

Theory of coherent measures

Miloš Kopa

Faculty of Mathematics and Physics
Charles University in Prague
Czech Republic
e-mail:kopa@karlin.mff.cuni.cz

Previous lecture

Measure of risk assigns a real number to any random variable L (loss).

Favorite risk measures:

- variance: $\text{var}(L) = \mathbb{E}(L - \mathbb{E}L)^2$
- standard deviation $sd(L) = (\mathbb{E}(L - \mathbb{E}L)^2)^{\frac{1}{2}}$
- semivariance: $r_s(L) = \mathbb{E} \left[\max(0, L - \mathbb{E}L)^2 \right]$
- mean absolute deviation: $r_a(L) = \mathbb{E}|L - \mathbb{E}L|$
- mean absolute semideviation: $r_{as}(L) = \mathbb{E}[\max(0, L - \mathbb{E}L)]$

- Value at Risk (VaR):

$$\text{VaR}_\alpha(L) = \inf \{ l \in \mathbb{R}, \mathbb{P}(L > l) \leq 1 - \alpha \}$$

- Conditional Value at Risk (CVaR):

$$\text{CVaR}_\alpha(L) = \inf \left\{ a \in \mathbb{R}, a + \frac{1}{1-\alpha} \mathbb{E}[\max(0, L - a)] \right\},$$

alternatively:

$$\text{CVaR}_\alpha(L) = \beta \mathbb{E}(L | L > \text{VaR}_\alpha(L)) + (1 - \beta) \text{VaR}_\alpha(L) \text{ with some } \beta \in [0, 1]$$

Risk measures:

- What are the “reasonable” properties that should have all “good” risk measures?
- Which of the considered measures has the properties?
- Is it possible to generalize a very well-known and popular standard deviation (variance)?
- What is the dual expression of measures with these properties?

Multiobjective optimization:

- How to formulate an optimization problem when multiple objective are considered?
- What are the best solutions of such problems?
- How to find all these best solutions?

Coherent risk measures

CRM: $\mathcal{R} : \mathcal{L}_2(\Omega) \rightarrow (-\infty, \infty]$ that satisfies

- (R1) Translation equivariance: $\mathcal{R}(L + C) = \mathcal{R}(L) + C$ for all X and constants C ,
- (R2) Positive homogeneity: $\mathcal{R}(0) = 0$, and $\mathcal{R}(\lambda L) = \lambda \mathcal{R}(L)$ for all L and all $\lambda > 0$,
- (R3) Subadditivity: $\mathcal{R}(L + M) \leq \mathcal{R}(L) + \mathcal{R}(M)$ for all L and M ,
- (R4) Monotonicity: $\mathcal{R}(L) \geq \mathcal{R}(M)$ when $L \geq M$.

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Strictly expectation bounded risk measures satisfy (R1), (R2), (R3), and

- (R5) $\mathcal{R}(L) > \mathbb{E}[L]$ for all nonconstant L , whereas $\mathcal{R}(L) = \mathbb{E}[L]$ for constant L .

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(R5) $\mathcal{R}(L) > \mathbb{E}[L]$ for all nonconstant L , whereas $\mathcal{R}(L) = \mathbb{E}[L]$ for constant L .

Note: (R2)&(R3) implies convexity of \mathcal{R} : for each $a \in [0, 1]$ we have:

$$\mathcal{R}(aL + (1-a)M) \leq \mathcal{R}(aL) + \mathcal{R}((1-a)M) = a\mathcal{R}(L) + (1-a)\mathcal{R}(M)$$

Other classes of risk measures and functionals: Follmer and Schied (2002), Pflug and Romisch (2007).

Coherent properties for the popular risk measures

- variance: none of (R1)-(R5)
- standard deviation: (R2)
- semivariance: none of (R1)-(R5)
- mean absolute deviation: (R2), (R3)
- mean absolute semideviation: (R2), (R3)
- Value at Risk (VaR): (R1),(R2), (R4)
- Conditional Value at Risk (CVaR): (R1)-(R5)

Dual representation of coherent risk measures

Consider a measurable space (Ω, \mathcal{A}) and the set \mathcal{P} of all probability measures on the space.

Definition

A set $\mathcal{Q} \subset \mathcal{P}$ is called a risk envelope if for each $Q \in \mathcal{Q}$ one has: $Q \geq 0$ and $\mathbb{E}Q = 1$.

Theorem

\mathcal{R} is a coherent risk measure if and only if there exists a risk envelope \mathcal{Q} such that:

$$\mathcal{R}(L) = \max_{Q \in \mathcal{Q}} E(QL)$$

and \mathcal{Q} can be chosen as a convex set.

Interpretation: A coherent risk measure can be understood as a worst-case expectation with respect to some class of probability distributions on (Ω, \mathcal{A}) . It means for some distribution P' . If the probability distribution of L is P then $Q = \frac{dP'}{dP}$.

Example - Risk envelope for CVaR

To simplify the situation consider a measurable space with M atoms (discrete distributions). Moreover let L has a discrete uniform distribution on the space - atoms are equiprobable, i.e. discrete distribution with M equiprobable scenarios l_j , $j = 1, 2, \dots, M$. Assume that $M(1 - \alpha)$ is an integer number. Then:

$$\begin{aligned} \text{CVaR}_\alpha(L) &= \min_{a, z_j} a + \frac{1}{(1 - \alpha) M} \sum_{j=1}^M z_j \\ \text{s. t. } z_j &\geq l_j - a, j = 1, \dots, M \\ z_j &\geq 0, j = 1, \dots, M \end{aligned}$$

Example - Risk envelope for CVaR

And dual problem:

$$\begin{aligned} \text{CVaR}_\alpha(L) &= \max_{y_j} \sum_{j=1}^M y_j l_j \\ \text{s. t. } &\sum_{j=1}^M y_j = 1 \\ &y_j \leq \frac{1}{(1-\alpha)M} \\ &y_j \geq 0, j = 1, \dots, M \end{aligned}$$

Note that optimal solution: $y_j^* = 0$ for $j = 1, 2, \dots, M\alpha$

$y_j^* = \frac{1}{(1-\alpha)M}$ for $j = M\alpha + 1, \dots, M$. Hence the risk envelope for CVaR is:

$$\mathcal{Q} = \left\{ Q : EQ = 1, 0 \leq Q \leq \frac{1}{(1-\alpha)} \right\}$$

A **return measure** is defined as a functional $\mathcal{E}(L) = -\mathcal{R}(L)$ for a coherent risk measure \mathcal{R} . It is obvious that the expectation belongs to this class.

General deviation measures

Rockafellar, Uryasev and Zabarankin (2006A, 2006B): GDM are introduced as an extension of *standard deviation* but they need not to be symmetric with respect to *upside* $X - \mathbb{E}X$ and *downside* $\mathbb{E}X - X$ of a random variable X .

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Any functional $\mathcal{D} : \mathcal{L}_2(\Omega) \rightarrow [0, \infty]$ is called a general deviation measure if it satisfies

(D1) $\mathcal{D}(X + C) = \mathcal{D}(X)$ for all X and constants C ,

(D2) $\mathcal{D}(0) = 0$, and $\mathcal{D}(\lambda X) = \lambda \mathcal{D}(X)$ for all X and all $\lambda > 0$,

(D3) $\mathcal{D}(X + Y) \leq \mathcal{D}(X) + \mathcal{D}(Y)$ for all X and Y ,

(D4) $\mathcal{D}(X) \geq 0$ for all X , with $\mathcal{D}(X) > 0$ for nonconstant X .

(D2) & (D3) \Rightarrow convexity

- **Standard deviation**

$$\mathcal{D}(X) = \sigma(X) = \sqrt{\mathbb{E} \|X - \mathbb{E}X\|_2}$$

- **Mean absolute deviation**

$$\mathcal{D}(X) = \mathbb{E}[|X - \mathbb{E}X|].$$

- **Mean absolute lower and upper semideviation**

$$\mathcal{D}_-(X) = \mathbb{E}[\min(0, X - \mathbb{E}X)], \quad \mathcal{D}_+(X) = \mathbb{E}[\max(0, X - \mathbb{E}X)].$$

- **Worst-case deviation**

$$\mathcal{D}(X) = \sup_{\omega \in \Omega} |X(\omega) - \mathbb{E}X|.$$

- See Rockafellar et al (2006 A, 2006 B) for another examples.

Mean absolute deviation from $(1 - \alpha)$ -th quantile

CVaR deviation

For any $\alpha \in (0, 1)$ a finite, continuous, lower range dominated deviation measure

$$\mathcal{D}_\alpha(X) = \text{CVaR}_\alpha(X - \mathbb{E}X). \quad (1)$$

The deviation is also called **weighted mean absolute deviation from the $(1 - \alpha)$ -th quantile**, see Ogryczak, Ruszczyński (2002), because it can be expressed as

$$\mathcal{D}_\alpha(X) = \min_{\xi \in \mathbb{R}} \frac{1}{1 - \alpha} \mathbb{E}[\max\{(1 - \alpha)(X - \xi), \alpha(\xi - X)\}] \quad (2)$$

with the minimum attained at any $(1 - \alpha)$ -th quantile. In relation with CVaR minimization formula, see Pflug (2000), Rockafellar and Uryasev (2000, 2002).

According to Proposition 4 in Rockafellar et al (2006 A):

- if $\mathcal{D} = \lambda \mathcal{D}_0$ for $\lambda > 0$ and a deviation measure \mathcal{D}_0 , then \mathcal{D} is a deviation measure.
- if $\mathcal{D}_1, \dots, \mathcal{D}_K$ are deviation measures, then
 - $\mathcal{D} = \max\{\mathcal{D}_1, \dots, \mathcal{D}_K\}$ is also deviation measure.
 - $\mathcal{D} = \lambda_1 \mathcal{D}_1 + \dots + \lambda_K \mathcal{D}_K$ is also deviation measure, if $\lambda_k > 0$ and $\sum_{k=1}^K \lambda_k = 1$.

Rockafellar et al (2006 A, B): Duality representation using *risk envelopes*, subdifferentiability and optimality conditions.

We say that general deviation measure \mathcal{D} is

(LSC) **lower semicontinuous** (lsc) if all the subsets of $\mathcal{L}_2(\Omega)$ having the form $\{X : \mathcal{D}(X) \leq c\}$ for $c \in \mathbb{R}$ (level sets) are closed;

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- (LSC) **lower semicontinuous** (lsc) if all the subsets of $\mathcal{L}_2(\Omega)$ having the form $\{X : \mathcal{D}(X) \leq c\}$ for $c \in \mathbb{R}$ (level sets) are closed;
- (D5) **lower range dominated** if $\mathcal{D}(X) \leq EX - \inf_{\omega \in \Omega} X(\omega)$ for all X .

Theorem 2 in Rockafellar et al (2006 A):

Theorem

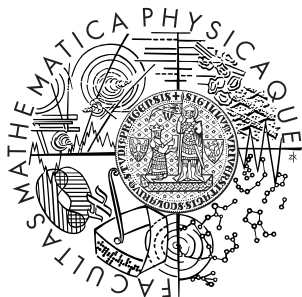
Deviation measures correspond one-to-one with strictly expectation bounded risk measures under the relations

- $\mathcal{D}(X) = \mathcal{R}(X - \mathbb{E}X)$
- $\mathcal{R}(X) = \mathbb{E}[-X] + \mathcal{D}(X)$

In this correspondence, \mathcal{R} is coherent if and only if \mathcal{D} is lower range dominated.

- Follmer, H., Schied, A.: **Stochastic Finance: An Introduction In Discrete Time**. Walter de Gruyter, Berlin, 2002.
- Pflug, G.Ch., Romisch, W.: **Modeling, measuring and managing risk**. World Scientific Publishing, Singapore, 2007.
- Rockafellar, R.T., Uryasev, S. (2002). **Conditional Value-at-Risk for General Loss Distributions**, *Journal of Banking and Finance* 26, 1443–1471.
- Rockafellar, R.T., Uryasev, S., Zabarankin M. (2006A). **Generalized Deviations in Risk Analysis**. *Finance and Stochastics* 10 , 51–74.
- Rockafellar, R.T., Uryasev, S., Zabarankin M. (2006B). **Optimality Conditions in Portfolio Analysis with General Deviation Measures**. *Mathematical Programming* 108, No. 2-3, 515–540.

Convergence of approximate solutions in mean-risk models



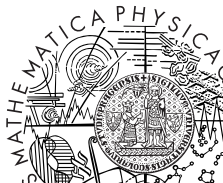
Václav Kozmík

Faculty of Mathematics and Physics
Charles University in Prague

Supervisor: Miloš Kopa

Introduction

- Mean-risk models
 - aim to find optimal portfolio of assets
 - analytical solutions for continuous distributions
 - solutions using generated scenarios
 - predetermined, for instance by historical data
 - generated with the assumption of continuous distribution
 - generated according to few moment estimators
- comparison of the approaches mentioned above
 - convergence and its properties
 - different continuous distributions
 - different risk measures
- computational part
 - data processing and generating the scenarios
 - optimization tasks in GAMS

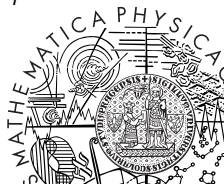


Efficient portfolios

- we consider portfolio based on N assets
- weights of the assets \mathbf{w} , $\sum_{i=1}^N w_i = 1$
- expected returns $u_{\mathbf{w}}$ (always using expectation)
- different risk measures $r_{\mathbf{w}}$
- minimal required returns u_e

Definition

Portfolio of given N assets with weights \mathbf{w} is (mean-risk) efficient, if there are no other weights w_1^, \dots, w_N^* such that $\sum_{i=1}^N w_i^* = 1$ and $u_{\mathbf{w}^*} \geq u_{\mathbf{w}}$ a $r_{\mathbf{w}^*} \leq r_{\mathbf{w}}$.*



Classical optimization task

Efficient portfolios can be obtained while solving following task:

$$\min_{\mathbf{w}} r_{\mathbf{w}}$$

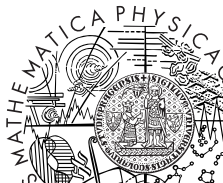
$$\text{s. t. } U_{\mathbf{w}} \geq U_e$$

$$\sum_{i=1}^N w_i = 1$$

$$w_i \in \mathbb{R}, i = 1, \dots, N.$$

Non-negativity condition:

$$w_1, \dots, w_N \geq 0.$$



Risk measures

- variance
- VaR
- cVaR
- absolute deviation
- semivariance

Definition

Let $\alpha \in (0, 1)$ be the threshold and L random variable which represents the loss from holding the portfolio. Then we define VaR_α as:

$$VaR_\alpha(L) = \inf \{l \in \mathbb{R}, P(L > l) \leq 1 - \alpha\}$$



Risk measures

Definition

$cVaR_\alpha$ is defined as:

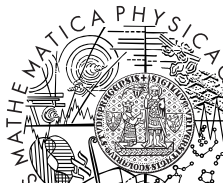
$$cVaR_\alpha(L) = \inf \left\{ a \in \mathbb{R}, a + \frac{1}{1-\alpha} E[\max(0, L - a)] \right\}.$$

Absolute deviation can be calculated as:

$$r_a(L) = E|L - EL|.$$

Semivariance can be calculated as::

$$r_s(L) = E \left[\max(0, L - EL)^2 \right]$$



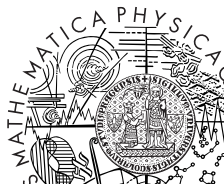
Elliptical distributions

- generalization of normal distribution
- include normal distribution, Student distribution, logistic elliptical distribution and others
- symmetrical around the mean
- simple analysis of linear combinations

Theorem

Let $\mathbf{X} \sim \mathbf{E}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \psi)$, $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$. Then it holds:

$$\mathbf{AX} + \mathbf{b} \sim \mathbf{E}\left(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^T, \psi\right).$$



Variance

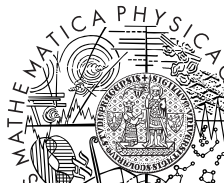
We get classical optimization task which can be used for all elliptical distributions:

$$\min_{\mathbf{w}} \mathbf{w}^T \mathbf{V} \mathbf{w}$$

$$\text{s. t. } \mathbf{w}^T \boldsymbol{\mu} \geq u_e$$

$$\sum_{i=1}^N w_i = 1$$

$$w_i \in \mathbb{R}, i = 1, \dots, N$$



Normal distribution

- VaR

$$\text{VaR}_\alpha(L) = -\mathbf{w}^T \boldsymbol{\mu} + q_\alpha \sqrt{\mathbf{w}^T \mathbf{V} \mathbf{w}}$$

- cVaR

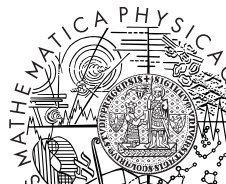
$$\text{cVaR}_\alpha(L) = -\mathbf{w}^T \boldsymbol{\mu} + \frac{\exp\left\{-\frac{q_\alpha^2}{2}\right\}}{(1-\alpha)\sqrt{2\pi}} \sqrt{\mathbf{w}^T \mathbf{V} \mathbf{w}}$$

- absolute deviation

$$r_a(L) = \sqrt{\frac{2}{\pi}} \sqrt{\mathbf{w}^T \mathbf{V} \mathbf{w}}$$

- semivariance

$$r_s(L) = \frac{1}{2} \text{E} \left[(L - \text{E}L)^2 \right]$$



Student distribution

- VaR

$$VaR_\alpha(L) = VaR_\alpha(L) = -\mathbf{w}^T \boldsymbol{\mu} + t_{\alpha, \nu} \sqrt{\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}}$$

- cVaR

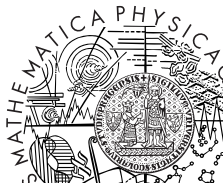
$$cVaR_\alpha(L) = -\mathbf{w}^T \boldsymbol{\mu} + \frac{\Gamma\left(\frac{\nu-1}{2}\right) \sqrt{\nu} \left(1 + \frac{t_{\alpha, \nu}^2}{\nu}\right)^{-\frac{\nu-1}{2}}}{\Gamma\left(\frac{\nu-2}{2}\right) (1-\alpha) (\nu-2) \sqrt{\pi}} \sqrt{\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}}$$

- absolute deviation

$$r_a(L) = \frac{2\sqrt{\nu}\Gamma\left(\frac{\nu+1}{2}\right)}{(\nu-1)\sqrt{\pi}\Gamma\left(\frac{\nu}{2}\right)} \sqrt{\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}}$$

- semivariance

$$r_s(L) = \frac{1}{2} \mathbb{E} \left[(L - \mathbb{E}L)^2 \right]$$



Variance - scenarios

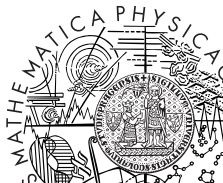
- suppose we have M scenarios of possible stock prices
- we can use mean and variance-covariance estimators $\hat{\mathbf{I}}$ and $\hat{\mathbf{V}}$ to minimize variance
- allows us to process estimators before running the optimization task and therefore is quick

$$\min_{\mathbf{w}} \mathbf{w}^T \hat{\mathbf{V}} \mathbf{w}$$

$$\text{s. t. } \mathbf{w}^T \hat{\mathbf{I}} \geq u_e$$

$$\sum_{i=1}^N w_i = 1$$

$$w_i \in \mathbb{R}, i = 1, \dots, N.$$



VaR - scenarios

- general case could be nonconvex
- we reformulate the task using integer programming
- still hardly computable - M binary variables

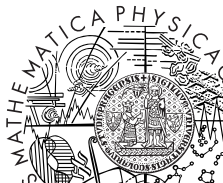
$$\min_{\nu, \mathbf{w}, \delta^j} \nu$$

$$\text{s. t. } -\mathbf{w}^T \mathbf{f}^j \leq \nu + K\delta^j, j = 1, \dots, M$$

$$\sum_{j=1}^M \delta^j = \lfloor (1 - \alpha) M \rfloor$$

$$\delta^j \in \{0, 1\}, j = 1, \dots, M$$

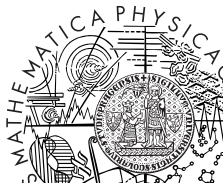
$$\frac{1}{M} \sum_{j=1}^M \mathbf{w}^T \mathbf{f}^j \geq u_e \dots$$



cVaR - scenarios

- linear programming task, can be solved quickly

$$\begin{aligned} \min_{a, \mathbf{w}, z^j} \quad & a + \frac{1}{(1-\alpha)M} \sum_{j=1}^M z^j \\ \text{s. t.} \quad & z^j \geq -\mathbf{w}^T \boldsymbol{\psi}^j - a, \quad j = 1, \dots, M \\ & z^j \geq 0, \quad j = 1, \dots, M \\ & \frac{1}{M} \sum_{j=1}^M \mathbf{w}^T \boldsymbol{\psi}^j \geq u_e \\ & \sum_{i=1}^N w_i = 1 \\ & w_i \in \mathbb{R}, \quad i = 1, \dots, N. \end{aligned}$$



Absolute deviation - scenarios

- linear programming task

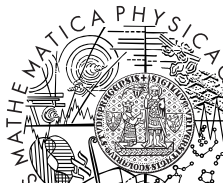
$$\min_{\mathbf{w}, z^j} \frac{1}{M} \sum_{j=1}^M z^j$$

$$\text{s. t. } \mathbf{w}^T \mathbf{v}^j - \frac{1}{M} \sum_{i=1}^M \mathbf{w}^T \mathbf{l}^i \leq z^j, j = 1, \dots, M$$

$$- \mathbf{w}^T \mathbf{v}^j + \frac{1}{M} \sum_{i=1}^M \mathbf{w}^T \mathbf{l}^i \leq z^j, j = 1, \dots, M$$

$$\frac{1}{M} \sum_{j=1}^M \mathbf{w}^T \mathbf{v}^j \geq u_e$$

...



Semivariance - scenarios

- quadratic programming task

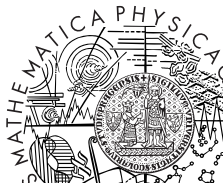
$$\min_{\mathbf{w}, z^j} \frac{1}{M} \sum_{j=1}^M (z^j)^2$$

$$\text{s. t. } z^j \geq -\mathbf{w}^T \mathbf{v}^j + \frac{1}{M} \sum_{i=1}^M \mathbf{w}^T \mathbf{v}^i, j = 1, \dots, M$$

$$z^j \geq 0, j = 1, \dots, M$$

$$\frac{1}{M} \sum_{j=1}^M \mathbf{w}^T \mathbf{v}^j \geq u_e$$

...



Conclusion

Thank you for your attention!

Miloš Kopa

e-mail:

kopa@karlin.mff.cuni.cz

