $S = \{3, 4\} \cup \{1, 2, 5, 6\}$ where $3, 4$ are non-periodic transient states and $1, 2, 5, 6$ are non-periodic positive recurrent states. The absorption probabilities are given by

$$\hat{\boldsymbol{U}} = \begin{pmatrix} u_{31} & u_{32} & u_{35} & u_{36} \\ u_{41} & u_{42} & u_{45} & u_{46} \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 3 & 1 & 3 & 1 \\ 1 & 3 & 1 & 3 \end{pmatrix}.$$

Answer to Exercise 41. The chain is not irreducible, its state space can be written as $S = \{0\} \cup \{1,2\} \cup \{4\} \cup \{3,5,6\}$ where 0 is a non-periodic transient state, $1,2$ are 2-periodic transient states, 4 is an absorbing state and $3,5,6$ are 3-periodic positive recurrent states. The absorption probabilities are given by

$$\hat{\boldsymbol{U}} = \begin{pmatrix} u_{03} & u_{04} & u_{05} & u_{06} \\ u_{13} & u_{14} & u_{15} & u_{16} \\ u_{23} & u_{24} & u_{25} & u_{26} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{7}{9} & \frac{1}{9} & \frac{1}{9} & 0 \\ \frac{8}{9} & \frac{1}{18} & \frac{1}{18} & 0 \end{pmatrix}.$$

Answer to Exercise 42. This is the same Markov chain as in Exercise 40.

Answer to Exercise 43. The state space of X is $S = \{0,1,2,3,4,5\}$ and its transition matrix is given by

$$\boldsymbol{P} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}.$$

S can be written as $S = \{2,4\} \cup \{0,1,3,5\}$ where $2,4$ are non-periodic, transient states and $0,1,3,5$ are non-periodic positive recurrent states. The absorption probabilities are given by

$$\hat{\boldsymbol{U}} = \begin{pmatrix} u_{20} & u_{21} & u_{23} & u_{25} \\ u_{40} & u_{41} & u_{43} & u_{45} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

This is not surprising at all when you look at the transition diagram for X (draw this yourself!).

Answer to Exercise 44. The states T, H, L are 2-periodic, transient and K and B are both absorbing. The absorption probabilities are given by

$$\hat{\boldsymbol{U}} = \begin{pmatrix} u_{TK} & u_{TB} \\ u_{HK} & u_{HB} \\ u_{LK} & u_{LB} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

Answer to Exercise 45. The states T, H, L are 2-periodic, transient and the states K, B and G are absorbing. The absorption probabilities are given by

$$\hat{\boldsymbol{U}} = \begin{pmatrix} u_{TK} & u_{TB} & u_{TG} \\ u_{HK} & u_{HB} & u_{HG} \\ u_{LK} & u_{LB} & u_{LG} \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \end{pmatrix}.$$

In particular, the probability that the mouse will escape the house before being caught in a trap when it starts at the toilet is $u_{TG} = 1/5$.

Answer to Exercise 46. Denote by (O) the state "outside". Then the states K, B and H are 2-periodic, transient and the states P and O are absorbing. The absorption probabilities are given by

$$\hat{\boldsymbol{U}} = \begin{pmatrix} u_{HP} & u_{HO} \\ u_{KP} & u_{KO} \\ u_{BP} & u_{BO} \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 3 & 5 \\ 4 & 4 \\ 5 & 3 \end{pmatrix}$$

and, in particular, $u_{BO} = 3/8$.

Answer to Exercise 47. The probability of ruining the gambler depends on his initial wealth \$$a$, the total wealth in the game \$$s$ and the probability of the gambler winning the game p and is given by

$$u_{a0} = \begin{cases} 1 - \frac{i}{s}, & p = q, \\ \frac{\left(\frac{q}{p}\right)^i - \left(\frac{q}{p}\right)^s}{1 - \left(\frac{q}{p}\right)^s}, & p \neq q, \end{cases} \qquad a \in \{1, 2, \ldots, s-1\}.$$

Answer to Exercise 48. The long-run proportion of time the mouse spends in the rooms T, H, K, B, L, is $1/10, 3/10, 2/10, 2/10, 2/10$, respectively.

Chapter 3

Continuous time Markov Chains

Definition 3.0.1 (Continuous time Markov chain)**.** A $\mathbb{Z}$-valued random process $X = (X_t, t \geq 0)$ is called a continuous time Markov chain with a state space S if

1. $S = \{i \in \mathbb{Z} : \exists\, t \geq 0 \text{ such that } \mathbb{P}(X_t = i) > 0\}$
2. and it holds that

$$\mathbb{P}(X_t = j | X_s = i, X_{s_n} = i_n, \dots, X_{s_1} = i_1) = \mathbb{P}(X_t = j | X_s = i),$$

   for all $n \in \mathbb{N}_0$ and all states $i, j, i_n, \dots, i_1 \in S$ and times $0 \leq s_1 < s_2 < \dots < s_n < s < t$ such that $\mathbb{P}(X_s = i, X_{s_n} = i_n, \dots, X_{s_1} = i_1) > 0$.

The first condition means that we only consider the effective states - those states we are able to reach at some time t and the second condition is called the Markov property. Again, roughly speaking, it says that to determine the probability distribution of the process at a future time, we only need to know the current state and not the whole preceding history.
Similarly as in the previous chapter, for a continuous time Markov chain X, we define the transition probability from state i at time s to state j at time $s + h$ as

$$p_{ij}(s, s+h) := \mathbb{P}(X_{s+h} = j | X_s = i)$$

where $i, j \in S$ and $s, h \geq 0$. With these, we can create the stochastic matrix $\boldsymbol{P(s, s+h)} = (p_{ij}(s, s+h))_{i,j \in S}$ which is called the transition matrix of X from time s to time $s + h$.

Definition 3.0.2. If for every $h \geq 0$, there is a stochastic matrix $\boldsymbol{P(h)}$ such that

$$\boldsymbol{P(s, s+h)} = \boldsymbol{P(h)}$$

for all $s \geq 0$, then X is called a (time) homogeneous continuous time Markov chain.

In the case of continuous time Markov chains, there is no natural notion of a "step". Hence, we cannot expect to obtain a formula analogous to $\boldsymbol{P(k)} = \boldsymbol{P}^k$ which we obtained for discrete time chains. However, it turns out that the fundamental property of transition matrices of a homogeneous continuous time Markov chain is the Chapman-Kolmogorov equality:

$$\boldsymbol{P(s+t)} = \boldsymbol{P(s)} \cdot \boldsymbol{P(t)}, \quad s,t \geq 0.$$

with $\boldsymbol{P(0)} := \boldsymbol{I}_S$. If we are interested in the probability distribution of X_t, denoted by $\boldsymbol{p(t)}$, we have that

$$\boldsymbol{p(t)}^T = \boldsymbol{p(0)}^T \boldsymbol{P(t)}, \quad t \geq 0,$$

where $\boldsymbol{p(0)}$ denotes the initial distribution of X, i.e. the vector which contains the probabilities $\mathbb{P}(X_0 = k) > 0$ for $k \in S$.

3.1 Generator $\boldsymbol{Q}$

So far, the definitions and properties are analogous to the discrete time case. The major exception is that for a homogeneous chain, we do not have only one matrix $\boldsymbol{P}$ but rather a whole system of matrices $(\boldsymbol{P(t)}, t \geq 0)$ which satisfies the Chapman-Kolmogorov equality[1]. Think about it. In order to describe the process X, we need to know $\boldsymbol{P(t)}$ for every $t \geq 0$. When we had the discrete time Markov chains, we could generate the system $(\boldsymbol{P(k)}, k \in \mathbb{N}_0)$ simply by multiplying the matrix $\boldsymbol{P}$. Is is possible to find one object which can be used to generate the continuous system $(\boldsymbol{P(t)}, t \geq 0)$? As it turns out, this really is the case.

Theorem 3.1.1. Assume that the transition probabilities are right continuous functions at 0 (i.e. $\lim_{t\to 0+} p_{ij}(t) = \delta_{ij}$). Then for all $i \in S$ there exists the (finite or infinite) limit

$$\lim_{h\to 0+} \frac{1 - p_{ii}(h)}{h} =: q_i =: -q_{ii}$$

and for all $i,j \in S$ there exists the (finite) limit

$$\lim_{h\to 0+} \frac{p_{ij}(h)}{h} =: q_{ij}.$$

Theorem 3.1.1 assures that we can define the matrix

$$\boldsymbol{Q} := \lim_{h\to 0+} \frac{\boldsymbol{P(h)} - \boldsymbol{I}_S}{h}$$

which is called the intensity matrix, transition rate matrix or (infinitesimal) generator[2] of the Markov chain X. The elements q_{ij} for $i \neq j$ are called transition intensities or transition

[1] A footnote useful if ever you will deal with general Markov processes with a general state space: Here we have a system of matrices (i.e. linear operators), a notion of a dot "$\cdot$" (i.e. matrix multiplication/operator composition) and a neutral element (i.e. $\boldsymbol{P(0)} = \boldsymbol{I}_S$) with the following properties:

1. $\boldsymbol{P(s+t)} = \boldsymbol{P(s)} \cdot \boldsymbol{P(t)}$ for all $s,t \geq 0$
2. $\|\boldsymbol{P(t)}\boldsymbol{v} - \boldsymbol{v}\|_{\mathbb{R}^{|S|}} \to 0$ as $t \to 0+$ for every $\boldsymbol{v} \in [0,1]^{|S|}$.

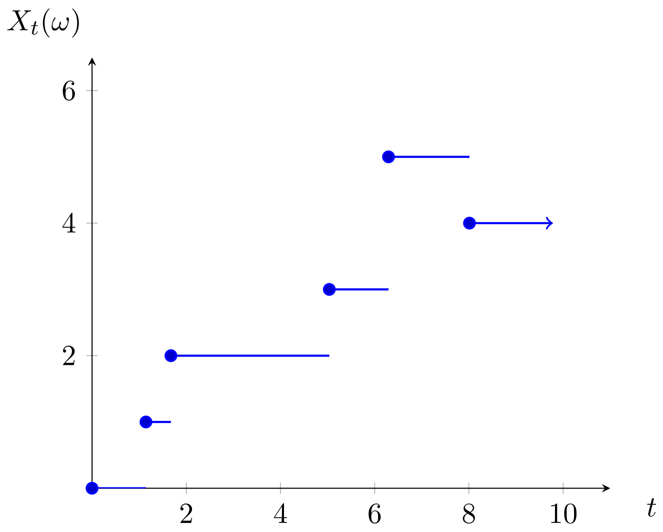
Such a system is called a strongly continuous semigroup of linear operators and in the case of Markov processes, this particular semigroup is called the Markov semigroup. Generally, strictly speaking, a strongly continuous semigroup is a representation of the semigroup $(\mathbb{R}_+, +)$ on some Banach space X (stochastic matrices in our case) that is continuous in the strong operator topology.

[2] The definition of $\boldsymbol{Q}$ might resemble the notion of derivative of the matrix-valued function $h \mapsto \boldsymbol{P(h)}$ since (recall) $\boldsymbol{I}_S = \boldsymbol{P(0)}$. This is not a coincidence.

rates from $i \in S$ to $j \in S$. The numbers q_i are called exit intensities/rates from the state $i \in S$. As we will see in the sequel, the matrix $\boldsymbol{Q}$ can be used to generate the whole semigroup $(\boldsymbol{P(t)}, t \geq 0)$ (via the Kolmogorov forward/backward equations).
For Theorem 3.1.1 and other technical reasons, we will always assume that our continuous-time Markov chain satisfies the following:

- X is homogeneous,
- transition probabilities are right continuous functions at 0 (i.e. $\lim_{t\to 0+} p_{ij}(t) = \delta_{ij}$),
- X is separable and measurable,
- X has right continuous sample paths,
- $\sum_{j\in S} q_{ij} = 0$ for all $i \in S$.

A typical sample path of a homogeneous continuous time Markov chain is the following:



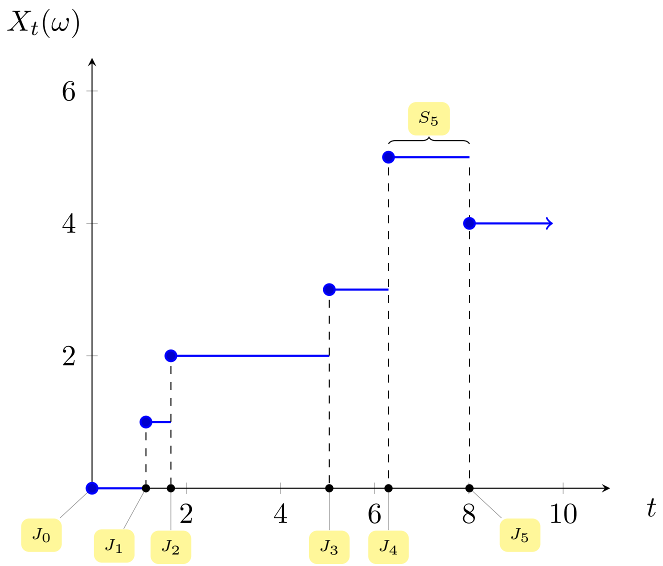
The picture suggests the following definitions: We will define

$$
\begin{aligned}
J_0 &:= 0 \\
J_{n+1} &:= \inf(t > 0 : X_t \neq X_{J_n}), \quad n \in \mathbb{N}_0.
\end{aligned}
$$

with the convention that $\inf \emptyset := \infty$. A careful reader should stop here. In general, there is no guarantee that X_{J_n} are random variables since J_n are themselves random variables. However, it can be shown that J_n are really well-defined. The J_n are called the jump times of X. Further, we can define for $n \in \mathbb{N}_0$ the following random variables:

$$
S_{n+1} := \begin{cases} J_{n+1} - J_n, & J_n < \infty \\ \infty, & \text{otherwise.} \end{cases}
$$

The random variables S_n are called the holding times.

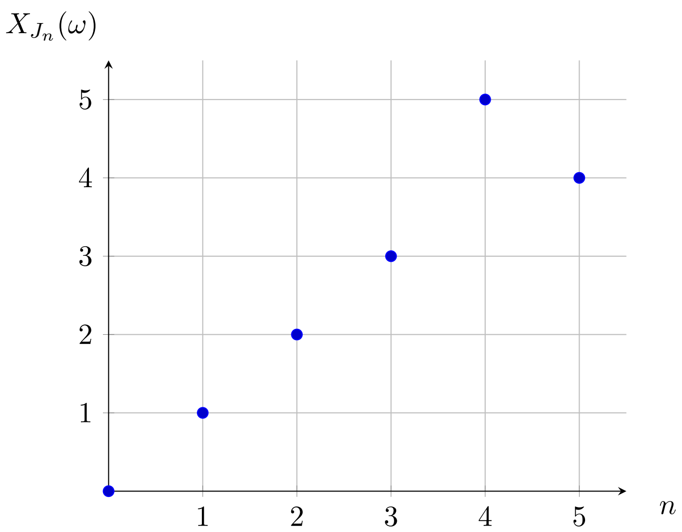


Finally, let us define the matrix

$$\boldsymbol{Q}^* = (q^*_{ij})_{i,j\in S}, \qquad \text{where} \qquad q^*_{ij} := \begin{cases} \frac{q_{ij}}{q_i}\mathbf{1}_{[i\neq j]}, & \text{if } q_i \neq 0 \\ \mathbf{1}_{[i=j]}, & \text{otherwise.} \end{cases}$$

Theorem 3.1.2. The stochastic process $Y = (Y_n, n \in \mathbb{N}_0)$ where $Y_n := X_{J_n}$ for $n \in \mathbb{N}_0$ is a homogeneous discrete time Markov chain whose transition matrix is $\boldsymbol{Q}^*$.

The process Y from Theorem 3.1.2 is called the embedded chain of X. Theorem 3.1.2 says in particular that if X is in the state i (and stays there for some time), then the probability that the state which X will reach when it jumps next will be a given state j is $\frac{q_{ij}}{q_i}$ (we assume that $i \neq j$ and $q_i \neq 0$). A sample path of the embedded chain corresponding to our example is as follows.



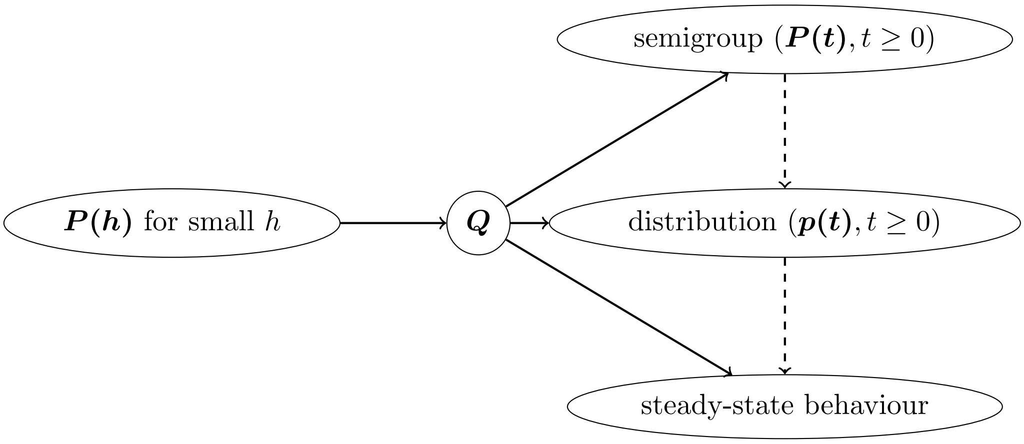
The next theorem answers the question how long X will stay in i.

Theorem 3.1.3. Let S_n be the holding times of a homogeneous continuous time Markov chain. Then

$$S_{n+1}|Y_0,\ldots,Y_n;S_1,\ldots,S_n;J_n<\infty \quad \sim \quad \mathrm{Exp}\,(q_{Y_n}).$$

In particular, Theorem 3.1.3 says that the holding times are all independent and exponentially distributed random variables. So, if you ever want to simulate paths of continuous time Markov chains, it suffices to generate independent exponentially distributed holding times (with appropriate parameters) and its embedded chain.

The analysis of continuous time Markov chains is done in more steps. First, we need to built a model. This is typically done in such a way that we define the approximate behaviour of the process on a small time interval, i.e. we are given the Taylor expansion of $p_{ij}(h)$ up to the first order (or we have to construct it). This is called the infinitesimal definition. In the next step, we have to find the generator $\boldsymbol{Q}$. Once we have $\boldsymbol{Q}$, we are able to obtain the whole system $(\boldsymbol{P(t)}, t \geq 0)$, which fully describes the process, or the distribution X_t for all $t \geq 0$ or we can analyse the limiting and steady-state behaviour of the process when $t \to \infty$.



Hence, we need to be able to find the generator $\boldsymbol{Q}$. This is practised in the following exercises. Also, since we will need some facts about the embedded chain in the sequel, you are also asked to find its transition matrix $\boldsymbol{Q^*}$.

Exercise 49 (Poisson process)**.** Consider certain events which repeatedly occur randomly in continuous time (e.g. incoming calls, customers arriving the the shop). We assume that number of events which occur in disjoint time intervals are independent random variables which only depend on the length of these time intervals. Consider a small time interval $(t, t+h]$ and a positive number $\lambda > 0$. We assume that there is exactly one event in the interval $(t, t+h]$ with probability $\lambda h + o(h)$, $h \to 0+$, two or more events with probability $o(h)$, $h \to 0+$ and no event with probability $1 - \lambda h + o(h)$, $h \to 0+$.

Let $N_0 := 0$ and let N_t be the number of events which occured in the interval $(0, t]$. Model the situation by a homogeneous continuous time Markov chain. Find its generator $\boldsymbol{Q}$ and find the transition matrix $\boldsymbol{Q^*}$ of its embedded chain.

Solution to Exercise 49. The formal definition of the model follows:

Definition 3.1.1. Let $\lambda > 0$. The stochastic process $N = (N_t, t \geq 0)$ of count random variables is called the Poisson process with parameter λ if

1. $N_0 = 0$,
2. the process N has independent and stationary increments,
3. $\mathbb{P}(N_{t+h} - N_t = 1) = \lambda h + o(h)$, $\mathbb{P}(N_{t+h} - N_t \geq 2) = o(h)$ for $h \to 0+$ for all $t \geq 0$.

The first step is to find the transition probabilities $p_{ij}(h)$. △

Exercise 50. Show that Definition 3.1.1 is equivallent to the following definition:

Definition 3.1.2. Let $\lambda > 0$. The stochastic process $N = (N_t, t \geq 0)$ of count random variables is called the Poisson process with parameter λ if

1. $N_0 = 0$,
2. the process N has independent increments,
3. $N_{t+s} - N_s \sim \text{Po}\,(\lambda s)$ for all times $s, t \geq 0$.

Solution to Exercise 50. Notice how remarkable the claim is: we claim that a process N which satisfies Definition 3.1.1 where only the Taylor expansion of the distribution of increments $N_{t+h} - N_t$ is known, already implies that the increments have to be distributed according to the Poisson law. △

Exercise 51 (Linear birth-death process)**.** Consider a population of identical reproducing and dying organisms. Assume that each organism dies within the time interval $(t, t+h]$ with probability $\mu h + o(h)$, $h \to 0+$. Similarly, in $(t, t+h]$, each organism gives birth to exactly one organism with probability $\lambda h + o(h)$, $\to 0+$, to two or more organisms with probability $o(h)$, $h \to 0+$ or it will not reproduce with probability $1 - \lambda h + o(h)$, $h \to 0+$. The fates of individual organisms are independent and $\mu > 0, \lambda > 0$.

Let X_t be the number of organisms in the population at time $t \geq 0$. Model the population dynamics by a homogeneous continuous time Markov chain $X = (X_t, t \geq 0)$, find its generator $\boldsymbol{Q}$ and the transition matrix of the embedded chain $\boldsymbol{Q^*}$.

Exercise 52 (Telephone switchboard)**.** Consider a telephone switchboard with N lines. Assume that there will be exactly one incoming phone call within the time interval $(t, t+h]$ with probability $\lambda h + o(h)$, $h \to 0+$ and $\lambda > 0$. The probability is the same for all $t \geq 0$. The probability that two or more calls will come within the same interval is $o(h)$, $h \to 0+$. No phone call will come with probability $1 - \lambda h + o(h)$, $h \to 0+$. All the incoming calls are independent of each other. If all the N lines are engaged, then the incoming phone call is lost. The length of one phone call is exponentially distributed with the expected value $1/\mu$, $\mu > 0$.

1. Compute the probability that a phone call ends within the time interval $(t, t+h]$ if we know that it has not ended by time t.
2. Model the number of engaged lines at time t by a homogeneous continuous time Markov chain $X = (X_t, t \geq 0)$. Find the generator $\boldsymbol{Q}$.
3. Find the transition matrix of the embedded chain $\boldsymbol{Q}^*$.

Exercise 53 (Factory machine)**.** A factory machine either works or does not work. The time for which it does work is exponentially distributed with the expected value $1/\lambda$, $\lambda > 0$. Then it breaks and has to be repaired. The time it takes to repair the machine is exponentially distributed with the expected value $1/\mu$, $\mu > 0$. Model the state of the machine (works/does not work) by a homogeneous continuous time Markov chain, find its generator $\boldsymbol{Q}$ and the transition matrix of its embedded chain $\boldsymbol{Q}^*$.

Exercise 54 (Unimolecular reaction)**.** Consider a chemical reaction during which the molecules of the compound A change irreversibly to the molecules of the compound B. Suppose the initial concentration of A is N molecules. If there are j molecules of A at time t, then each of these molecules changes to a B molecule within the time interval $(t, t+h]$ with probability $qh + o(h)$, $h \to 0+$ where $q > 0$. Model the number of A molecules by a homogeneous continuous time Markov chain, find its generator and the transition matrix of its embedded chain $\boldsymbol{Q}^*$.

Exercise 55 (Bimolecular reaction)**.** Consider a liquid solution of two chemical compounds A and B. The chemical reaction is described as follows:

$$\mathrm{A} + \mathrm{B} \to \mathrm{C}$$

i.e. one A molecule reacts with one B molecule and they produce one C molecule. Let the initial concentrations of the compounds A,B and C be a, b, and 0 molecules and denote $N := \min\{a, b\}$. If there are j molecules of C at time t, then the reaction will produce exactly one molecule of C within the time interval $(t, t+h]$ with probability

$$q(a-j)(b-j)h + o(h), \quad h \to 0+$$

for $j = 0, 1, \ldots, N$. Model the concentration of C at time t by a homogeneous continuous time Markov chain $X = (X_t, t \geq 0)$, find its generator $\boldsymbol{Q}$ and the transition matrix of its embedded chain $\boldsymbol{Q}^*$.

3.2 Classification of states

In this section, we briefly revisit the concept of irreducibility and transience/recurrence of a Markov chain.

Definition 3.2.1. Let X be a homogeneous continuous time Markov chain with the state space S. We say that the state $j \in S$ is accessible from the state $i \in S$ (and write $i \to j$) if there is $t > 0$ such that

$$\mathbb{P}_i(X_t = j) = p_{ij}(t) > 0.$$

If the states i, j are mutually accessible, then we say that they communicate (and write $i \leftrightarrow j$). X is called irreducible if all its states communicate..

It can be shown that $i \to j$ in X if and only if $i \to j$ in its embedded chain Y.

Definition 3.2.2. Let X be a homogeneous continuous time Markov chain with the state space S. Denote

$$T_i(1) := \inf(t \geq J_1 : X_t = i),$$

the first return time to the state $i \in S$. The state i is called

- recurrent if either $(q_i = 0)$ or $(q_i > 0$ and $\mathbb{P}_i(T_i(1) < \infty) = 1)$,
- transient if $(q_i > 0$ and $\mathbb{P}_i(T_i(1) = \infty) > 0)$.

If $i \in S$ is recurrent, we call it

- positive if either $(q_i = 0)$ or $(q_i > 0$ and $\mathbb{E}_i T_i(1) < \infty)$,
- null if $(q_i > 0$ and $\mathbb{E}_i T_i(1) = \infty)$.

The intuition behind these definitions is the same as in the discrete time case. If a state is recurrent, the chain will revisit it in a finite time almost surely. If it is transient, it might happen (with positive probability), that the state will never be visited again.

In fact, if we consider a chain X which describes a moving particle (on, say, $\mathbb{N}_0$), then if the chain turns out to have only transient states, then the particle drifts to the infinity almost surely. In this case, the state space should be enriched with the state ∞, an absorbing state, which may be reached in finite or infinite time. More precisely, for any irreducible homogeneous continuous time Markov chain X on $\mathbb{N}_0$ with transient states, we have for all $i \in S$ that

$$\mathbb{P}_i(\lim_{t \to \infty} X_t = \infty) = 1.$$

Recurrent states, on the other hand, are visited infinitely many times. However, the average time it may take the process to revisit a given state can be either finite or infinite. These two options correspond to either positive recurrence or null recurrence of the chain.

The following definition characterizes states according to how long the chain stays in them.

Definition 3.2.3. Let X be a homogeneous continuous time Markov chain with the state space S. Let $i \in S$.

- If $q_i = 0$, then we call i absorbing.
- If $q_i \in (0, \infty)$, then we call i stable.
- If $q_i = \infty$, then we call i unstable.

Obviously, if $q_i = 0$, then from the definition of the embedded chain we can see, that $q^*_{ii} = 1$ and this means that i is absorbing (in the embedded chain). The interpretation of q_i is the rate at which the chain jumps to a different state. The higher q_i, the less likely the chain is to stay in i.

Naturally, we may ask what is the relationship between the states of a continuous time Markov chain and the states of its embedded chain. This is answered by the following theorem and exercise below.

Theorem 3.2.1. Let X be a homogeneous continuous time Markov chain with the state space S and let Y be its embedded chain. Then a state $i \in S$ is recurrent in X if and only if i is recurrent in Y.

However, the statement is not true for positivity of the recurrent states. Indeed, the following example shows that a continuous time Markov chain can have positive recurrent states (in the sense of Definition 3.2.2) while its embedded chain has null recurrent states (in the sense of Definition 2.2.1).

Exercise 56. Let X be a homogeneous continuous time Markov chain with the state space $S = \mathbb{N}_0$ and the generator

$$\boldsymbol{Q} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & 0 & \ddots \\ 0 & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & 0 & \ddots \\ 0 & 0 & \frac{1}{9} & -\frac{2}{9} & \frac{1}{9} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Use Theorem **??** to show that X has only positive recurrent states but its embedded chain has null recurrent states.

Solution to Exercise 56. If we look at the structure of the generator $\boldsymbol{Q}$ we see that the process is a birth-death process with $\mu_0 = 0, \lambda_0 = 1$ and $\mu_n = \frac{1}{n^2} = \lambda_n$ for $n \in \mathbb{N}$. We have that

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n \rho_n} = \sum_{n=1}^{\infty} \frac{\mu_1 \cdot \ldots \cdot \mu_n}{\lambda_0 \lambda_1 \cdot \ldots \cdot \lambda_n} = \sum_{n=1}^{\infty} \frac{1}{\lambda_0} = \infty$$

so that the chain X has recurrent states (by Theorem **??**) and

$$\sum_{n=1}^{\infty} \rho_n = \sum_{n=1}^{\infty} \frac{\lambda_0 \cdot \ldots \cdot \lambda_{n-1}}{\mu_1 \cdot \ldots \cdot \mu_n} = \sum_{n=1}^{\infty} \frac{\lambda_0}{\mu_n} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

which assures that the states are positive recurrent. However, the transition matrix of the embedded chain is

$$\boldsymbol{Q}^* = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \ddots \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

which is the transition matrix of a symmetric random walk on $\mathbb{N}_0$ with reflecting boundary at 0. We have already shown that this random walk has null recurrent states. △

3.3 Kolmogorov differential equations

In this section, we will be able to construct the transition semigroup $(\boldsymbol{P}(\boldsymbol{t}), t \geq 0)$ of a given homogeneous continuous time Markov chain. The main ingredient is its generator $\boldsymbol{Q}$ as follows from the next theorem.

Theorem 3.3.1 (Kolmogorov differential equations (KDE))**.** Let $X = (X_t, t \geq 0)$ be a homogeneous continuous time Markov chain with the generator $\boldsymbol{Q}$ and the state space S for which we have that $q_i < \infty$ for all $i \in S$.

1. The transition probabilities $p_{ij}(\cdot)$ of X are differentiable in $(0, \infty)$ and it holds that

$$\boldsymbol{P}'(\boldsymbol{t}) = \boldsymbol{Q}\boldsymbol{P}(\boldsymbol{t}), \quad t > 0. \tag{3.3.1}$$

2. If, moreover, for all $j \in S$, the convergence $\frac{p_{ij}(h)}{h} \to q_{ij}$ as $h \to 0+$ is uniform in $i \in S$, then it also holds that

$$\boldsymbol{P}'(\boldsymbol{t}) = \boldsymbol{P}(\boldsymbol{t})\boldsymbol{Q}, \quad t > 0. \tag{3.3.2}$$

The first differential equation, (3.3.1), always holds for chains with stable states (i.e. $q_i < \infty$ for all $i \in S$) and it is called the Kolmogorov backward equation. The second one, called the Kolmogorov forward equation holds for example in chains with finite state space (and stable states).[3] Sometimes, in particular when we are given a chain with finite, yet general, state space, it can be useful to use Theorem 3.3.1 directly and use recursion to compute the transition probabilities $\boldsymbol{P}(\boldsymbol{t})$. However, for finite spaces, we know the explicit solution to both these equations.

Theorem 3.3.2 (Solution to KDE)**.** Let X be a homogeneous continuous time Markov chain with a finite state space S and stable states. Then both equations (3.3.1) and (3.3.2) have exactly one solution which satisfies $\boldsymbol{P}(\boldsymbol{0}) = \boldsymbol{I}_S$. The solution is the same for both equations and can be written as the matrix exponential

$$\boldsymbol{P}(\boldsymbol{t}) = \boldsymbol{e}^{\boldsymbol{Q}\boldsymbol{t}} = \sum_{k=0}^{\infty} \frac{t^k \boldsymbol{Q}^k}{k!}, \quad t \geq 0.$$

Now, if we want to use Theorem 3.3.2, we have two options. We can either use the definition of matrix exponential (as an infinite sum) for which we will need to know $\boldsymbol{Q}^k$ for all $k \in \mathbb{N}_0$ or we can use the fact that the solution is matrix exponential directly (for which we will need the Perron's formula for holomorphic functions).

Theorem 3.3.3 (Perron's formula for holomorphic functions)**.** Let $f : U(0, R) \to \mathbb{C}$ be a holomorphic function on some R-neighbourhood of 0, $0 < R \leq \infty$ and let A be a

[3] To remember which one is forward/backward one, just notice that in (3.3.1), the letters Q and P are in the backward alphabetical order whereas in (3.3.2), the letters go as P, Q (i.e. forward).

square matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$ whose multiplicities are $m_1, m_2, \dots, m_k$. If $|\lambda_j| < R$ for all $j = 1, 2, \dots, k$, then

$$f(\boldsymbol{A}) = \sum_{j=1}^{k} \frac{1}{(m_j - 1)!} \frac{d^{m_j - 1}}{d\lambda^{m_j - 1}} \left[f(\lambda) \frac{(\boldsymbol{\lambda I - A})}{\psi_j(\lambda)} \right]_{\lambda = \lambda_j}$$

where

$$\psi_j(\lambda) = \frac{\det(\boldsymbol{\lambda I - A})}{(\lambda - \lambda_j)^{m_j}}.$$

In particular, if $\boldsymbol{A}$ is a $N \times N$-matrix with only simple eigenvalues (i.e. $m_j = 1$) then the formula takes the simpler form

$$f(\boldsymbol{A}) = \sum_{j=1}^{N} f(\lambda_j) \frac{\mathbf{Adj}\,(\boldsymbol{\lambda_j I - A})}{\psi_j(\lambda_j)}.$$

To sum it up, in order to compute the semigroup $(\boldsymbol{P(t)}, t \geq 0)$, we can use the following:

- forward/backward equation
- $\boldsymbol{P(t)} = \sum_{k=0}^{\infty} \frac{t^k \boldsymbol{Q}^k}{k!}$ (we need $\boldsymbol{Q}^k$)
- $\boldsymbol{P(t)} = \boldsymbol{e^{Qt}}$ (we need Perron's formula for $f(z) = e^{tz}$)

Exercise 57. Let X be a homogeneous continuous time Markov chain with the state space $S = \{0, 1\}$ and the generator

$$\boldsymbol{Q} = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$$

where $\lambda > 0$ and $\mu > 0$. Compute the transition semigroup $(\boldsymbol{P(t)}, t \geq 0)$ of X and also the distribution of X_t for all $t > 0$ if the initial distribution of X on S is $\boldsymbol{p(0)}^T = (p, q)$ where $p = 1 - q > 0$.

Exercise 58. Let X be a homogeneous continuous time Markov chain with the state space $S = \{0, 1, 2\}$ and the generator

$$\boldsymbol{Q} = \begin{pmatrix} -3 & 3 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & -2 \end{pmatrix}.$$

Compute the transition semigroup $(\boldsymbol{P(t)}, t \geq 0)$ of X and also the distribution of X_t for all $t > 0$ if the initial distribution of X is uniform on S.

Exercise 59. Let X be a homogeneous continuous time Markov chain with the state space $S = \{0, 1, 2\}$ and the generator

$$\boldsymbol{Q} = \begin{pmatrix} -2 & 2 & 0 \\ 1 & -3 & 2 \\ 0 & 2 & -2 \end{pmatrix}.$$

Compute the transition semigroup $(\boldsymbol{P(t)}, t \geq 0)$ of X and also the distribution of X_t for all $t > 0$ if the initial distribution of X is uniform on S.

3.4 Explosive and non-explosive chains

In this section, we will have a closer look at the number of jumps that can occur within a finite time interval. Naturally, we distinguish two cases - either the number of jumps is finite, or infinite with the latter being considered "badly-behaved".

Definition 3.4.1 (Regular chains)**.** Let X be a homogeneous continuous time Markov chain with stable states. Denote

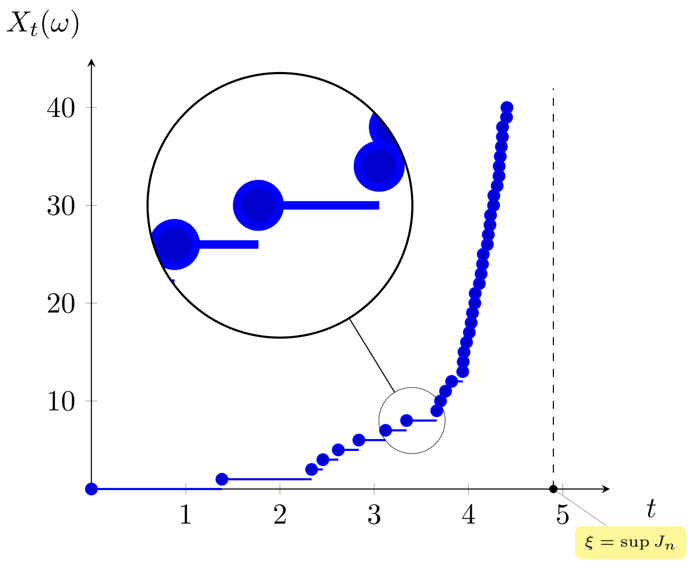
$$\xi := \sup J_n = \sum_{k=1}^{\infty} S_k.$$

We say that X is regular or non-explosive, if

$$\mathbb{P}_i(\xi = \infty) = 1$$

for all $i \in S$. The $N_0 \cup \{\infty\}$-valued random variable ξ is called the time to explosion.

If a chain X is regular, then within a finite time interval only a finite number of jumps can occur. For intuition, this is what a sample path of an explosive chain may look like.



Theorem 3.4.1 (Characterization of non-explosive chains)**.** Let X be a homogeneous continuous time Markov chain with the state space $S = \mathbb{N}_0$, the generator $\boldsymbol{Q}$ and with

the embedded chain Y. Then X is non-explosive if and only if

$$\mathbb{P}_i\left(\sum_{k=0}^{\infty}\frac{1}{q_{Y_k}}=\infty\right)=1$$

for all $i \in S$.

Exercise 60. Decide whether the Poisson process $N = (N_t, t \geq 0)$ is a regular process or not.

Exercise 61. Let X be a homogeneous continuous time Markov chain with the state space $S = \mathbb{N}_0$ and generator

$$\boldsymbol{Q} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & -4 & 4 & 0 & 0 & \ddots \\ 0 & 0 & -9 & 9 & 0 & \ddots \\ 0 & 0 & 0 & -16 & 16 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Decide whether X is explosive or non-explosive.

3.5 Steady-state and long-term behaviour

In order to analyse the long-term behaviour of homogeneous continuous time Markov chains, we will need an analogue of the stationary distribution discussed in the discrete time case. A natural definition would be the following.

Definition 3.5.1 (Invariant measure and stationary distribution)**.** Let $X = (X_t, t \geq 0)$ be a homogeneous continuous time Markov chain with the state space S and the transition semigroup $(\boldsymbol{P(t)}, t \geq 0)$.

- A vector $\boldsymbol{\eta}^T = (\eta_i)_{i\in S}$ such that $\eta_i \geq 0$ for all $i \in S$ and $\boldsymbol{\eta}^T = \boldsymbol{\eta}^T\boldsymbol{P(t)}$ for all $t \geq 0$, is called an invariant measure of X.
- An invariant measure $\boldsymbol{\eta}$ is called the stationary distribution of X, if it is also a probability distribution on S, i.e. $\eta_i \in [0, 1]$ for all $i \in S$ and $\sum_{i\in S}\eta_i = 1$.

As opposed to the discrete time case, the situation is more complicated now. In the discrete time case, it sufficed to find a probability vector $\boldsymbol{pi}^T$ such that $\boldsymbol{\pi}^T = \boldsymbol{\pi}^T\boldsymbol{P}$ since then we could show by induction that also

$$\boldsymbol{\pi}^T = \boldsymbol{\pi}^T\boldsymbol{P(n)}$$

for all $n \in \mathbb{N}_0$. In the continuous time case, we have to use the generator $\boldsymbol{Q}$. To motivate the theorems in this section, we are going to give a very informal derivation which will give us some insight into the formulas used in exercises. Consider the equation for invariant measure $\boldsymbol{\eta}^T = \boldsymbol{\eta}^T\boldsymbol{P(h)}$ written as

$$\eta_j = \sum_{i\in S}\eta_i p_{ij}(h), \quad h \geq 0.$$

This can be rewritten for $h > 0$ as

$$\begin{aligned}
\eta_j - \eta_j p_{jj}(h) &= \sum_{i\in S, i\neq j} \eta_i p_{ij}(h) \\
\eta_j(1 - p_{jj}(h)) &= \sum_{i\in S, i\neq j} \eta_i p_{ij}(h) \\
\eta_j \frac{1 - p_{jj}(h)}{h} &= \sum_{i\in S, i\neq j} \eta_i \frac{p_{ij}(h)}{h}
\end{aligned}$$

Now, taking the limit $h \to 0+$, (!!!), we obtain

$$\begin{aligned}
\eta_j q_{jj} &= \sum_{i\in S, i\neq j} \eta_i q_{ij} \\
\sum_{i\in S} \eta_i q_{ij} &= 0
\end{aligned}$$

which is the same as

$$\boldsymbol{\eta}^T \boldsymbol{Q} = \mathbf{0}^T. \tag{3.5.1}$$

Hence, we see that a vector $\boldsymbol{\eta}$ satisfies $\boldsymbol{\eta}^T = \boldsymbol{\eta}^T \boldsymbol{P(h)}$ at least for all small $h > 0$ if and only if it also satisfies the equation $\boldsymbol{\eta}^T \boldsymbol{Q} = \mathbf{0}^T$. This means that in order to find an invariant measure of X, we need to solve one equation only. In the case when the chain is regular and has only recurrent states, the invariant measure indeed exists as proposed by the following theorem.

Theorem 3.5.1 (Existence of invariant measure)**.** Let X be a regular homogeneous continuous time Markov chain with the generator $\boldsymbol{Q}$ and the embedded chain Y. Assume that Y is irreducible and all its states are recurrent. Then X has an invariant measure $\boldsymbol{\eta}$ which is the unique (up to a multiplicative constant) positive solution to

$$\boldsymbol{\eta}^T \boldsymbol{Q} = \mathbf{0}^T.$$

Furthermore, if $\sum_{i\in S} \eta_i < \infty$, then $\boldsymbol{\pi} = (\pi_i)_{i\in S}$ where

$$\pi_i = \frac{\eta_i}{\sum_{k\in S} \eta_k}, \quad i \in S,$$

is the stationary distribution of X.

Exercise 62. Consider a Markov chain X with the state space $S = \{0, 1\}$ and the generator

$$\boldsymbol{Q} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Show that the stationary distribution of X exists and find it.

Exercise 63. There are $N \in \mathbb{N}$ workers in a factory, each of whom uses a certain electrical device. A worker who is not using the device at time t will start using it within the time interval $(t, t+h]$ with probability $\lambda h + o(h)$, $h \to 0+$, independently of the other workers. A worker who is using the device at time t will stop using it within the interval $(t, t+h]$ with probability $\mu h + o(h)$, $h \to 0+$. Here, $\lambda > 0$ and $\mu > 0$. Model the number of workers using the device at time t by a homogeneous continuous time Markov chain. Find its generator $\boldsymbol{Q}$, show that its stationary distribution exists and find it.

Exercise 64. Let X be a homogeneous continuous time Markov chain with the state space $S = \mathbb{N}_0$ and the generator

$$\boldsymbol{Q} = \begin{pmatrix} -1 & 1 & 0 & 0 & \cdots \\ pq & -p & p^2 & 0 & \cdots \\ p^2 q & 0 & -p^2 & p^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

where $0 < p = 1 - q < 1$. Decide, whether the stationary distribution of X exists or not and if it does, find it.

Exercise 65. Let X be a homogeneous continuous time Markov chain with the state space $S = \mathbb{N}_0$ and the generator

$$\mathbf{Q} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & -3 & 2 & 0 & 0 & \cdots \\ 1 & 0 & -4 & 3 & 0 & \cdots \\ 1 & 0 & 0 & -5 & 4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Decide, whether the stationary distribution of X exists or not and if it does, find it.

The situation becomes a lot more complicated when the chain is either explosive, or non-explosive but has transient states. In the following examples, we will see that we cannot say anything about the existence of the solution to (3.5.1) without additional assumptions unless the chain is non-explosive and recurrent.

Exercise 66. Let X be a homogeneous continuous time Markov chain with the state space $S = \mathbb{N}_0$ and the generator

$$\boldsymbol{Q} = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & \cdots \\ \lambda(1-p_1) & -\lambda & \lambda p_1 & 0 & 0 & \ddots \\ \lambda(1-p_2) & 0 & -\lambda & \lambda p_2 & 0 & \ddots \\ \lambda(1-p_3) & 0 & 0 & -\lambda & \lambda p_3 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

where $\lambda > 0$ and $p_k \in (0,1)$ are such that $\lim_{k\to\infty} \prod_{i=1}^{k} p_i \neq 0$. Show that X is a non-explosive, transient chain (i.e. irreducible chain whose all states are transient) and show that there is no positive solution to $\boldsymbol{\eta}^T \boldsymbol{Q} = \mathbf{0}^T$.

Solution to Exercise 66. X is non-explosive: We have that $q_j = \lambda$ for all $j \in \mathbb{N}_0$. Hence

$$\mathbb{P}_i\left(\sum_{k=0}^{\infty} \frac{1}{q_{Y_k}} = \infty\right) = \mathbb{P}_i\left(\sum_{k=0}^{\infty} \frac{1}{\lambda} = \infty\right) = 1$$

and we may appeal to Theorem 3.4.1.
X is transient: The chain X is recurrent if and only if its embedded chain is recurrent. The transition matrix of the embedded chain is given by

$$\boldsymbol{Q}^* = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 1-p_1 & 0 & p_1 & 0 & \ddots \\ 1-p_2 & 0 & 0 & p_2 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Obviously, the embedded chain is irreducible. In a standard way, we can use the reduced system $\boldsymbol{x} = \boldsymbol{Rx}$ to show that the embedded chain is recurrent if and only if

$$\lim_{k\to\infty} \prod_{i=1}^{k} p_i = 0.$$

This is not surprising, since $\prod_{i=1}^{k} p_i$ is the probability that the chain will not return to the state zero in $(k+1)$ transitions. Hence, if the limit is zero, we will revisit 0 in a finite time, which is precisely the definition of a recurrent state. Hence, 0 is recurrent and irreducibility assures that the remaining states are also recurrent. However, in our case we know, that the limit is not zero which implies that the embedded chain is transient and so is X.
There is no invariant measure: We will show that there is no solution to the equation

$$\boldsymbol{\eta}^T \boldsymbol{Q} = \boldsymbol{0}^T.$$

We obtain

$$\begin{aligned} -\lambda\eta_0 + \lambda\sum_{k=1}^{\infty}(1-p_k)\eta_k &= 0 \\ \lambda\eta_0 - \lambda\eta_1 &= 0 \\ \lambda p_1\eta_1 - \lambda\eta_2 &= 0 \\ \lambda p_2\eta_2 - \lambda\eta_3 &= 0 \\ &\vdots \\ \lambda p_k\eta_k - \lambda\eta_{k+1} &= 0, \quad k = 2, 3, \ldots \end{aligned}$$

The second and the consecutive equations yield

$$\begin{aligned} \eta_1 &= \eta_0 \\ \eta_k &= \prod_{i=1}^{k-1} p_i\eta_0, \quad k = 2, 3, \ldots \end{aligned}$$

Writing $p_0 := 1$ and plugging this in into the first equation, we obtain

$$\eta_0 = \sum_{k=1}^{\infty}(1 - p_k)\left(\prod_{i=0}^{k-1} p_i\right)\eta_0.$$

Hence, the equation admits a solution if and only if

$$\sum_{k=1}^{\infty}\left(\prod_{i=0}^{k-1} p_i - \prod_{i=0}^{k} p_i\right) = 1.$$

Writing $a_k := \prod_{i=0}^{k} p_i$, we can take a partial sum

$$\sum_{k=1}^{N}(a_{k-1} - a_k) = a_0 - a_1 + a_1 - a_2 + \ldots + a_{N-1} - a_N = a_0 - a_N = 1 - \prod_{i=0}^{N} p_i.$$

Now taking the limit $N \to \infty$ and using the fact that $\lim_{k\to\infty}\prod_{i=1}^{k} p_i \neq 0$, we see that $\boldsymbol{\eta}^T\boldsymbol{Q} = \boldsymbol{0}^T$ does not admit a solution (in fact, there is a solution if and only if $\lim_{k\to\infty}\prod_{i=1}^{k} p_i = 0$). △

Exercise 67. Let X be a homogeneous continuous time Markov chain with the state space $S = \mathbb{Z}$ and the generator $\boldsymbol{Q}$ given by

$$\begin{aligned} q_{i,i+1} &= \lambda p \\ q_{i,i} &= -\lambda \\ q_{i,i-1} &= \lambda q \end{aligned}$$

and $q_{ij} = 0$ otherwise, $i, j \in \mathbb{Z}$. Assume that $\lambda > 0$ and that $p \in (0, 1)$, $q := 1 - p$ and that $p \neq \frac{1}{2}$. Show that X is a non-explosive, transient Markov chain and find all solutions to $\boldsymbol{\eta}^T\boldsymbol{Q} = \boldsymbol{0}^T$.

Exercise 68. Let X be a homogeneous continuous time Markov chain with the state space $S = \mathbb{N}_0$ and the generator

$$\boldsymbol{Q} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & \cdots \\ (1-p_1) & -1 & p_1 & 0 & 0 & \ddots \\ 4(1-p_2) & 0 & -4 & 4p_2 & 0 & \ddots \\ 9(1-p_3) & 0 & 0 & -9 & 9p_3 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

where $p_k \in (0, 1)$ are such that $\lim_{k\to\infty}\prod_{i=1}^{k} p_i \neq 0$. Show that X is an explosive chain and show that there is no positive solution to $\boldsymbol{\eta}^T\boldsymbol{Q} = \boldsymbol{0}^T$.

Exercise 69. Let $\frac{1}{2} < p < 1$ and let $q := 1 - p$. Let X be a homogeneous continuous time Markov chain with the state space $S = \mathbb{Z}$ and the generator $\boldsymbol{Q}$ given by

$$\begin{aligned} q_{i,i+1} &= \lambda_i p \\ q_{i,i} &= -\lambda_i \\ q_{i,i-1} &= \lambda_i q \end{aligned}$$

and $q_{ij} = 0$ otherwise, $i, j \in \mathbb{Z}$. Here λ_i is chosen such that

$$\lambda_i = \begin{cases} q, & i = -1, -2, \dots, \\ \left(\frac{p}{q}\right)^{2i}, & i = 0, 1, 2, \dots \end{cases}$$

Show that X is a an explosive Markov chain and find all solutions to $\boldsymbol{\eta}^T \boldsymbol{Q} = \boldsymbol{0}^T$.

Exercise 70. Let X be a homogeneous continuous time Markov chain with the state space $S = \mathbb{N}_0$ and the generator

$$\boldsymbol{Q} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2}\cdot 4 & -\frac{3}{2}\cdot 4 & 4 & 0 & 0 & \ddots \\ 0 & \frac{1}{2}\cdot 4^2 & -\frac{3}{2}\cdot 4^2 & 4^2 & 0 & \ddots \\ 0 & 0 & \frac{1}{2}\cdot 4^3 & -\frac{3}{2}\cdot 4^3 & 4^3 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Show that X is an explosive chain, find the unique positive solution $\boldsymbol{\eta}$ to $\boldsymbol{\eta}^T \boldsymbol{Q} = \boldsymbol{0}^T$ and show that it converges (i.e. show that $\sum_{i=0}^{\infty} \eta_i < \infty$).

Exercise 71. Let $\frac{1}{2} < p < 1$ and let $q := 1 - p$. Let X be a homogeneous continuous time Markov chain with the state space $S = \mathbb{Z}$ and the generator $\boldsymbol{Q}$ given by

$$\begin{aligned} q_{i,i+1} &= \lambda_i p \\ q_{i,i} &= -\lambda_i \\ q_{i,i-1} &= \lambda_i q \end{aligned}$$

and $q_{ij} = 0$ otherwise, $i, j \in \mathbb{Z}$. Here $\lambda_i = \left(\frac{p}{q}\right)^{2|i|}$ for $i \in \mathbb{Z}$. Show that X is a an explosive Markov chain find all solutions to $\boldsymbol{\eta}^T \boldsymbol{Q} = \boldsymbol{0}^T$ and show that every such solution is convergent.

In the table below, there are various cases we have encountered. When we write "solution", we always mean positive (i.e. $\eta_i > 0$ for all $i \in S$) solution to (3.5.1). When we write "unique", we always mean uniqueness up to a multiplicative constant. When we write "convergent", we mean, that $\sum_{i \in S} \eta_i < \infty$. Unless explicitly stated, all solutions in the table are non-convergent.

Chain X and its states			Solution to $\boldsymbol{\eta}^T \boldsymbol{Q} = \boldsymbol{0}^T$
Non-explosive	positive recurrent	$\Longrightarrow$	unique convergent solution
	null recurrent	$\Longrightarrow$	unique solution
	transient	$\Longrightarrow$	no solution
			non-unique solution
			unique solution
Explosive		$\Longrightarrow$	no solution
			non-unique solution
			unique solution
			unique convergent solution
			non-unique convergent solution

3.6 Answers to exercises

Answer to Exercise 49. The state space of the Poisson proces N is $S = \mathbb{N}_0$, its generator $\boldsymbol{Q}$ and the transition matrix of its embedded chain $\boldsymbol{Q}^*$ are

$$\boldsymbol{Q} = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \cdots \\ 0 & -\lambda & \lambda & 0 & \ddots \\ 0 & 0 & -\lambda & \lambda & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \quad \boldsymbol{Q}^* = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \ddots \\ 0 & 0 & 0 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Answer to Exercise 50. The definitions are equivalent.

Answer to Exercise 51. The state space of the linear birth-death process is $S = \mathbb{N}_0$ and the generator

$$\boldsymbol{Q} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ \mu & -(\mu+\lambda) & \lambda & 0 & 0 & \ddots \\ 0 & 2\mu & -2(\mu+\lambda) & 2\lambda & 0 & \ddots \\ 0 & 0 & 3\mu & -3(\mu+\lambda) & 3\lambda & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Its embedded chain is an asymmetric random walk with the absorbing state 0 whose transition matrix is

$$\boldsymbol{Q}^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{\mu}{\mu+\lambda} & 0 & \frac{\lambda}{\mu+\lambda} & 0 & 0 & \ddots \\ 0 & \frac{\mu}{\mu+\lambda} & 0 & \frac{\lambda}{\mu+\lambda} & 0 & \ddots \\ 0 & 0 & \frac{\mu}{\mu+\lambda} & 0 & \frac{\lambda}{\mu+\lambda} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Answer to Exercise 52. The probability that one incoming call ends within the interval $(t, t+h]$ for small $h > 0$, is $1 - e^{-\mu h}$. The state space of X is $S = \{0, 1, \ldots, N\}$ and its generator is

$$\boldsymbol{Q} = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \cdots & 0 & 0 & 0 \\ \mu & -(\mu+\lambda) & \lambda & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2\mu & -(2\mu+\lambda) & \lambda & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (N-1)\mu & -((N-1)\mu+\lambda) & \lambda \\ 0 & 0 & 0 & 0 & \cdots & 0 & N\mu & -N\mu \end{pmatrix}.$$

The embedded chain of X is a random walk on S with reflecting boundaries and its transition matrix is

$$\boldsymbol{Q}^* = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{\mu}{\mu+\lambda} & 0 & \frac{\lambda}{\mu+\lambda} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{2\mu}{2\mu+\lambda} & 0 & \frac{\lambda}{2\mu+\lambda} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{(N-1)\mu}{(N-1)\mu+\lambda} & 0 & \frac{\lambda}{(N-1)\mu+\lambda} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

Answer to Exercise 53. The state space of X is $S = \{0, 1\}$ where 0 represents "the machine does not work" and 1 represents "the machine works". The generator and the transition matrix of the embedded chain are

$$\boldsymbol{Q} = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}, \quad \boldsymbol{Q}^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Answer to Exercise 54. The state space of X is $S = \{0, 1, \ldots, N\}$ and its embedded chain is the left-shift process with absorbing state 0. The generator and transition matrix of the embedded chain are

$$\boldsymbol{Q} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ q & -q & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2q & -2q & 0 & \cdots & 0 & 0 \\ 0 & 0 & 3q & -3q & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & Nq & -Nq \end{pmatrix}, \quad \boldsymbol{Q}^* = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

Answer to Exercise 55. The state space of X is $S = \{0, 1, \ldots, N\}$ and its embedded chain is the right-shift with the absorbing state N. The generator of X is

$$\boldsymbol{Q} = \begin{pmatrix} -abq & abq & 0 & 0 & \cdots & 0 & 0 \\ 0 & -(a-1)(b-1)q & (a-1)(b-1)q & 0 & \cdots & 0 & 0 \\ 0 & 0 & -(a-2)(b-2)q & (a-2)(b-2)q & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

and the transition matrix of the embedded chain is

$$\boldsymbol{Q}^* = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Answer to Exercise 56. The chain X is a birth-death process with positive recurrent states. Its embedded chain is the symmetric random walk which has null-recurrent states.

Answer to Exercise 57. The transition semigroup $(\boldsymbol{P(t)}, t \geq 0)$ is

$$\boldsymbol{P(t)} = \frac{1}{\lambda+\mu} \begin{pmatrix} \mu + \lambda e^{-(\lambda+\mu)t} & \lambda - \lambda e^{-(\lambda+\mu)t} \\ \mu - \mu e^{-(\lambda+\mu)t} & \lambda + \mu e^{-(\lambda+\mu)t} \end{pmatrix}.$$

The distribution of X_t is

$$\boldsymbol{p(t)}^T = \frac{1}{\lambda+\mu} \left(\mu + (p\lambda - q\mu)e^{-(\lambda+\mu)}, \lambda - (p\lambda - q\mu)e^{-(\lambda+\mu)} \right).$$

Answer to Exercise 58. The transition semigroup $(\boldsymbol{P(t)}, t \geq 0)$ is

$$\boldsymbol{P(t)} = \begin{pmatrix} e^{-3t} & 1 - e^{-3t} & 0 \\ 0 & 1 & 0 \\ e^{-2t} - e^{-3t} & 1 + e^{-3t} - 2e^{-2t} & e^{-2t} \end{pmatrix}.$$

The distribution of X_t is

$$\boldsymbol{p(t)}^T = \frac{1}{3} \left(e^{-2t}, 3 - 2e^{-2t}, e^{-2t} \right).$$

Answer to Exercise 59. The transition semigroup $(\boldsymbol{P(t)}, t \geq 0)$ is

$$\boldsymbol{P(t)} = \begin{pmatrix} \frac{1}{5} + \frac{2}{3}e^{-2t} + \frac{2}{15}e^{-5t} & \frac{2}{5} - \frac{2}{5}e^{-5t} & \frac{2}{5} - \frac{2}{3}e^{-2t} + \frac{4}{15}e^{-5t} \\ \frac{1}{5} - \frac{1}{5}e^{-5t} & \frac{2}{5} - \frac{2}{5}e^{-5t} & \frac{2}{5} - \frac{2}{5}e^{-5t} \\ \frac{1}{5} - \frac{1}{3}e^{-2t} + \frac{2}{15}e^{-5t} & \frac{2}{5} - \frac{2}{5}e^{-5t} & \frac{2}{5} + \frac{1}{3}e^{-2t} + \frac{4}{15}e^{-5t} \end{pmatrix}.$$

The distribution of X_t is

$$\boldsymbol{p(t)}^T = \frac{1}{3} \left(\frac{3}{5} + \frac{1}{3}e^{-2t} + \frac{1}{15}e^{-5t}, \frac{6}{5} - \frac{1}{5}e^{-5t}, \frac{6}{5} - \frac{1}{3}e^{-2t} + \frac{2}{15}e^{-5t} \right).$$

Answer to Exercise 60. Yes, the Poisson process is regular.

Answer to Exercise 61. X is an explosive chain.

Answer to Exercise 62. The stationary distribution exists (finite, irreducible chain) and is given by $\boldsymbol{\pi}^T = \left(\frac{1}{2}, \frac{1}{2}\right)$.

Answer to Exercise 63. The intensity matrix Q of X is given by

$$\begin{pmatrix} -N\lambda & N\lambda & 0 & 0 & \cdots & 0 & 0 & 0 \\ \mu & -(\mu + (N-1)\lambda) & (N-1)\lambda & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2\mu & -(2\mu + (N-2)\lambda) & (N-2)\lambda & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (N-1)\mu & -((N-1)\mu + \lambda) & \lambda \\ 0 & 0 & 0 & 0 & \cdots & 0 & N\mu & -N\mu \end{pmatrix},$$

its stationary distribution exists (finite, irreducible chain) and is given by

$$\pi_k = \binom{N}{k}\left(\frac{\lambda}{\mu}\right)^k\left(1+\frac{\lambda}{\mu}\right)^{-N}, \quad k=0,\dots,N.$$

Answer to Exercise ??. The intensity matrix of X is given by

$$\boldsymbol{Q} = \begin{pmatrix} -2 & 2 & 0 & 0 \\ 3 & -5 & 2 & 0 \\ 0 & 6 & -8 & 2 \\ 0 & 0 & 9 & -9 \end{pmatrix},$$

its stationary distribution exists (finite, irreducible chain) and is given by $\pi_0 = \frac{81}{157}$ which is the probability that neither of the shop assistants is occupied by a customer. Further, the stationary distribution of X exists and is given by $\pi_1 = \frac{2}{3}\pi_0, \pi_2 = \frac{2}{9}\pi_0, \pi_3 = \frac{4}{81}\pi_0$.

Answer to Exercise ??. The intensity matrix of X is given by

$$\boldsymbol{Q} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{3} & -\frac{4}{3} & 1 & 0 \\ 0 & \frac{2}{3} & -\frac{5}{3} & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix},$$

its stationary distribution exists and is given by $\boldsymbol{\pi}^T = (1,0,0,0)$. The mean length of time spent in state 1 is $\frac{3}{4}$.

Answer to Exercise ??. The stationary distribution of X exists if and only if $\lambda < 2\mu$. In this case, we have that $\pi_0 = \left(1+\frac{2\lambda}{2\mu-\lambda}\right)^{-1}$ and $\pi_k = \pi_0\frac{\lambda}{\mu}\left(\frac{\lambda}{2\mu}\right)^{k-1}$ for $k=1,2,\dots$. The mean number of customers in the system in equilibrium is $\frac{4\mu\lambda}{4\mu^2-\lambda^2}$.

Answer to Exercise ??. The stationary distribution of X exists and if we denote by $\rho := \frac{\lambda}{\mu}$, it takes the form $\pi_k = \frac{\rho^k}{k!}\left(\sum_{j=0}^{N}\frac{\rho^j}{j!}\right)^{-1}$ for $k=0,\dots,N$.

Answer to Exercise 64. The chain X admits a non-convergent invariant measure $\boldsymbol{\eta}$ with $\eta_i = \frac{1}{p}\eta_0$ for $i=1,2,\dots$. Hence, there is no stationary distribution of X.

Answer to Exercise 65. The chain X admits the stationary distribution which is given by $\pi_0 = \frac{1}{2}$ and $\pi_k = \frac{1}{(k+1)(k+2)}$ for $k=1,2,\dots$