Theory of coherent measures

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Measure of risk assigns a real number to any random variable L (loss).
Favorite risk mesures:

- variance: $var(L) = \mathbb{E}(L - \mathbb{E}L)^2$
- standard deviation $sd(L) = (\mathbb{E}(L - \mathbb{E}L)^2)^{\frac{1}{2}}$
- semivariance: $r_s(L) = \mathbb{E}\left[\max(0, L - \mathbb{E}L)^2\right]$
- mean absolute deviation: $r_a(L) = \mathbb{E}|L - \mathbb{E}L|$
- mean absolute semideviation: $r_{as}(L) = \mathbb{E}\left[\max(0, L - \mathbb{E}L)\right]$
- Value at Risk (VaR):
  $VaR_\alpha(L) = \inf\{l \in \mathbb{R}, \mathrm{P}(L > l) \leq 1 - \alpha\}$
- Conditional Value at Risk (CVaR):
  $CVaR_\alpha(L) = \inf\left\{a \in \mathbb{R}, a + \frac{1}{1-\alpha}\mathbb{E}\left[\max(0, L - a)\right]\right\}$,
  alternatively:
  $CVaR_\alpha(L) = \beta E(L|L > VaR_\alpha(L)) + (1 - \beta)VaR_\alpha(L)$ with some $\beta \in [0, 1]$

Questions for today

Risk measures:

- What are the “reasonable” properties that should have all “good” risk measures?
- Which of the considered measures has the properties?
- Is it possible to generalize a very well-known and popular standard deviation (variance)?
- What is the dual expression of measures with these properties?

Multiobjective optimization:

- How to formulate an optimization problem when multipple objective are considered?
- What are the best solutions of such problems?
- How to find all these best solutions?

Coherent risk measures

CRM: $\mathcal{R} : \mathcal{L}_2(\Omega) \to (-\infty, \infty]$ that satisfies

- (R1) Translation equivariance: $\mathcal{R}(L + C) = \mathcal{R}(L) + C$ for all X and constants C,
- (R2) Positive homogeneity: $\mathcal{R}(0) = 0$, and $\mathcal{R}(\lambda L) = \lambda \mathcal{R}(L)$ for all L and all $\lambda > 0$,
- (R3) Subaditivity: $\mathcal{R}(L + M) \leq \mathcal{R}(L) + \mathcal{R}(M)$ for all L and M,
- (R4) Monotonicity: $\mathcal{R}(L) \geq \mathcal{R}(M)$ when $L \geq M$.

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Strictly expectation bounded risk measures satisfy (R1), (R2), (R3), and

(R5) $\mathcal{R}(L) > \mathbb{E}[L]$ for all nonconstant L, whereas $\mathcal{R}(L) = \mathbb{E}[L]$ for constant L.

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Note: (R2)&(R3) implies convexity of $\mathcal{R}$: for each $a \in [0, 1]$ we have:

$$\mathcal{R}(aL+(1-a)M) \leq \mathcal{R}(aL)+\mathcal{R}((1-a)M) = a\mathcal{R}(L)+(1-a)\mathcal{R}(M)$$

Other classes of risk measures and functionals: Follmer and Schied (2002), Pflug and Romisch (2007).

Coherent properties for the popular risk mesures

- variance: none of (R1)-(R5)
- standard deviation: (R2)
- semivariance: none of (R1)-(R5)
- mean absolute deviation: (R2), (R3)
- mean absolute semideviation: (R2), (R3)
- Value at Risk (VaR): (R1),(R2), (R4)
- Conditional Value at Risk (CVaR): (R1)-(R5)

Dual representation of coherent risk measures

Consider a measurable space $(\Omega, \mathcal{A})$ and the set $\mathcal{P}$ of all probability measures on the space.

Definition

A set $\mathcal{Q} \subset \mathcal{P}$ is called a risk envelope if for each $Q \in \mathcal{Q}$ one has: $Q \geq 0$ and $\mathbb{E}Q = 1$.

Theorem

$\mathcal{R}$ is a coherent risk measure if and only if there exists a risk envelope $\mathcal{Q}$ such that:

$$\mathcal{R}(L) = \max_{Q \in \mathcal{Q}} E(QL)$$

and $\mathcal{Q}$ can be chosen as a convex set.

Interpretation: A coherent risk measure can be understood as a worst-case expectation with respect to some class of probability distributions on $(\Omega, \mathcal{A})$, It means for some distribution P'. If the probability distribution of L is P then $Q = \frac{dP'}{dP}$.

Example - Risk envelope for CVaR

To simplify the situation consider a measurable space with M atoms (discrete distributions). Moreover let L has a discrete uniform distribution on the space - atoms are equiprobable, i.e. discrete distribution with M equiprobable scenarios l_j, $j = 1, 2, ..., M$. Assume that $M(1-\alpha)$ is an integer number. Then:

$$CVaR_\alpha(L) = \min_{a,z_j} \; a + \frac{1}{(1-\alpha)\,M} \sum_{j=1}^{M} z_j$$

$$\text{s. t. } z_j \geq l_j - a, \; j = 1, .., M$$

$$z_j \geq 0, \; j = 1, .., M$$

Example - Risk envelope for CVaR

And dual problem:

$$CVaR_\alpha(L) = \max_{y_j} \sum_{j=1}^{M} y_j l_j$$

$$\text{s. t.} \sum_{j=1}^{M} y_j = 1$$

$$y_j \leq \frac{1}{(1-\alpha)M}$$

$$y_j \geq 0, j = 1, .., M$$

Note that optimal solution: $y_j^* = 0$ for $j = 1, 2, ..., M\alpha$
$y_j^* = \frac{1}{(1-\alpha)M}$ for $j = M\alpha + 1, ..., M$. Hence the risk envelope for CVaR is:

$$\mathcal{Q} = \{Q : EQ = 1, 0 \leq Q \leq \frac{1}{(1-\alpha)}\}$$

Coherent risk and return measures

A **return measure** is defined as a functional $\mathcal{E}(L) = -\mathcal{R}(L)$ for a coherent risk measure $\mathcal{R}$. It is obvious that the expectation belongs to this class.

Rockafellar, Uryasev and Zabarankin (2006A, 2006B): GDM are introduced as an extension of *standard deviation* but they need not to be symmetric with respect to *upside* $X - \mathbb{E}X$ and *downside* $\mathbb{E}X - X$ of a random variable X.

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Any functional $\mathcal{D} : \mathcal{L}_2(\Omega) \rightarrow [0, \infty]$ is called a general deviation measure if it satisfies

(D1) $\mathcal{D}(X + C) = \mathcal{D}(X)$ for all X and constants C,

(D2) $\mathcal{D}(0) = 0$, and $\mathcal{D}(\lambda X) = \lambda \mathcal{D}(X)$ for all X and all $\lambda > 0$,

(D3) $\mathcal{D}(X + Y) \leq \mathcal{D}(X) + \mathcal{D}(Y)$ for all X and Y,

(D4) $\mathcal{D}(X) \geq 0$ for all X, with $\mathcal{D}(X) > 0$ for nonconstant X.

(D2) & (D3) $\Rightarrow$ convexity

Deviation measures

- **Standard deviation**

$$\mathcal{D}(X) \quad = \quad \sigma(X) = \sqrt{\mathbb{E}\,\|X - \mathbb{E}X\|_2}$$

- **Mean absolute deviation**

$$\mathcal{D}(X) = \mathbb{E}\big[|X - \mathbb{E}X|\big].$$

- **Mean absolute lower and upper semideviation**

$$\mathcal{D}_-(X) = \mathbb{E}\big[\min(0, X - \mathbb{E}X)\big],\ \mathcal{D}_+(X) = \mathbb{E}\big[\max(0, X - \mathbb{E}X)\big].$$

- **Worst-case deviation**

$$\mathcal{D}(X) = \sup_{\omega \in \Omega} |X(\omega) - \mathbb{E}X|.$$

- See Rockafellar et al (2006 A, 2006 B) for another examples.

For any $\alpha \in (0,1)$ a finite, continuous, lower range dominated deviation measure

$$\mathcal{D}_\alpha(X) = CVaR_\alpha(X - \mathbb{E}X). \tag{1}$$

The deviation is also called **weighted mean absolute deviation from the** $(1-\alpha)$-th **quantile**, see Ogryczak, Ruszczynski (2002), because it can be expressed as

$$\mathcal{D}_\alpha(X) = \min_{\xi \in \mathbb{R}} \frac{1}{1-\alpha} \mathbb{E}[\max\{(1-\alpha)(X-\xi), \alpha(\xi - X)\}] \tag{2}$$

with the minimum attained at any $(1-\alpha)$-th quantile. In relation with CVaR minimization formula, see Pflug (2000), Rockafellar and Uryasev (2000, 2002).

According to Proposition 4 in Rockafellar et al (2006 A):

- if $\mathcal{D} = \lambda \mathcal{D}_0$ for $\lambda > 0$ and a deviation measure $\mathcal{D}_0$, then $\mathcal{D}$ is a deviation measure.
- if $\mathcal{D}_1, \dots, \mathcal{D}_K$ are deviation measures, then
  - $\mathcal{D} = \max\{\mathcal{D}_1, \dots, \mathcal{D}_K\}$ is also deviation measure.
  - $\mathcal{D} = \lambda_1 \mathcal{D}_1 + \dots + \lambda_K \mathcal{D}_K$ is also deviation measure, if $\lambda_k > 0$ and $\sum_{k=1}^{K} \lambda_k = 1$.

Rockafellar et al (2006 A, B): Duality representation using *risk envelopes* , subdifferentiability and optimality conditions.

General deviation measures

We say that general deviation measure $\mathcal{D}$ is

(LSC) **lower semicontinuous** (lsc) if all the subsets of $\mathcal{L}_2(\Omega)$ having the form $\{X : \mathcal{D}(X) \leq c\}$ for $c \in \mathbb{R}$ (level sets) are closed;

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(D5) **lower range dominated** if $\mathcal{D}(X) \leq EX - \inf_{\omega \in \Omega} X(\omega)$ for all X.

Theorem 2 in Rockafellar et al (2006 A):

Theorem

Deviation measures correspond one-to-one with strictly expectation bounded risk measures under the relations

- $\mathcal{D}(X) = \mathcal{R}(X - \mathbb{E}X)$
- $\mathcal{R}(X) = \mathbb{E}[-X] + \mathcal{D}(X)$

In this correspondence, $\mathcal{R}$ is coherent if and only if $\mathcal{D}$ is lower range dominated.

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Convergence of approximate solutions in mean-risk models

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Introduction

- Mean-risk models
  - aim to find optimal portfolio of assets
  - analytical solutions for continuous distributions
  - solutions using generated scenarios
    - predetermined, for instance by historical data
    - generated with the assumption of continuous distribution
    - generated according to few moment estimators
- comparison of the approaches mentioned above
  - convergence and its properties
  - different continuous distributions
  - different risk measures
- computational part
  - data processing and generating the scenarios
  - optimization tasks in GAMS

Efficient portfolios

- we consider portfolio based on N assets
- weights of the assets $\mathbf{w}$, $\sum_{i=1}^{N} w_i = 1$
- expected returns $u_{\mathbf{w}}$ (always using expectation)
- different risk measures $r_{\mathbf{w}}$
- minimal required returns u_e

Definition

Portfolio of given N assets with weights $\mathbf{w}$ is (mean-risk) efficient, if there are no other weights $w_1^*, .., w_N^*$ such that $\sum_{i=1}^{N} w_i^* = 1$ and $u_{\mathbf{w}^*} \geq u_{\mathbf{w}}$ a $r_{\mathbf{w}^*} \leq r_{\mathbf{w}}$.

Classical optimization task

Efficient portfolios can be obtained while solving following task:

$$\min_{\mathbf{w}} \; r_{\mathbf{w}}$$
$$\text{s. t.} \; u_{\mathbf{w}} \geq u_e$$
$$\sum_{i=1}^{N} w_i = 1$$
$$w_i \in \mathbb{R}, \; i = 1, .., N.$$

Non-negativity condition:

$$w_1, .., w_N \geq 0.$$

Risk measures

- variance
- VaR
- cVaR
- absolute deviation
- semivariance

Definition

Let $\alpha \in (0,1)$ be the threshold and L random variable which represents the loss from holding the portfolio. Then we define VaR_α as:

$$VaR_\alpha(L) = \inf \{ l \in \mathbb{R}, \mathrm{P}(L > l) \leq 1 - \alpha \}$$

Risk measures

Definition

$cVaR_\alpha$ is defined as:

$$cVaR_\alpha(L) = \inf\left\{a \in \mathbb{R}, a + \frac{1}{1-\alpha}\mathrm{E}\left[\max\left(0, L - a\right)\right]\right\}.$$

Absolute deviation can be calculated as:

$$r_a(L) = \mathrm{E}|L - \mathrm{E}L|.$$

Semivariance can be calculated as::

$$r_s(L) = \mathrm{E}\left[\max\left(0, L - \mathrm{E}L\right)^2\right]$$

Elliptical distributions

- generalization of normal distribution
- include normal distribution, Student distribution, logistic elliptical distribution and others
- symmetrical around the mean
- simple analysis of linear combinations

Theorem

Let $\mathbf{X} \sim \mathbf{E}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \psi\right)$, $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^{m}$. *Then it holds:*

$$\mathbf{AX} + \mathbf{b} \sim \mathbf{E}\left(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^{T}, \psi\right).$$

Variance

We get classical optimization task which can be used for all elliptical distributions:

$$\min_{\mathbf{w}} \mathbf{w}^T \mathbf{V} \mathbf{w}$$

$$\text{s. t. } \mathbf{w}^T \boldsymbol{\mu} \geq u_e$$

$$\sum_{i=1}^{N} w_i = 1$$

$$w_i \in \mathbb{R},\ i = 1, .., N$$

Normal distribution

- VaR

$$VaR_\alpha(L) = -\mathbf{w}^T\boldsymbol{\mu} + q_\alpha\sqrt{\mathbf{w}^T\mathbf{V}\mathbf{w}}$$

- cVaR

$$cVaR_\alpha(L) = -\mathbf{w}^T\boldsymbol{\mu} + \frac{\exp\left\{-\frac{q_\alpha^2}{2}\right\}}{(1-\alpha)\sqrt{2\pi}}\sqrt{\mathbf{w}^T\mathbf{V}\mathbf{w}}$$

- absolute deviation

$$r_a(L) = \sqrt{\frac{2}{\pi}}\sqrt{\mathbf{w}^T\mathbf{V}\mathbf{w}}$$

- semivariance

$$r_s(L) = \frac{1}{2}\mathrm{E}\left[(L-\mathrm{E}L)^2\right]$$

Student distribution

- VaR

$$VaR_{\alpha}(L) = VaR_{\alpha}(L) = -\mathbf{w}^T\boldsymbol{\mu} + t_{\alpha,\nu}\sqrt{\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}}$$

- cVaR

$$cVaR_{\alpha}(L) = -\mathbf{w}^T\boldsymbol{\mu} + \frac{\Gamma\left(\frac{\nu-1}{2}\right)\sqrt{\nu}\left(1+\frac{t_{\alpha,\nu}^2}{\nu}\right)^{-\frac{\nu-1}{2}}}{\Gamma\left(\frac{\nu-2}{2}\right)(1-\alpha)(\nu-2)\sqrt{\pi}}\sqrt{\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}}$$

- absolute deviation

$$r_a(L) = \frac{2\sqrt{\nu}\Gamma\left(\frac{\nu+1}{2}\right)}{(\nu-1)\sqrt{\pi}\Gamma\left(\frac{\nu}{2}\right)}\sqrt{\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}}$$

- semivariance

$$r_s(L) = \frac{1}{2}\mathsf{E}\left[(L-\mathsf{E}L)^2\right]$$

Variance - scenarios

- suppose we have M scenarios of possible stock prices
- we can use mean and variance-covariance estimators $\hat{\mathbf{l}}$ and $\hat{\mathbf{V}}$ to minimize variance
- allows us to process estimators before running the optimization task and therefore is quick

$$
\begin{aligned}
\min_{\mathbf{w}} \ & \mathbf{w}^T \hat{\mathbf{V}} \mathbf{w} \\
\text{s. t. } & \mathbf{w}^T \hat{\mathbf{l}} \geq u_e \\
& \sum_{i=1}^{N} w_i = 1 \\
& w_i \in \mathbb{R}, \ i = 1, .., N.
\end{aligned}
$$

VaR - scenarios

- general case could be nonconvex
- we reformulate the task using integer programming
- still hardly computable - M binary variables

$$\min_{\nu,\mathbf{w},\delta^j} \ \nu$$

$$\text{s. t.} \ -\mathbf{w}^T\mathbf{y}^j \leq \nu + K\delta^j, \ j = 1, .., M$$

$$\sum_{j=1}^{M} \delta^j = \lfloor (1-\alpha) M \rfloor$$

$$\delta^j \in \{0, 1\}, \ j = 1, .., M$$

$$\frac{1}{M}\sum_{j=1}^{M} \mathbf{w}^T\mathbf{y}^j \geq u_e \ldots$$

cVaR - scenarios

- linear programming task, can be solved quickly

$$\min_{a,\mathbf{w},z^j} \; a + \frac{1}{(1-\alpha)\,M} \sum_{j=1}^{M} z^j$$

$$\text{s. t. } z^j \geq -\mathbf{w}^T \mathbf{r}^j - a,\; j = 1, .., M$$

$$z^j \geq 0,\; j = 1, .., M$$

$$\frac{1}{M} \sum_{j=1}^{M} \mathbf{w}^T \mathbf{r}^j \geq u_e$$

$$\sum_{i=1}^{N} w_i = 1$$

$$w_i \in \mathbb{R},\; i = 1, .., N.$$

Absolute deviation - scenarios

- linear programming task

$$\min_{\mathbf{w},z^j} \frac{1}{M}\sum_{j=1}^{M} z^j$$

$$\text{s. t. } \mathbf{w}^T\mathbf{l}^j - \frac{1}{M}\sum_{i=1}^{M}\mathbf{w}^T\mathbf{l}^i \leq z^j,\ j=1,..,M$$

$$-\mathbf{w}^T\mathbf{l}^j + \frac{1}{M}\sum_{i=1}^{M}\mathbf{w}^T\mathbf{l}^i \leq z^j,\ j=1,..,M$$

$$\frac{1}{M}\sum_{j=1}^{M}\mathbf{w}^T\mathbf{l}^j \geq u_e$$

...

Semivariance - scenarios

- quadratic programming task

$$\min_{\mathbf{w}, z^j} \frac{1}{M} \sum_{j=1}^{M} \left(z^j\right)^2$$

$$\text{s. t. } z^j \geq -\mathbf{w}^T \mathbf{l}^j + \frac{1}{M} \sum_{i=1}^{M} \mathbf{w}^T \mathbf{l}^i,\ j = 1, .., M$$

$$z^j \geq 0,\ j = 1, .., M$$

$$\frac{1}{M} \sum_{j=1}^{M} \mathbf{w}^T \mathbf{l}^j \geq u_e$$

...

Conclusion

Thank you for your attention!

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