Weak dependence in stochastic programming

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I.

Coefficients of weak dependence and convergence of integrated empirical process

Weak dependence
Notation

- $(\Omega, \mathcal{A}, \mathbb{P})$ ... probability space
- $(X, \mathbb{B})$ ... measurable space (value space)
- $\{\xi_t\}_{-\infty}^{+\infty}$ ... X-valued stochastic process with discrete or continuous time
- $\mathcal{B}_a^b$ ... σ-algebra generated by events

$$\{\xi_{t_1} \in A_{t_1}, \dots, \xi_{t_n} \in A_{t_n}\}$$

  where $(a \leq)\ t_1 \leq \cdots \leq t_n\ (\leq b)$, n are arbitrary, $A_{t_1}, \dots, A_{t_n}$ are $\mathcal{B}$-measurable sets
- $\mathcal{B}_1, B_2$ ... two arbitrary σ-algebras of subsets of Ω

Definition

The process $\{\xi_t\}$ is *m*-dependent if $\mathcal{B}^{a}_{-\infty}$ and $\mathcal{B}^{+\infty}_{b}$ are independent when $b - a > m$

Example: moving average process MA(m) is $(m+1)$-dependent (but not *m*-dependent)

Strong mixing (α-mixing) coefficient

$$\alpha(\mathcal{B}_1, \mathcal{B}_2) = \sup_{A \in \mathcal{B}_1, B \in \mathcal{B}_2} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|$$

$$\alpha(t) = \sup_s \alpha(\mathcal{B}^s_{-\infty}, \mathcal{B}^{+\infty}_{s+t})$$

Definition

The process $\{\xi_t\}$ is α-mixing if $\alpha(t) \to 0$ as $t \to +\infty$

- α-coefficient measures direct covariance dependence
- range: $\alpha(\mathcal{B}_1, \mathcal{B}_2) \leq 1/4$

Example: autoregressive process AR(m) with normal increments is strong mixing (but not with binomial increments)

Absolute regularity (β-mixing) coefficient

$$\beta(\mathcal{B}_1, \mathcal{B}_2) = \mathbb{E} \operatorname*{esssup}_{B \in \mathcal{B}_2} |\mathbb{P}(B|\mathcal{B}_1) - \mathbb{P}(B)|$$

$$\beta(t) = \sup_s \beta(\mathcal{B}^s_{-\infty}, \mathcal{B}^{+\infty}_{s+t})$$

Definition

The process $\{\xi_t\}$ is β-mixing if $\beta(t) \to 0$ as $t \to +\infty$

- $\beta(\mathcal{B}_1, \mathcal{B}_2) = \sup_{\{A_i\},\{B_i\}} \frac{1}{2} \sum_{i=1}^{I} \sum_{j=1}^{J} |\mathbb{P}(A_i \cap B_j) - \mathbb{P}(A_i)\mathbb{P}(B_j)|$

  $\{A_i\} \subset \mathcal{B}_1, \{B_i\} \subset \mathcal{B}_2$ are partitions of Ω

*-mixing (φ-mixing) coefficient

$$\varphi(\mathcal{B}_1, \mathcal{B}_2) = \sup_{A \in \mathcal{B}_1, B \in \mathcal{B}_2} \left| \frac{\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)}{\mathbb{P}(A)\mathbb{P}(B)} \right|$$

$$\varphi(t) = \sup_s \varphi(\mathcal{B}^s_{-\infty}, \mathcal{B}^{+\infty}_{s+t})$$

Definition

The process $\{\xi_t\}$ is φ-mixing if $\varphi(t) \to 0$ as $t \to +\infty$

Uniform mixing (ϕ-mixing) coefficient

$$\phi(\mathcal{B}_1, \mathcal{B}_2) = \sup_{A \in \mathcal{B}_1, B \in \mathcal{B}_2} \left| \frac{\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)}{\mathbb{P}(A)} \right|$$

$$\phi(t) = \sup_s \phi(\mathcal{B}_{-\infty}^s, \mathcal{B}_{s+t}^{+\infty})$$

Definition

The process $\{\xi_t\}$ is ϕ-mixing if $\phi(t) \to 0$ as $t \to +\infty$

Complete regularity (ρ-mixing) coefficient

$$\rho(\mathcal{B}_1, \mathcal{B}_2) = \sup_{\eta_1, \eta_2} \left| \frac{\mathbb{E}\eta_1\eta_2 - \mathbb{E}\eta_1\mathbb{E}\eta_2}{\sqrt{\mathrm{var}\,\eta_1\,\mathrm{var}\,\eta_2}} \right|$$

$$\phi(t) = \sup_a \rho(\{\xi_s, s \leq a\}, \{\xi_s, s \geq a + t\})$$

η_1, η_2 are $\mathcal{B}_1$-, $\mathcal{B}_2$-measurable random variables

Definition

The process $\{\xi_t\}$ is ϕ-mixing if $\phi(t) \to 0$ as $t \to +\infty$

Relationships among mixing conditions

General relationships:

$$\varphi \;\Rightarrow\; \phi \;\Rightarrow\; \begin{array}{c}\beta \\ \\ \rho\end{array} \;\Rightarrow\; \alpha$$

Strictly stationnary Gaussian sequences:

$$\begin{array}{c}m\text{-dep.} \\ \Updownarrow \\ \varphi \\ \Updownarrow \\ \phi\end{array} \;\Rightarrow\; \beta \;\Rightarrow\; \begin{array}{c}\rho \\ \Updownarrow \\ \alpha\end{array}$$

Relationships among mixing conditions

Various limiting theorems remain valid with weakly dependent sequences. Example of CLT:

Theorem (Mori, Yoshihara (1986))

Let

- $\{\xi_i\}$ *. . . strong mixing sequence with* $\alpha(n)$
- $\mathbb{E}\xi_1 = 0$, $\mathbb{E}\xi_1^2 < +\infty$
- $S_0 = 0$, $S_n = \sum_{j=1}^{n} \xi_j$
- $s_n^2 = ES_n^2$

Then

$$\frac{S_n}{s_n} \longrightarrow^d N(0;1)$$

iif $\left\{(\frac{S_n}{s_n})^2\right\}_{n=1}^{+\infty}$ *is uniformly integrable, i.e.,*

$$\lim_{a\to+\infty} \sup_{n\geq 1} \int_{|\frac{S_n}{s_n}|>a} \frac{S_n^2}{s_n^2} \mathrm{d}\mathbb{P} = 0$$

$$\sqrt{N}\, W(\mu_N, \mu) = \int_{-\infty}^{+\infty} \sqrt{N}\, |F_N(t) - F(t)|\, \mathrm{d}t \tag{1}$$

- $F_N(t) = \frac{1}{N} \sum_{i=1}^{N} I_{(-\infty;t]}(\xi_i), \quad t \in \mathbb{R}$ ... empirical distribution function
- μ_N ... corresponding probability measure
- μ ... probability measure with finite first moment and distribution function F
- $\xi_1, \ldots, \xi_N$ ... iid sample from μ

Classical result for μ uniform distribution on $[0; 1]$:

$$\int_0^1 \sqrt{N} \left| \frac{1}{N} \sum_{i=1}^{N} I_{(0;t]}(\xi_i) - F(t) \right| \mathrm{d}t \to_d \int_0^1 |\mathbb{U}(t)| \mathrm{d}t \tag{2}$$

Distribution of RHS is known explicitly in this case.
Shorack, Wellner (1986)

General distribution

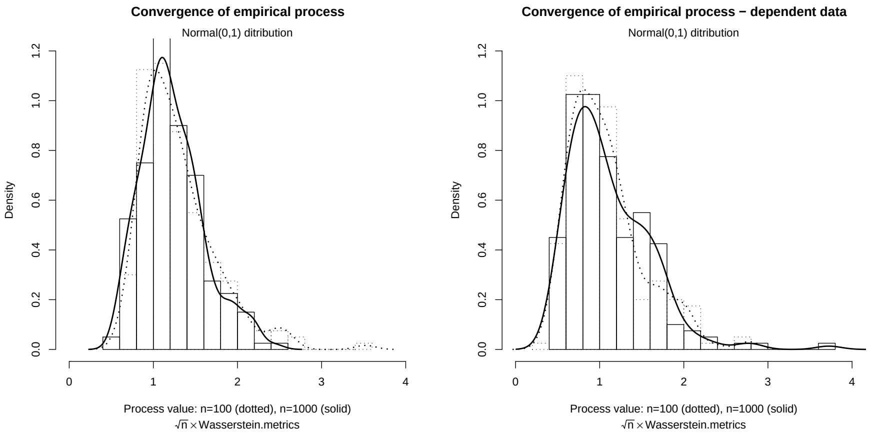
$$\int_{-\infty}^{+\infty} \sqrt{N} \left| \frac{1}{N} \sum_{i=1}^{N} I_{(-\infty;t]}(\xi_i) - F(t) \right| \mathrm{d}t \to_d \int_{-\infty}^{+\infty} |\mathbb{U}(F(t))| \mathrm{d}t. \qquad (3)$$

Del Barrio, Giné, Matrán (1999): (3) is valid if (and only if)

$$\int_{-\infty}^{+\infty} \sqrt{F(t)(1 - F(t))} \mathrm{d}t < +\infty$$

(In fact: if some processes Y_N converge weakly in $L_1(\mathbb{R})$ to Y, then, among others, $||Y_N||_{L_1} \to_d ||Y||_{L_1}$ where $||g||_{L_1} = \int_{-\infty}^{\infty} g(t)\mathrm{d}t$ for each non-negative $g \in L_1(\mathbb{R})$.)

- idea: convergence is proved for iid data, but some weak dependence property would not make difficulties (as CLT is valid with weak dependence)
- illustration: on simple MA(1) process $\xi_k := 0.5\zeta_k + 0.5\zeta_{k-1}$ with normal distribution – comparison of independent and weakly dependent and samples
- still to do: AR process (α-mixing for some class of continuous distributions)
- still to do: modify the proof of Del Bario et al. (1999) – involving the appropriate condition from theory of weakly dependent sequences



Convergence of empirical process
Normal(0,1) ditribution
Density
0.0
0.2
0.4
0.6
0.8
1.0
1.2
0
1
2
3
4
Process value: n=100 (dotted), n=1000 (solid)
√n × Wasserstein.metrics
Convergence of empirical process − dependent data
Normal(0,1) ditribution
Density
0.0
0.2
0.4
0.6
0.8
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4
Process value: n=100 (dotted), n=1000 (solid)
√n × Wasserstein.metrics

II.

Convexity of chance-constrained programs – independent and dependent rows

$$\min F_0(x) \quad \text{subject to} \quad \mathbb{P}\big(h(x;\xi) \geq 0)\big) \geq p \tag{4}$$

- $x \in \mathbb{R}^m$ ... decision vector
- $\xi : \Omega \to \mathbb{R}^s$ ... s-dim. random vector defined on $(\Omega, \mathcal{A}, \mathbb{P})$
- $h : \mathbb{R}^m \times \mathbb{R}^s \to \mathbb{R}^d$ ... vector-valued mapping
- $p \in [0;1]$ ... (prescribed) probability level

Denote

- $\mu = \mathbb{P} \circ \xi^{-1}$ ... distribution of ξ
- $F = F_\mu$ ... distribution function of ξ
- $H(x) = \{\xi \in \mathbb{R}^m : h(x;\xi) \geq 0)\}$
- $M(p) = \{x \in \mathbb{R}^m : \mathbb{P}(H(x)) = \mu(H(x)) \geq p\}$ ... set of feasible decisions

$$\min F_0(x) \quad \text{subject to} \quad \int_{\mathbb{R}^s} p - \chi_{H(x)}(\xi)\ \mu(\mathrm{d}\xi) \le 0 \tag{5}$$

- form adapted to the general stability theorem – Henrion, Römisch (1999)

$$\min F_0(x) \quad \text{subject to} \quad \mathbb{P}(h(x;\xi) < 0) \le \varepsilon \tag{6}$$

- $\varepsilon = 1 - p$ … (admissible) level of violation of the constraints
- $M(1-\varepsilon)$ … set of ε-feasible solutions (used in robust programming)

Convexity of chance constraints: known results

PRÉKOPA (1995)

Key question

When the set $M(p)$ of feasible solutions is convex?

1. Trivial result

   If $h(\cdot, \xi)$ is convex for all ξ, $M(0)$, $M(1)$ are convex
2. $M(p)$ is convex if
   - μ is a log-concave (or r-concave for $r \geq -1/s$) measure (implied by a log-concave, or $\frac{r}{1-rs}$-concave density)
   - components of h are quasi-concave (in both variables)

Parameterization of the concavity

Definition

$f : \mathbb{R}^d \to (0; +\infty)$ is *r-concave* for $r \in [-\infty; +\infty]$ if

$$f(\lambda x + (1-\lambda)y) \geq [\lambda f^r(x) + (1-\lambda) f^r(y)]^{1/r}$$

- cases $r = -\infty, 0, +\infty$ by continuity
$r = +\infty$	...	$\mathrm{RHS} = \max\{f(x), f(y)\}$
$r \in (1; +\infty)$	...	f^r is concave
$r = 1$	...	f is concave
$r = 0$	...	f is *log-concave* ($\log f$ is concave): $f(\lambda x + (1-\lambda)y) \geq f^{\lambda}(x) f^{1-\lambda}(y)$
$r < 0$	...	f^r is convex
$r = -\infty$	...	f is *quasi-concave*: $\mathrm{RHS} = \min\{f(x), f(y)\}$
- for all $r \leq r^*$, f is r-concave if it is r^*-concave
- interesting cases: $r \leq 1$

Special case: random RHS

$$\min F_0(x) \quad \text{subject to} \quad \mathbb{P}\big(g(x) \geq \xi)\big) \geq p \qquad (7)$$

- $h(x;\xi) = g(x) - \xi$
- $M(p) = \{x \in \mathbb{R}^n : F(g(x)) \geq p\}$

Required condition: components of h are quasi-concave
Problem: quasi-concavity is not preserved under addition
$\Rightarrow$ we require $g(x)$ to be convex

Idea of Henrion, Strugarek (2006):

- relax concavity condition of g;
- make more stringent concavity condition on μ.

Parameterization of the function decrease

Definition

$f : \mathbb{R} \to \mathbb{R}$ is *r-decreasing* for $r \in \mathbb{R}$ if

- it is continuous on $(0; +\infty)$, and
- there exists a threshold $t^* > 0$ such that $t^r f(t)$ is *strictly decreasing* for all $t > t^*$

- $r = 0$ ... strictly decreasing (in the classical sense)
- for all $r \leq r^*$, nonnegative f is r-decreasing if it is r^*-decreasing
- key property for marginal densities from the chance-constrained problem
- lemma: if F is distribution function with $(r+1)$-decreasing density, then $z \mapsto F(z^{-1/r})$ is concave on $(0; (t^*)^{-r})$

Convexity of the problem with RHS
Henrion, Strugarek (2006)

Theorem (Henrion, Strugarek (2006))

If

1. g_i *are* $-r_i$*-concave,*
2. ξ_i *have* $(r_i + 1)$*-decreasing densities,*
3. ξ_i *are independent,*

then $M(p)$ *is convex for* $p \geq \max F_i(t_i^*)$

Notation ($i = 1, \ldots, s$):

g_i ... components of g
ξ_i ... components of ξ
F_i ... components of F
t_i^* ... threshold of $r_i + 1$-decreasing density