On Extracting Information Implied in Options

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Black-Scholes (BS) price - call option

$$C_t(S_t, K, \tau, r, \sigma) = S_t \Phi(d_1) - K e^{-r\tau} \Phi(d_2)$$

S_t	...	stock price at date t
K	...	strike price
τ	...	time-to-maturity
r	...	interest rate
σ	...	unknown and constant volatility parameter
Φ	...	standard normal cumulative distribution function

$$d_1 = \frac{\ln(S_t/K) + (r + 0.5\sigma^2)\tau}{\sigma\sqrt{\tau}} \ , \quad d_2 = d_1 - \sigma\sqrt{\tau}$$

Implied volatility

The implied volatility (IV) $\tilde{\sigma}$ is defined as the volatility σ, which matches the Black-Scholes (BS) price C_t with the price $\tilde{C}_t$ actually observed on the market.

The aim of this paper is to propose a direct method of estimating the IV function from observed noisy IVs. Denoting IVs observed on K_i and τ by $\tilde{\sigma}_{K_i,\tau}$ and the corresponding "true" IV function by $\sigma(K_i, \tau)$, $i = 1, \ldots, n$ we assume following model:

$$\tilde{\sigma}_{K_i,\tau} = \sigma(K_i, \tau) + \varepsilon_{i,\tau} \ , \qquad (1)$$

State price density function

State price density (SPD):

$$q_{t,S_t}(x,\tau) = e^{r\tau} \frac{\partial^2 C_t(K,T)}{\partial K^2}\bigg|_{K=x}. \qquad (2)$$

After some algebra we obtain:

$$\begin{aligned} q_{t,S_t}(x,\tau) &= e^{r\tau} S_t \sqrt{\tau}\varphi(d_1(x,\tau)) \left\{ \frac{1}{x^2\sigma(K,\tau)\tau} + \frac{2d_1(x,\tau)}{x\sigma(x,\tau)\sqrt{\tau}} \frac{\partial\sigma(K,\tau)}{\partial K}\bigg|_{K=x} \right. \\ &\quad \left. + \frac{d_1(x,\tau)d_2(x,\tau)}{\sigma(x,\tau)} \left(\frac{\partial\sigma}{\partial K}\bigg|_{K=x} \right)^2 + \frac{\partial^2\sigma}{\partial K^2}\bigg|_{K=x} \right\}. \qquad (3) \end{aligned}$$

$$d_1(x,\tau) = \frac{\ln(S_t/x)+(r+0.5\sigma^2(x,\tau))\tau}{\sigma(x,\tau)\sqrt{\tau}},\ d_2 = d_1(x,\tau) - \sigma(x,\tau)\sqrt{\tau}$$

and φ is a p.d.f. of N(0,1).

Arbitrage-free conditions

For fixed time-to-maturity τ:

- SPD is positive:
  $q_{t,S_t}(x,\tau) \geq 0$ for all $x > 0$.

For varying maturity:

- SPD is positive:
  $q_{t,S_t}(x,\tau) \geq 0$ for all $x, \tau > 0$.
- Total variance is strictly increasing in time-to-maturity:
  $\frac{\partial}{\partial \tau}\sigma^2(K,\tau)\tau > 0$ for all $K, \tau > 0$.

Local polynomial estimation for fixed τ

The local quadratic estimator of the regression function $\hat{\sigma}$ in the point K is defined as minimizer of the following local least squares criterion:

$$\min_{\alpha_0,\alpha_1,\alpha_2} \sum_{i=1}^{n_\tau} \Bigg\{ \tilde{\sigma}_i - \underbrace{\left[\alpha_0 + \alpha_1(K_i - K) + \alpha_2(K_i - K)^2\right]}_{\hat{\sigma}(K)} \Bigg\}^2 \mathcal{K}_h(K - K_i$$

where $\mathcal{K}_h(K - K_i) = \frac{1}{h}\mathcal{K}\left(\frac{K-K_i}{h}\right)$ and $\mathcal{K}$ is a so-called kernel function – typically a symmetric density function with compact support, e.g. $\mathcal{K}(u) = \frac{3}{4}(1-u^2)I(|u| \leq 1)$ (Epanechnikov kernel) and h is called bandwidth.

Choice of bandwidth

The choice of h governs the trade-off between bias and variance of $\hat{\sigma}(K)$

- large h yields a small variance but the large bias
- small h yields a small bias but the large variance.

Since arbitrage-free conditions are typically violated for small h we choose as small bandwidth as possible to have enough data in each bandwidth.

Taylor expansion

Comparing $\hat{\sigma}$ with the Taylor expansion of σ yields

$$\alpha_0 = \hat{\sigma}(K_i),\ \alpha_1 = \hat{\sigma}'(K_i),\ 2\alpha_2 = \hat{\sigma}''(K_i),$$

which makes the estimation of the regression function and its first two derivatives ($\hat{\sigma}'$ and $\hat{\sigma}''$ respectively) possible.

$$\hat{q}(K,\tau) = S_t e^{r\tau}\sqrt{\tau}\varphi(d_1)\left\{\frac{1}{K^2\alpha_0\tau} + \frac{2d_1}{K\alpha_0\sqrt{\tau}}\alpha_1 + \frac{d_1 d_2}{\alpha_0}(\alpha_1)^2 + 2\alpha_2\right.$$

where:
$d_1 = \frac{\ln(S/K)+(r+0.5(\alpha_0)^2)\tau}{\alpha_0\sqrt{\tau}}$, $d_2 = d_1 - \alpha_0\sqrt{\tau}$.

Optimization problem

The optimization problem can be written as:

$$\min_{\alpha_0,\alpha_1,\alpha_2} \sum_{i=1}^{n_\tau} \left\{\tilde{\sigma}_i - \alpha_0 - \alpha_1(K_i - K) - \alpha_2(K_i - K)^2\right\}^2 \mathcal{K}_h(K - K_i) \tag{4}$$

subject to

$$S_t e^{r\tau}\sqrt{\tau}\varphi(d_1)\left\{\frac{1}{K^2\alpha_0\tau} + \frac{2d_1}{K\alpha_0\sqrt{\tau}}\alpha_1 + \frac{d_1 d_2}{\alpha_0}(\alpha_1)^2 + 2\alpha_2\right\} \geq 0$$

$$d_1 = \frac{\ln(S_t/K) + (r + 0.5(\alpha_0)^2)\tau}{\alpha_0\sqrt{\tau}}$$

$$d_2 = d_1 - \alpha_0\sqrt{\tau}.$$

Standardization: arbitrage-free conditi

We express the IV as a function of *futures moneyness* $\kappa = \frac{K}{F_t}$ where $F_t = S_t e^{r\tau}$. The IV function located at $\kappa = 1$ is called at-the-money (ATM).

SPD:

$$q(\kappa,\tau) = \frac{\sqrt{\tau}\varphi(d_1)}{F}\left\{\frac{1}{\kappa^2\sigma\tau} + \frac{2d_1}{\kappa\sigma\sqrt{\tau}}\frac{\partial\sigma}{\partial\kappa} + \frac{d_1 d_2}{\sigma}\left(\frac{\partial\sigma}{\partial\kappa}\right)^2 + \frac{\partial^2\sigma}{\partial\kappa^2}\right\}. \quad (5$$

$$d_1 = \frac{-\ln(\kappa e^{r\tau}) + (r + 0.5\sigma^2)\tau}{\sigma\sqrt{\tau}} = \frac{\sigma^2\tau/2 - \ln(\kappa)}{\sigma\sqrt{\tau}}$$

$$d_2 = d_1 - \sigma\sqrt{\tau} = \frac{-\sigma^2\tau/2 - \ln(\kappa)}{\sigma\sqrt{\tau}}.$$

Standardization: optimization problem

$$\min_{\alpha_0,\alpha_1,\alpha_2} \sum_{i=1}^{n_\tau} \left\{\tilde{\sigma}_i - \alpha_0 - \alpha_1(\kappa_i - \kappa) - \alpha_2(\kappa_i - \kappa)^2\right\}^2 \mathcal{K}_h(K - K_i).$$

subject to:

$$\frac{\sqrt{\tau}\varphi(d_1)}{F}\left\{\frac{1}{\kappa^2\alpha_0\tau} + \frac{2d_1}{\kappa\alpha_0\sqrt{\tau}}\alpha_1 + \frac{d_1 d_2}{\alpha_0}(\alpha_1)^2 + 2\alpha_2\right\} \geq 0$$

$$\begin{aligned} d_1 &= \frac{\alpha_0^2\tau/2 - \ln(\kappa)}{\alpha_0\sqrt{\tau}} \\ d_2 &= \frac{-\alpha_0^2\tau/2 - \ln(\kappa)}{\alpha_0\sqrt{\tau}}. \end{aligned}$$

Estimating the IV-Surface

The idea of local polynomial estimation in higher dimensions is a straight-forward generalization of the one-dimensional case. A standard – unconstrained two-dimensional local quadratic estimator $\hat{\sigma}(\kappa, \tau)$ is given by minimizer of:

$$\min_{\alpha} \sum_{i=1}^{n} \mathcal{K}_H(\kappa - \kappa_i, \tau - \tau_i)\{\tilde{\sigma}_i - \alpha_0 - \alpha_1(\kappa_i - \kappa) - \alpha_2(\tau_i - \tau) \\ -\alpha_{1,1}(\kappa_i - \kappa)^2 - \alpha_{1,2}(\kappa_i - \kappa)(\tau_i - \tau) - \alpha_{2,2}(\tau_i - \tau)^2\}^2$$

where $\mathcal{K}_H(u) = \frac{1}{\det H}\mathcal{K}(H^{-1}u)$ is a (bivariate) kernel function with bandwidths (matrix) H.

IV-Surface: Taylor expansion

Comparing it with a truncated bi-variate Taylor expansion of $\sigma(\kappa,\tau)$ shows $\alpha_0 = \hat{\sigma}(\kappa,\tau)$, $\alpha_1 = \frac{\partial\hat{\sigma}}{\partial\kappa}(\kappa,\tau)$, $\alpha_2 = \frac{\partial\hat{\sigma}}{\partial\tau}(\kappa,\tau)$, $\alpha_{1,1} = \frac{\partial^2\hat{\sigma}}{2\partial\kappa^2}(\kappa,\tau)$, $\alpha_{2,2} = \frac{\partial^2\hat{\sigma}}{2\partial\tau^2}(\kappa,\tau)$, $\alpha_{1,2} = \frac{\partial^2\hat{\sigma}}{\partial\kappa\partial\tau}(\kappa,\tau)$.

Since in our application it is typical to have small number of design points in the τ direction, the idea is to construct a local smoother $\hat{\sigma}(\kappa,\tau)$ quadratic in κ and linear in τ, given by the solution of:

$$\min_{\alpha} \sum_{i=1}^{n} \mathcal{K}_H(\kappa-\kappa_i,\tau-\tau_i)\left\{\tilde{\sigma}_i - \alpha_0 - \alpha_1(\kappa_i-\kappa)\right.$$
$$\left.-\alpha_2(\tau_i-\tau) - \alpha_{1,1}(\kappa_i-\kappa)^2 - \alpha_{1,2}(\kappa_i-\kappa)(\tau_i-\tau)\right\}^2.$$

IV-Surface: Arbitrage free conditions

- SPD $\hat{q}(\kappa, \tau)$ is positive for all $\kappa, \tau > 0$:
  $$\sqrt{\tau}\varphi(d_1)\left\{\frac{1}{\kappa^2\alpha_0\tau} + \frac{2d_1}{\kappa\alpha_0\sqrt{\tau}}\alpha_1 + \frac{d_1 d_2}{\alpha_0}\alpha_1^2 + 2\alpha_{1,1}\right\} \geq 0$$
- total variance locally strictly increasing in τ:
  $$\frac{\partial\hat{w}}{\partial\tau} > 0 \longrightarrow 2\tau_l\alpha_0\alpha_2 + \alpha_0^2 > 0, \qquad l = 1, 2, ..., L$$
- total variance strictly increasing on a grid:
  $\hat{w}(\kappa, \tau_l,) < \hat{w}(\kappa, \tau_{l'}),$ for all $\tau_l < \tau_l' \longrightarrow$
  $\hat{\sigma}^2(\kappa, \tau_l)\tau_l < \hat{\sigma}^2(\kappa, \tau_{l'})\tau_{l'}$ for all $\tau_l < \tau_l' \longrightarrow$
  $\alpha_0^2(l)\tau_l < \alpha_0^2(l+1)\tau_{l+1}, \quad l = 1, 2, ..., L$

IV-Surface: Optimization problem

$$\min_{\alpha(l)} \sum_{l=1}^{L} \sum_{i=1}^{n} \mathcal{K}_H(\kappa - \kappa_i, \tau_l - \tau_i) \{\tilde{\sigma}_i - \alpha_0(l) - \alpha_1(l)(\kappa_i - \kappa)$$

$$-\alpha_2(l)(\tau_i - \tau) - \alpha_{1,1}(l)(\kappa_i - \kappa)^2 - \alpha_{1,2}(l)(\kappa_i - \kappa)(\tau_i - \tau)\}^2$$

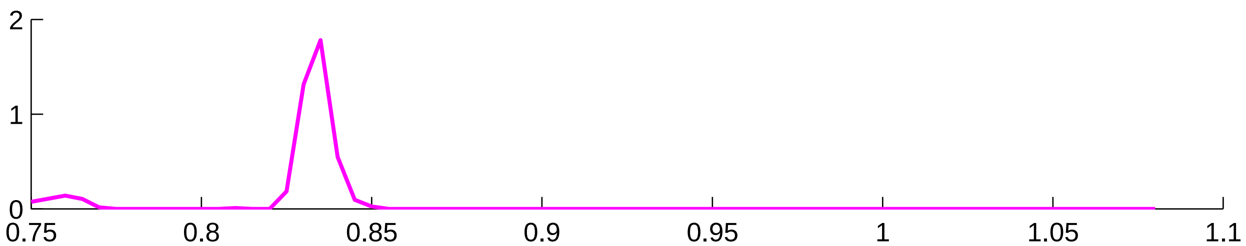
subject to

$$\sqrt{\tau_l}\varphi(d_1(l)) \left\{ \frac{1}{\kappa^2 \alpha_0(l)\tau_l} + \frac{2d_1(l)}{\kappa\alpha_0(l)\sqrt{\tau_l}}\alpha_1(l) + \frac{d_1(l)d_2(l)}{a_0(l)}\alpha_1^2(l) + 2\alpha_{1,1}(l) \right\} \geq 0,$$

$$d_1(l) = \frac{\alpha_0^2(l)\tau_l/2 - \ln(\kappa)}{\alpha_0(l)\sqrt{\tau_l}}, \quad d_2(l) = d_1(l) - a_0(l)\sqrt{\tau_l}, \quad l = 1, \quad \ldots \quad , L$$

$$2\tau_l\alpha_0(l)\alpha_2(l) + \alpha_0^2(l) > 0 \quad l = 1, \quad \ldots \quad , L$$

$$\alpha_0^2(l)\tau_l < \alpha_0^2(l+1)\tau_{l+1}, \quad l = 1, \quad \ldots \quad , L.$$



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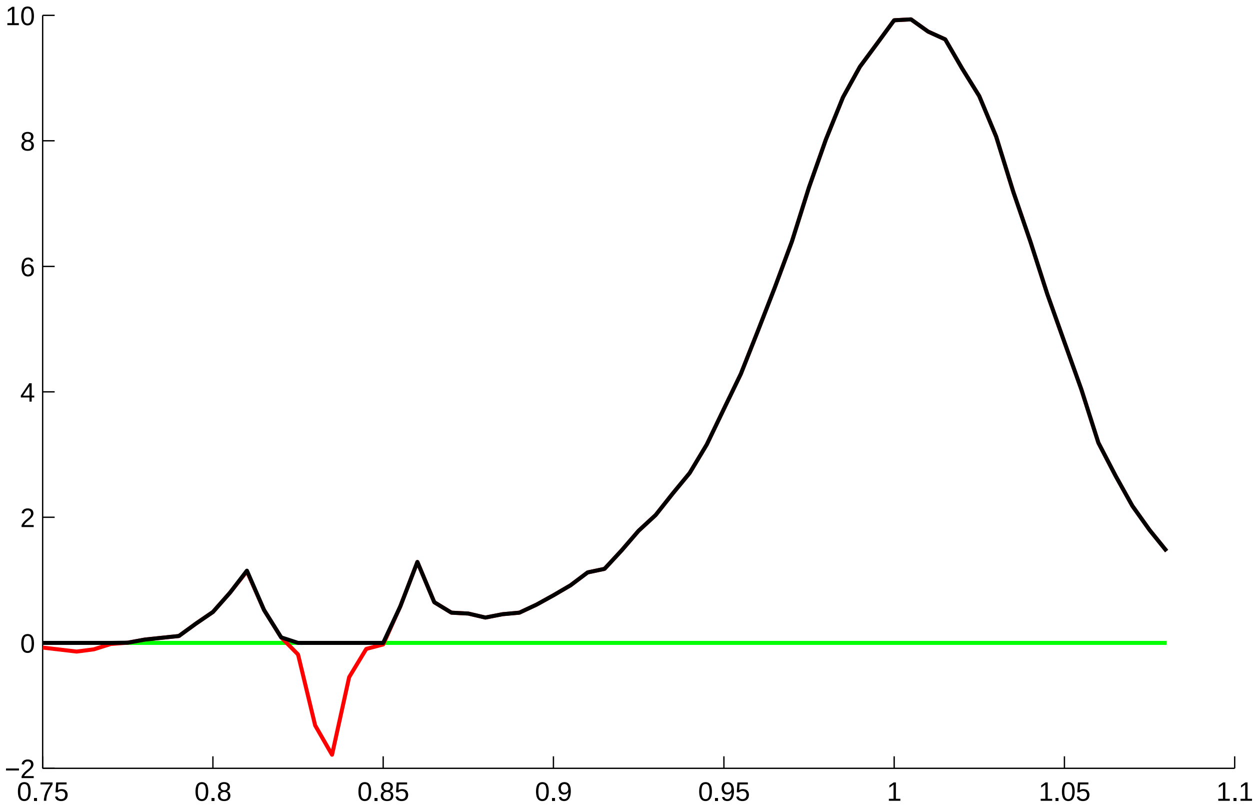

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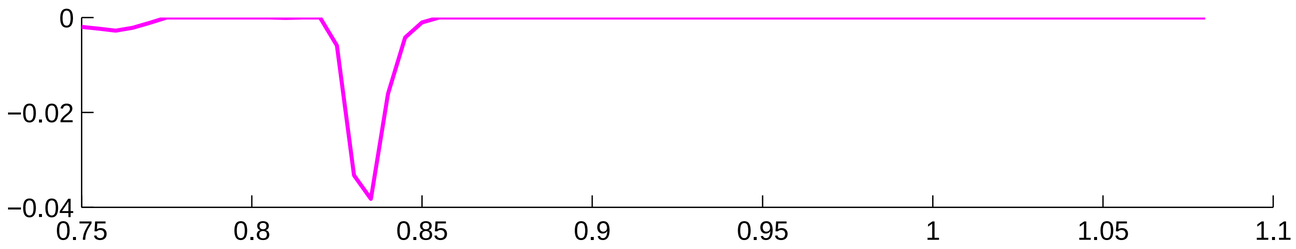
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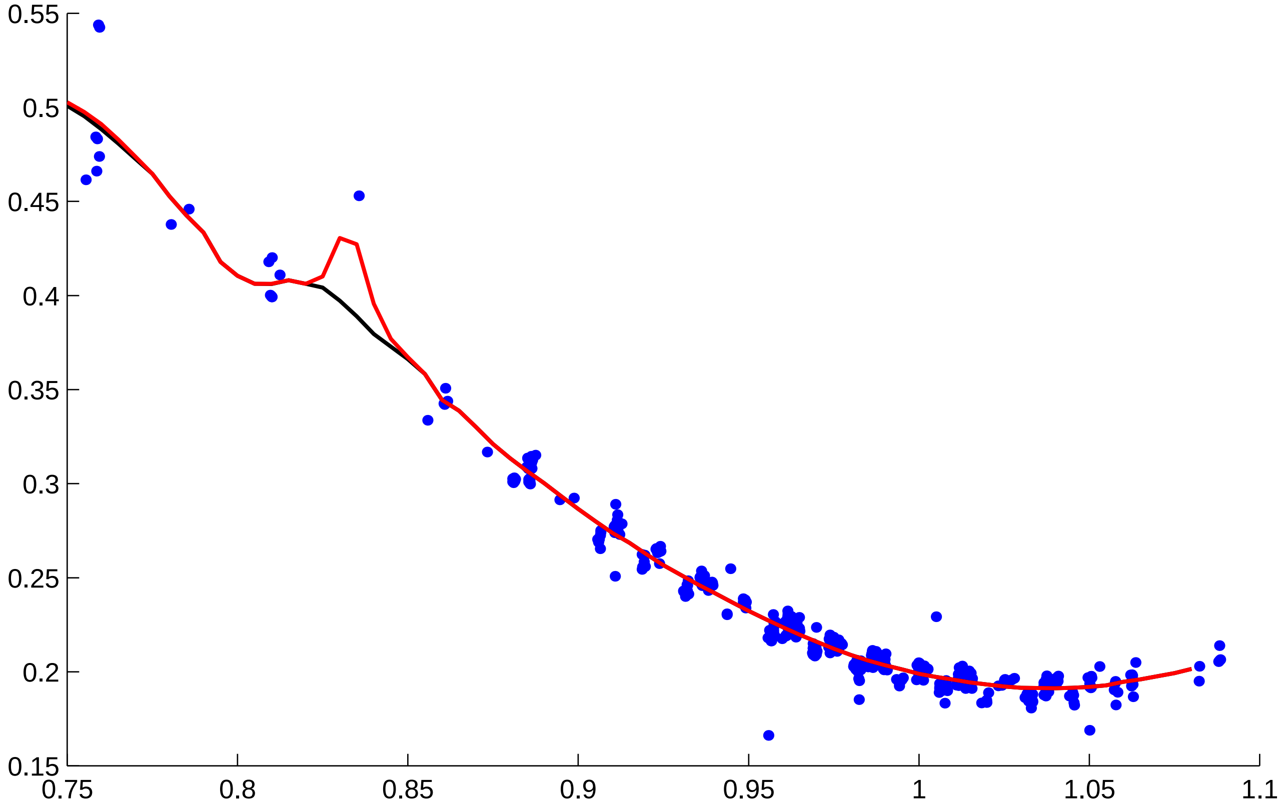
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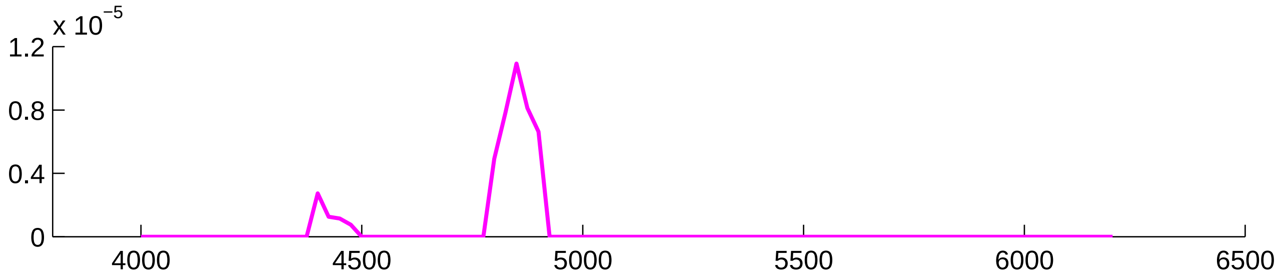
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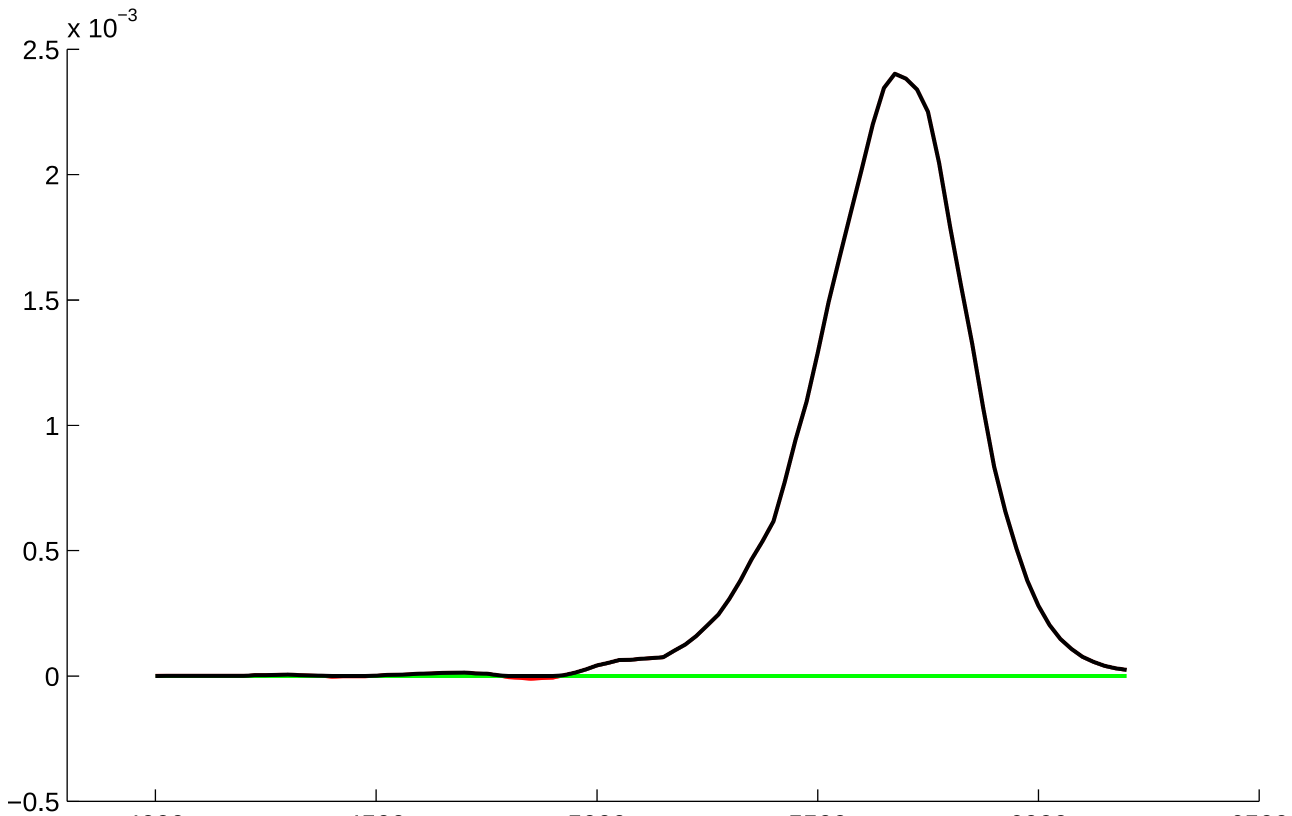
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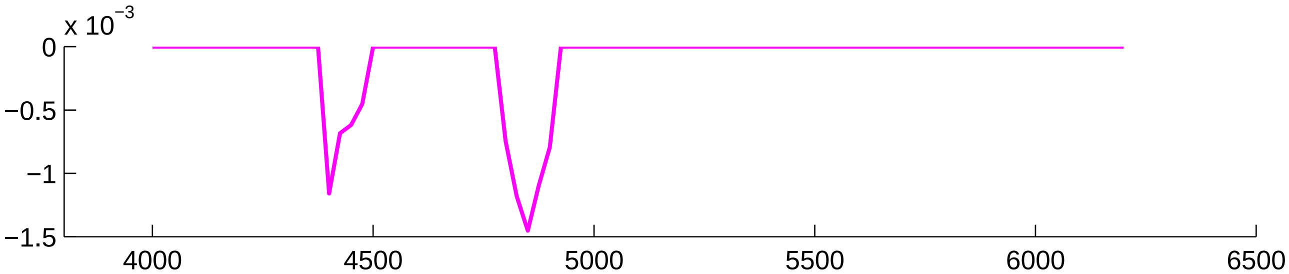

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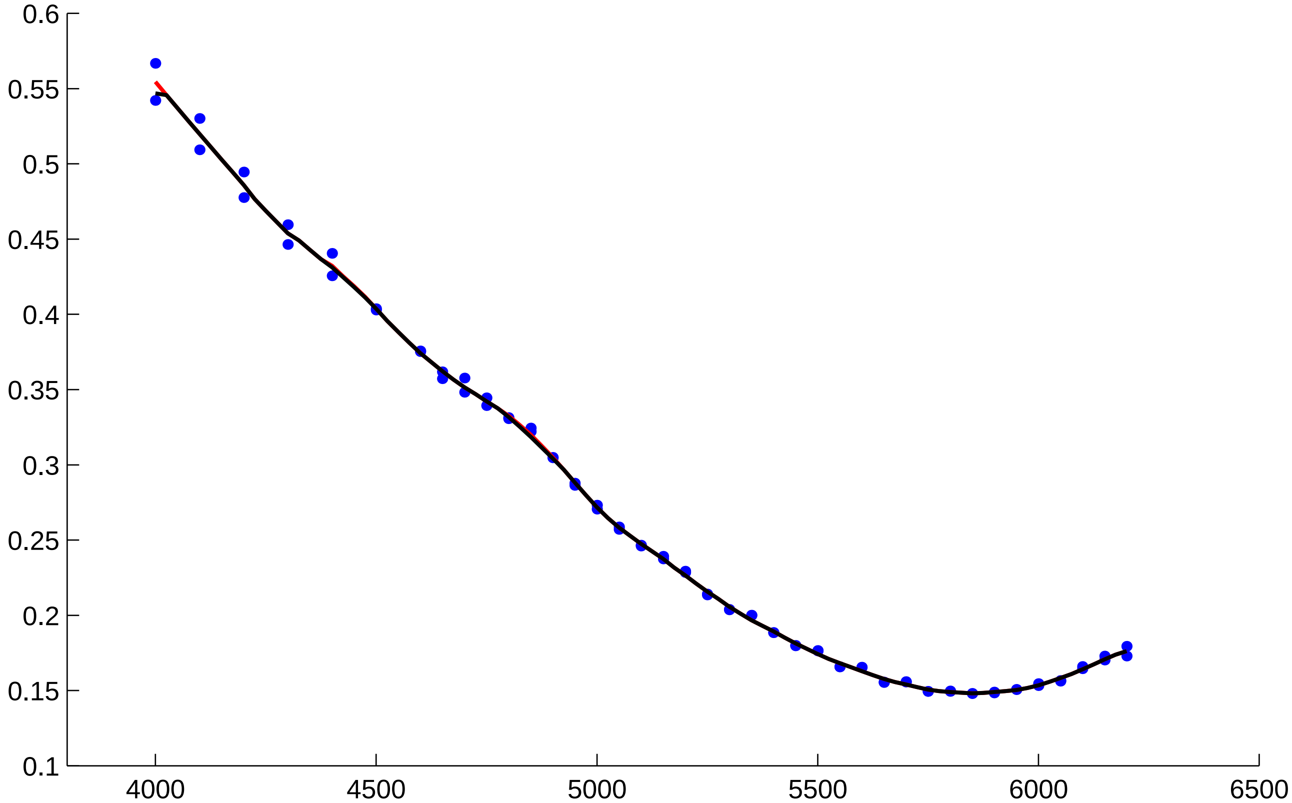
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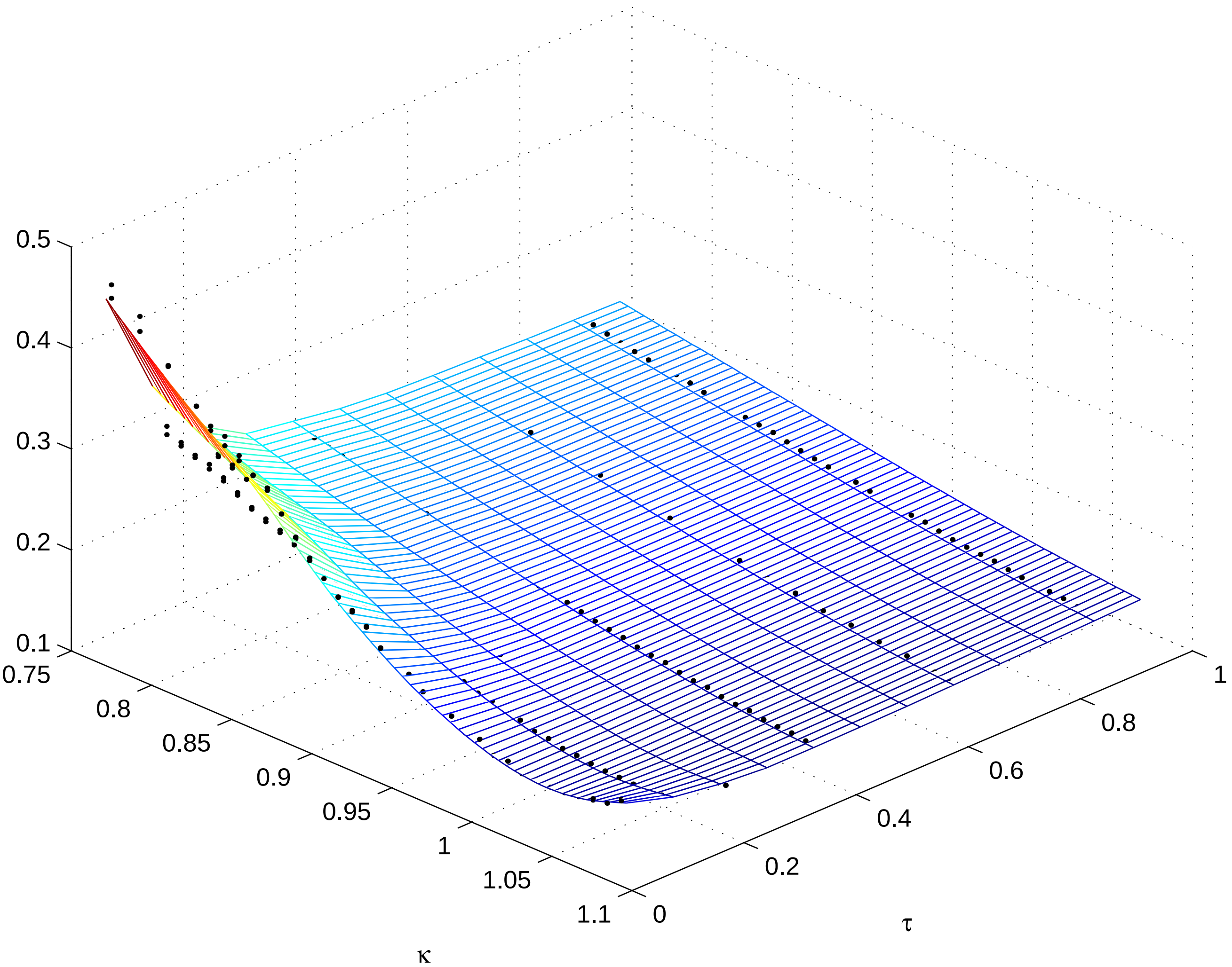
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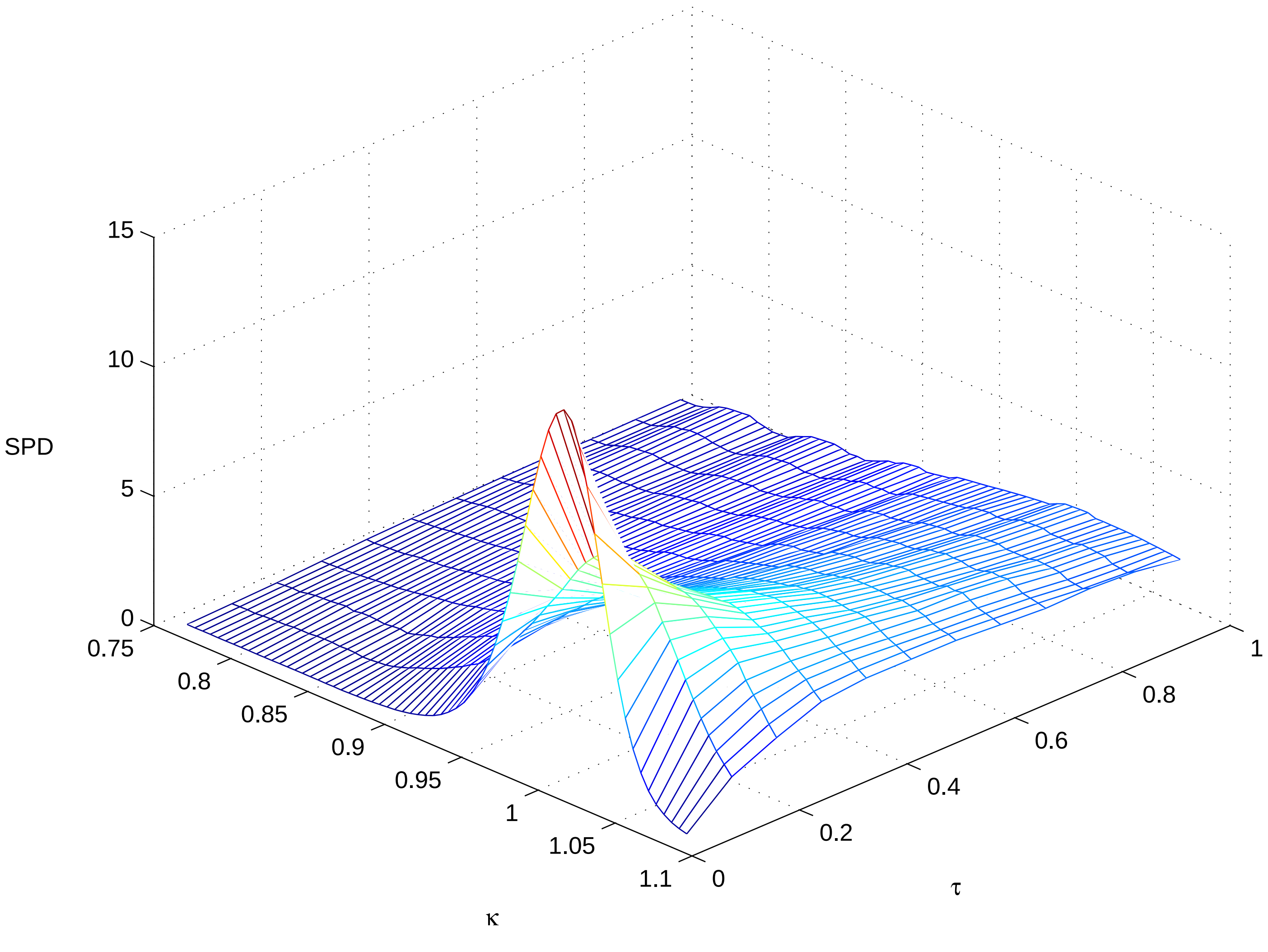

SPD
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