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UNIFORM VALIDITY OF DISCRETE FRIEDRICHS' INEQUALITY FOR GENERAL NONCONFORMING FINITE ELEMENT SPACES

Petr Knobloch

Faculty of Mathematics and Physics, Charles University,
Sokolovská 83, 186 75 Praha 8, Czech Republic
E-mail: knobloch@karlin.mff.cuni.cz

ABSTRACT

We prove the uniform validity of discrete Friedrichs' inequality for general nonconforming finite element spaces defined over triangulations consisting of triangles and/or quadrilaterals. The result is valid for arbitrary polygonal domains and also for locally refined triangulations.

Key Words: Nonconforming finite element method; Discrete Friedrichs' inequality; Discrete Korn's inequality.

AMS Subject Classification: 65N30.

1. INTRODUCTION

In the theory of second order partial differential equations (cf. e.g. [26]), a significant role is played by the Friedrichs inequality (often also called Poincaré's inequality)

$$\|v\|_{0,\Omega} \leq C|v|_{1,\Omega} \quad \forall v \in H_0^1(\Omega), \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^d$, $d \geq 2$. The notation $\|\cdot\|_{0,\Omega}$ denotes the norm in the Lebesgue space $L^2(\Omega)$ and $|\cdot|_{1,\Omega}$ is the seminorm in the Sobolev space $H^1(\Omega)$. This space consists of functions from $L^2(\Omega)$ whose generalized

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first derivatives are also in $L^2(\Omega)$. The space $H_0^1(\Omega)$ is formed by functions from $H^1(\Omega)$ having zero traces on the boundary $\partial\Omega$ of the domain Ω. We refer to [1] or [23] for details on Sobolev spaces.

For problems where Dirichlet boundary conditions are only prescribed on a part Γ^D of the boundary $\partial\Omega$, the Friedrichs inequality

$$\|v\|_{0,\Omega} \leq C|v|_{1,\Omega} \quad \forall\, v \in V \tag{2}$$

with

$$V = \{v \in H^1(\Omega);\ v = 0 \text{ on } \Gamma^D\}$$

is needed. This inequality holds as long as $\text{meas}_{d-1}(\Gamma^D) \neq 0$ (cf. e.g. [23, p. 20]). Particularly, for $\Gamma^D = \partial\Omega$, the inequality (2) is identical with (1). The importance of (2) consists in the fact that it implies an equivalence in V between the $H^1(\Omega)$ norm $\|\cdot\|_{1,\Omega}$ and the seminorm $|\cdot|_{1,\Omega}$.

A powerful numerical method for solving second order partial differential equations is the finite element method (cf. e.g. [6]). An important feature of the finite element method is that it is based on week formulations of the respective equations, which enables to investigate the existence and convergence of discrete solutions employing various approaches of the functional analysis and the theory of Sobolev spaces. Thus, it is not surprising that one needs a discrete analogue of the Friedrichs inequality (2) (shortly discrete Friedrichs inequality).

If the above space V is approximated by a space V_h defined using the conforming finite element method, without committing any variational crimes in the sense of [29], then $V_h \subset V$. Therefore, in this case, we automatically get the Friedrichs inequality also for the space V_h. Often, however, some variational crimes are committed and, consequently, $V_h \not\subset V$. In this case, it is usually obvious that a discrete Friedrichs inequality holds for each space V_h with some constant C_h but it is not clear how these constants C_h depend on the discretization parameter h.

There are two types of variational crimes which can cause that $V_h \not\subset V$. The first one is an approximation of the boundary of Ω, which was thoroughly investigated in [13], [18], [31]. In these papers, it was shown that the constants C_h are bounded independently of h.

The second type of variational crimes is an approximation of V by functions having jumps across element edges, which leads to nonconforming finite element spaces. The nonconforming finite element method has been already investigated for more than three decades and during this time, many various nonconforming finite elements have been developed (cf. e.g. [5], [8], [9], [15], [16], [21], [22], [25]). Although the mentioned jumps of nonconforming finite element functions cause additional difficulties in theoretical investigations of the corresponding finite element discretizations, the application of nonconforming elements can be justified since they possess several favourable properties. First, nonconforming finite elements are more suitable for a parallel implementation than conforming elements since they lead to a cheap local communication between processors. Nowadays, with the increasing importance of parallel computers for scientific computations, this feature becomes still more and more important. Another important feature of nonconforming finite elements is that they usually fulfil an inf-sup condition so that they are very attractive for solving problems describing incompressible or nearly incompressible materials.

Like for the first type of variational crimes, it is again not obvious whether, for a given family of nonconforming finite element spaces, a discrete Friedrichs inequality holds uniformly with respect to the discretization parameter h (i.e., with a constant C independent of h). In the literature, results on the uniform validity of the discrete Friedrichs inequality for nonconforming finite element spaces are rather rare and they are mostly established under some restrictive assumptions like convexity of Ω or the validity of an inverse assumption on the triangulation (cf. [10], [14], [17], [30]). A proof for quadrilateral elements which does not use these two restrictive assumptions but is still much less general than the proof in the present paper can be found in [28]. The little attention paid to investigations of the discrete Friedrichs inequality for nonconforming finite elements is probably due to the fact that finite element discretizations can be seemingly analyzed without using a uniform discrete Friedrichs inequality. Namely, for proving the existence of a discrete solution, the h-dependent Friedrichs inequality is sufficient and the discrete Friedrichs inequality is often not needed at all for proving a convergence in the discrete analogue of the seminorm $|\cdot|_{1,\Omega}$. Moreover, convergence in the norm $\|\cdot\|_{0,\Omega}$ can often be proven by applying the Aubin–Nitsche duality argument (cf. e.g. [5], [9], [25]), again without using a discrete Friedrichs inequality. However, the regularity of the dual problem needed for applying the Aubin–Nitsche trick is often known only for special problems (e.g. convex domains, Dirichlet boundary conditions etc.) so that, in fact, an estimate in the $L^2(\Omega)$ norm is generally not available without applying the Friedrichs inequality.

The aim of the present paper is to prove that a discrete Friedrichs inequality holds uniformly with respect to the discretization parameter h for general nonconforming finite element spaces V_h approximating the space V. For simplicity we shall confine ourselves to the two-dimensional case and to triangulations consisting of triangles and/or quadrilaterals, however, it will be obvious that our considerations can be easily generalized to three dimensions. Let us emphasize that our proof is valid for general polygonal domains and for triangulations satisfying a shape regularity assumption only. Particularly, we allow locally refined meshes containing elements with arbitrarily large ratios of their diameters, which is very important for applying adaptive refinement techniques. In addition, quadrilateral elements may be non-convex and they are allowed to degenerate to triangles. Finally, let us mention that for spaces V_h consisting of functions satisfying the patch test of order 1, the present paper implies the validity of the discrete Friedrichs inequality with a constant independent of the polynomial degrees of functions from V_h. Thus, the discrete Friedrichs inequality can also be applied to nonconforming spaces constructed by means of the *hp* finite element method (cf. [27] for the conforming case).

The paper is organized in the following way. In the next section we summarize notation and all the assumptions made in this paper. In Section 3 we introduce a general set W_h containing any usual nonconforming finite element space. This generalization will facilitate the proof of the discrete Friedrichs inequality. The idea of the proof is to map the set W_h into the space V using a Clément type interpolation operator (cf. [7], [2], [3]) and then to apply the Friedrichs inequality (2). The interpolation operator is described and investigated in Section 4. Then, in Section 5, we prove that the set W_h satisfies the uniform discrete Friedrichs inequality. Finally, in Section 6, we compare the results of this paper with some recent results concerning the discrete Korn inequality.

2. NOTATION AND ASSUMPTIONS

We suppose that Ω is a bounded domain in $\mathbb{R}^2$ having a polygonal boundary $\partial\Omega$ and that $\Gamma^D \subset \partial\Omega$ is a relatively open set satisfying $\mathrm{meas}_1(\Gamma^D) \neq 0$.

We assume that we are given a family of triangulations $\mathcal{T}_h$ of the domain Ω consisting of triangular and/or quadrilateral elements K having the usual compatibility properties (see e.g. [6]) and satisfying $h_K \equiv \mathrm{diam}(K) \leq h$ for any $K \in \mathcal{T}_h$. We assume that any edge of $\mathcal{T}_h$ lying on $\partial\Omega$ belongs either to $\overline{\Gamma^D}$ or to $\partial\Omega \backslash \Gamma^D$.

For each triangulation $\mathcal{T}_h$, we introduce a triangulation $\mathcal{T}_h^*$ obtained from $\mathcal{T}_h$ by dividing each quadrilateral element of $\mathcal{T}_h$ into two triangles. This construction of $\mathcal{T}_h^*$ is not unique unless $\mathcal{T}_h$ only consists of triangles in which case we have $\mathcal{T}_h^* = \mathcal{T}_h$. Thus, we assume that, for each triangulation $\mathcal{T}_h$, one of the possible triangulations $\mathcal{T}_h^*$ has been fixed. We require that this family of triangulations $\mathcal{T}_h^*$ is regular, i.e., there exists a constant σ independent of h satisfying

$$\frac{h_K}{\varrho_K} \leq \sigma \quad \forall\, K \in \mathcal{T}_h^*, \tag{3}$$

where ϱ_K is the maximum diameter of circles inscribed into K. Note that our assumptions do not exclude the case when some quadrilateral elements of $\mathcal{T}_h$

degenerate to triangles. Also, quadrilateral elements of $\mathcal{T}_h$ are allowed to be non-convex.

We denote by $\mathcal{E}_h$ the set of the edges E of $\mathcal{T}_h$ and by $\mathcal{E}_h^i$ and $\mathcal{E}_h^D$ the subsets of $\mathcal{E}_h$ consisting of inner edges and boundary edges lying on Γ^D, respectively. For any edge E, we choose a fixed unit normal vector $\boldsymbol{n}_E$. If $E \subset \partial\Omega$, then $\boldsymbol{n}_E$ coincides with the outer normal vector to the boundary of Ω.

For any inner edge E, we define the jump of a function v across E by

$$[v]_E = (v|_K)\big|_E - \big(v|_{\tilde{K}}\big)\big|_E, \tag{4}$$

where K, $\tilde{K}$ are the two elements adjacent to E denoted in such a way that $\boldsymbol{n}_E$ points into $\tilde{K}$. If an edge $E \in \mathcal{E}_h$ lies on the boundary of Ω, then we set

$$[v]_E = v|_E.$$

Throughout the paper we use standard notation $L^2(G), H^k(G) = W^{k,2}(G), P_k(G), C(\bar{G})$, etc. for the usual function spaces defined on a set G, see e.g. [6]. The norm and seminorm in the Sobolev space $H^k(G)$ will be denoted by $\|\cdot\|_{k,G}$ and $|\cdot|_{k,G}$, respectively. In addition, we define discrete analogues of $\|\cdot\|_{1,\Omega}$ and $|\cdot|_{1,\Omega}$ by

$$\|v\|_{1,h} = \left(\sum_{K\in\mathcal{T}_h} \|v\|_{1,K}^2\right)^{1/2}, \quad |v|_{1,h} = \left(\sum_{K\in\mathcal{T}_h} |v|_{1,K}^2\right)^{1/2}.$$

Finally, for notational convenience, we set $|\cdot|_{0,h} = \|\cdot\|_{0,\Omega}$.

3. GENERALIZATION OF NONCONFORMING FINITE ELEMENT SPACES

Nonconforming finite element spaces defined over the triangulation $\mathcal{T}_h$ and approximating the space V introduced in Section 1 have typically the form

$$V_h = \left\{v \in L^2(\Omega);\ v|_K \in R(K) \quad \forall K \in \mathcal{T}_h, \varphi_E\big([v]_E\big) = 0 \quad \forall E \in \mathcal{E}_h^i \cup \mathcal{E}_h^D\right\}.$$

Here, $R(K) \subset H^1(K)$ are finite-dimensional spaces of functions defined on the elements K, and φ_E are functions which determine how strongly functions from neighbouring elements are coupled on the common edges E. The functions φ_E also determine in which sense the homogenous Dirichlet boundary condition on Γ^D is approximated.

The strongest possible coupling is the continuity requirement which can be achieved, e.g., by setting

$$\varphi_E(v) = |v|.$$

Then $V_h \subset V$, i.e., V_h is a conforming space.

For nonconforming finite element spaces, the values $\varphi_E(v)$ usually represent either integral moments of v on E or values of v at some Gaussian points on E (cf. e.g. [5], [9], [16], [21], [25]). Thus, for example, for first order spaces, we typically have

$$\varphi_E(v) = \frac{1}{\mathrm{meas}_1(E)} \int_E v \, \mathrm{d}\sigma$$

or

$$\varphi_E(v) = v(C_E),$$

where C_E is the midpoint of E. These two choices of φ_E often lead to the same finite element space (like e.g. for the linear triangular Crouzeix–Raviart element [9]), but in some cases they are not equivalent (e.g. for the rotated bilinear element [25]). Denoting by ψ_E a linear mapping which transforms the reference edge $\hat{E} \equiv [0,\ 1] \times \{0\}$ onto the edge E, the above mappings φ_E are defined by $\varphi_E(v) = \varphi(v \circ \psi_E)$ with

$$\varphi(\hat{v}) = \int_{\hat{E}} \hat{v} \, \mathrm{d}\hat{\sigma} \tag{5}$$

or

$$\varphi(\hat{v}) = \hat{v}(C_{\hat{E}}), \tag{6}$$

where $C_{\hat{E}}$ is the midpoint of $\hat{E}$ and $\hat{v}$ is a function defined on $\hat{E}$.

Let us first consider a space V_h defined using functions φ_E representing integral moments on edges E. Then, on any edge $E \in \mathcal{E}_h^i \cup \mathcal{E}_h^D$, the functions $v \in V_h$ satisfy the patch test of order 1, i.e.,

$$\int_E [v]_E \, \mathrm{d}\sigma = 0 \quad \forall E \in \mathcal{E}_h^i \cup \mathcal{E}_h^D, \quad v \in V_h. \tag{7}$$

Thus, the space V_h is a subspace of the space

$$W_h = \left\{ v \in L^2(\Omega);\ v|_K \in H^1(K) \quad \forall K \in \mathcal{T}_h, \quad \varphi\big([v]_E \circ \psi_E\big) = 0 \quad \forall E \in \mathcal{E}_h^i \cup \mathcal{E}_h^D \right\},$$

where φ is defined by (5). More generally, we can state that the space V_h is a subset of the above set W_h defined using a suitable function φ possessing the following two properties:

$$\varphi \in C(L^2(\hat{E}), \mathbb{R}), \tag{8}$$

$$\forall\, \hat{v} \in P_0(\hat{E}): \quad \varphi(\hat{v}) = 0 \quad \Leftrightarrow \quad \hat{v} = 0. \tag{9}$$

The first property means that

$$\hat{v}_n \to \hat{v} \quad \text{in } L^2(\hat{E}) \Rightarrow \varphi(\hat{v}_n) \to \varphi(\hat{v}) \quad \text{in } \mathbb{R}. \tag{10}$$

Note that the properties (8) and (9) do not guarantee that W_h has a linear structure. Nevertheless, they are sufficient for proving that W_h satisfies the discrete Friedrichs inequality

$$\|v\|_{0,\Omega} \le C|v|_{1,h} \quad \forall\, v \in W_h \tag{11}$$

with a constant C depending only on Ω, Γ^D, φ and σ from (3). This result, which will be proven in Section 5, implies that any nonconforming finite element space V_h satisfying (7) (and hence contained in W_h defined using φ from (5)) fulfils the discrete Friedrichs inequality with a constant independent of both h and the polynomial degrees of functions from V_h.

However, as we mentioned above, the functions φ_E in the definition of V_h may also represent values of their arguments at some points on the edges E, which does not necessarily imply the validity of (7). Nevertheless, the space V_h is still contained in W_h for some suitable function φ satisfying (8) and (9).

To see this, let us for simplicity consider a space V_h consisting of functions continuous in the midpoints of edges, i.e., φ is given by (6). Further, for simplicity, we assume that the elements of $\mathcal{T}_h$ are of the same type (i.e., either triangles or quadrilaterals). The spaces $R(K)$ in the definition of V_h are typically constructed by specifying a finite–dimensional space $R(\hat{K}) \subset C(\hat{K}) \cap H^1(\hat{K})$ of (usually polynomial) functions defined on a reference element $\hat{K}$ and by transforming functions from $R(\hat{K})$ into functions defined on the elements K. The transformations are performed via regular linear or bilinear mappings which map $\hat{K}$ onto K. Since these mappings are linear along the edges of $\hat{K}$, we deduce that there exists a finite–dimensional space $S(\hat{E}) \subset C(\hat{E})$ such that $P_0(\hat{E}) \subset S(\hat{E})$ and $[v]_E \circ \psi_E \in S(\hat{E})$ for any $v \in V_h$ and any $E \in \mathcal{E}_h$. Denoting by P the orthogonal projection of $L^2(\hat{E})$ onto $S(\hat{E})$ and replacing the function φ by $\varphi \circ P$, we obtain a function satisfying (8) and (9) and leading to a set W_h containing the space V_h.

Analogously we can proceed if $\varphi_E(v)$ represents values of v at other points on E or if $\mathcal{T}_h$ contains both triangles and quadrilaterals. Moreover, we can also treat cases when various coupling conditions are used to define the space V_h. For example, if on some edges the coupling is given by φ from (5) and on other edges by φ from (6), we can set

$$\varphi(\hat{v}) = \hat{v}\big(C_{\hat{E}}\big) \int_{\hat{E}} \hat{v}\, \mathrm{d}\hat{\sigma}$$

and extend this function to $L^2(\hat{E})$ as above. Then the corresponding set W_h again contains the respective space V_h.

However, in all the above cases where the function φ has to be extended to $L^2(\hat{E})$, the resulting set W_h depends on the polynomial degrees of functions from V_h and hence we derive discrete Friedrichs inequalities which are uniform in h but not necessarily in the polynomial degrees of functions from V_h.

Thus, we can conclude that, for any usual nonconforming finite element space V_h, there exists a function φ independent of h and satisfying (8) and (9) such that the corresponding set W_h contains the space V_h. Therefore, to derive a discrete Friedrichs inequality for V_h, it suffices to investigate the validity of (11). For this, we shall first introduce a suitable interpolation operator which maps W_h into V.

4. AN INTERPOLATION OPERATOR

Usual interpolation operators, like e.g. Lagrange's interpolation operators, require some smoothness of the interpolated functions, e.g. the continuity. However, functions from the set W_h have jumps across interelement edges and are only of class H^1 in the interiors of elements. Thus, we need an interpolation operator defined for functions having a very low regularity only.

Such an interpolation operator was introduced by Clément [7] and the basic idea was to replace the point values used in the definition of a Lagrange interpolation operator by values obtained by projecting the interpolated function into spaces of polynomials on macroelements. This technique was further developed in [2], where the projections were constructed on reference macroelements, and also in [3], where piecewise polynomial functions on reference macroelements were considered. Further modifications of the Clément operator can be found e.g. in [4].

Here we shall consider the interpolation operator of [3] (however, those ones of [7] and [2] could also be used). This operator is defined for functions from $L^1(\Omega)$ and has usual optimal approximation properties if the interpolated functions are smooth enough. We shall present a particular case of this operator for which the interpolates are piecewise linear. This will serve our purpose.

We consider any of the triangulations $\mathcal{T}_h^*$ introduced in Section 2 and denote by $a_1, \ldots, a_{N_h}$ the vertices of the elements of this triangulation. We assume that vertices lying on $\overline{\Gamma^D}$ are numbered first, i.e., denoting by N_h^D the number of vertices on $\overline{\Gamma^D}$, we have $a_1, \ldots, a_{N_h^D} \in \overline{\Gamma^D}$ and $a_{N_h^D+1}, \ldots, a_{N_h} \notin \overline{\Gamma^D}$. For any vertex a_i, we denote by Δ_i the macroelement consisting of elements of $\mathcal{T}_h^*$ that share this vertex a_i (cf. Fig. 1). It follows from (3) that the number of elements contained in any macroelement is bounded by a constant depending only on σ from (3). Therefore, we can introduce reference macroelements $\hat{\Delta}_1, \ldots, \hat{\Delta}_L$ (with L depending on σ only) such that any macroelement Δ_i is the image of some reference macroelement $\hat{\Delta}_j$ under a continuous and invertible mapping F_i which is affine on the triangles $\hat{K}$ making up $\hat{\Delta}_j$ (cf. Fig. 2). For definiteness, we assume that $a_i = F_i(0)$ and that each reference macroelement $\hat{\Delta}_j$ contains the point $(1, 0)$ and consists of equal isosceles triangles whose vertices different from 0 lie on the unit circle with the centre at 0. This does not determine $\hat{\Delta}_j$ uniquely if $a_i \in \partial\Omega$. Therefore, in this case, we further assume that $\hat{\Delta}_j \subset [0, 1] \times [0, 1]$ and that the point (0, 1) lies in $\hat{\Delta}_j$.

For any $j \in \{1, \ldots, L\}$, we denote

$$\Theta(\hat{\Delta}_j) = \{\hat{v} \in C(\hat{\Delta}_j);\ \hat{v}|_{\hat{K}} \in P_1(\hat{K}) \quad \forall \hat{K} \subset \hat{\Delta}_j\}$$

and we define a projection operator $\hat{r}_j \in \mathcal{L}(L^1(\hat{\Delta}_j), \Theta(\hat{\Delta}_j))$ by

$$\int_{\hat{\Delta}_j} (\hat{v} - \hat{r}_j \hat{v})\hat{\theta}\, \mathrm{d}\hat{x} = 0 \quad \forall\, \hat{v} \in L^1(\hat{\Delta}_j),\, \hat{\theta} \in \Theta(\hat{\Delta}_j).$$

Further, if $i \in \{1, \ldots, N_h\}$ and $\Delta_i = F_i(\hat{\Delta}_j)$ for some $j \in \{1, \ldots, L\}$, then we set

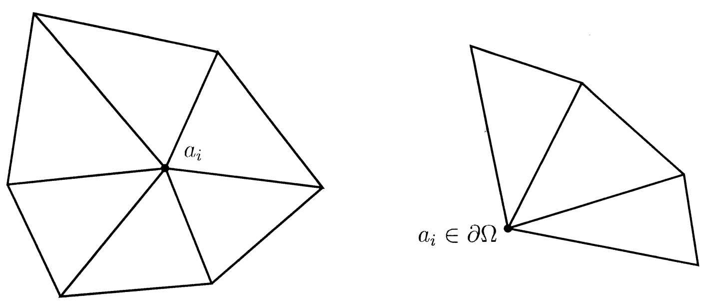

Figure 1. Examples of macroelements Δ_i.

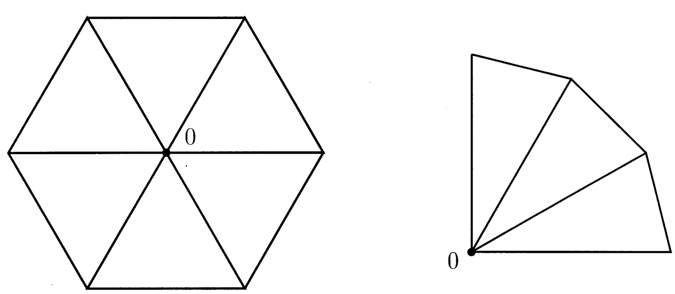

Figure 2. Reference macroelements $\hat{\Delta}_j$ corresponding to the macroelements from Fig. 1.

$$r_i v = \hat{r}_j(v \circ F_i) \circ F_i^{-1} \quad \forall\, v \in L^1(\Delta_i).$$

This gives an operator $r_i \in \mathcal{L}\big(L^1(\Delta_i), \Theta(\Delta_i)\big)$, where

$$\Theta(\Delta_i) = \big\{v \in C(\Delta_i); \quad v|_K \in P_1(K) \quad \forall\, K \subset \Delta_i\big\}.$$

Now, let

$$\Theta_h = \big\{v \in C(\bar{\Omega}); \quad v|_K \in P_1(K) \quad \forall\, K \in \mathcal{T}_h^*\big\}$$

and let $\varphi_1, \ldots, \varphi_{N_h} \in \Theta_h$ be the usual basis functions, i.e., $\varphi_k(a_k) = 1$, $\varphi_k(a_l) = 0$ for any k, $l \in \big\{1, \ldots, N_h\big\}$, $k \neq l$. Setting

$$R_h v = \sum_{i=N_h^D+1}^{N_h} (r_i v)(a_i)\varphi_i \quad \forall\, v \in L^1(\Omega),$$

we get an interpolation operator $R_h \in \mathcal{L}\big(L^1(\Omega),\ \Theta_h \cap V\big)$. It was shown in [3] that, for $v \in V \cap H^2(\Omega)$, the operator R_h has the same approximation properties as the corresponding Lagrange interpolation operator (for which $(r_i v)(a_i) = v(a_i)$). Moreover, it was shown there that

$$\big|v - R_h v\big|_{m,\Omega} \leq C\, h^{1-m} \big|v\big|_{1,\Omega} \quad \forall\, v \in V, \quad m = 0,\ 1,$$

where C only depends on σ. Our aim is to show that this estimate also holds for functions from the set W_h introduced in the preceding section.

First, however, let us consider the case when no conditions are imposed on the boundary of Ω. Then, instead of W_h, we use the set

$$\bar{W}_h = \big\{v \in L^2(\Omega);\, v|_K \in H^1(K) \quad \forall K \in \mathcal{T}_h,\, \varphi\big([v]_E \circ \psi_E\big) = 0 \quad \forall E \in \mathcal{E}_h^i\big\} \tag{12}$$

and an appropriate interpolation operator is defined by

$$\bar{R}_h v = \sum_{i=1}^{N_h} (r_i v)(a_i)\varphi_i \quad \forall\, v \in L^1(\Omega).$$

Clearly, the operator $\bar{R}_h$ is a continuous linear mapping from $L^1(\Omega)$ into Θ_h. To investigate its approximation properties, we shall need the following lemma.

Lemma 1. *Let $\hat{\Delta}_j$ be any of the above-introduced reference macroelements and let $\hat{E}_1, \ldots, \hat{E}_n$ be edges of elements $\hat{K} \subset \hat{\Delta}_j$ which do not lie on the boundary of $\hat{\Delta}_j$. For any $i = 1, \ldots, n$,* let *$\psi_{\hat{E}_i}$ be a linear transformation of the reference edge $\hat{E}$ onto $\hat{E}_i$ and let $[\cdot]_{\hat{E}_i}$ be a jump across $\hat{E}_i$ defined* *analogously as in (4). Finally, let*

$$W(\hat{\Delta}_j) = \Big\{ \hat{v} \in L^2(\hat{\Delta}_j);\, \hat{v}|_{\hat{K}} \in H^1(\hat{K}) \quad \forall\, \hat{K} \subset \hat{\Delta}_j, \\ \varphi([\hat{v}]_{\hat{E}_i} \circ \psi_{\hat{E}_i}) = 0 \quad \forall i = 1, \dots, n \Big\},$$

where φ is any function satisfying (8) and (9). Then there exists a constant C such that

$$\inf_{\hat{q} \in P_0(\hat{\Delta}_j)} \|\hat{v} + \hat{q}\|_{1,\hat{\Delta}_j,*} \le C |\hat{v}|_{1,\hat{\Delta}_j,*} \quad \forall\, \hat{v} \in W(\hat{\Delta}_j), \tag{13}$$

where

$$\|\hat{v}\|_{1,\hat{\Delta}_j,*} = \left(\sum_{\hat{K} \subset \hat{\Delta}_j} \|\hat{v}\|^2_{1,\hat{K}} \right)^{1/2}, \quad |\hat{v}|_{1,\hat{\Delta}_j,*} = \left(\sum_{\hat{K} \subset \hat{\Delta}_j} |\hat{v}|^2_{1,\hat{K}} \right)^{1/2}.$$

Proof. We shall prove by contradiction that there exists a constant C such that

$$\|\hat{v}\|_{1,\hat{\Delta}_j,*} \le C \left(|\hat{v}|_{1,\hat{\Delta}_j,*} + \left| \int_{\hat{\Delta}_j} \hat{v}\, \mathrm{d}\hat{x} \right| \right) \quad \forall\, \hat{v} \in W(\hat{\Delta}_j). \tag{14}$$

Then, for any $\hat{v} \in W(\hat{\Delta}_j)$ and for $\hat{q} = -\int_{\hat{\Delta}_j} \hat{v}\, \mathrm{d}\hat{x} / \mathrm{meas}_2(\hat{\Delta}_j)$, we have

$$\|\hat{v} + \hat{q}\|_{1,\hat{\Delta}_j,*} \le C |\hat{v}|_{1,\hat{\Delta}_j,*},$$

which gives (13). The proof will be similar as in [6, p. 120], where (13) was proved for $\hat{v} \in H^1(\hat{\Delta}_j)$.

Let us assume that (14) does not hold. Then there exists a sequence $\{\hat{v}_k\}_{k=1}^\infty \subset W(\hat{\Delta}_j)$ such that

$$\|\hat{v}_k\|_{1,\hat{\Delta}_j,*} = 1, \quad |\hat{v}_k|_{1,\hat{\Delta}_j,*} + \left| \int_{\hat{\Delta}_j} \hat{v}_k\, \mathrm{d}\hat{x} \right| < \frac{1}{k} \quad \forall\, k \ge 1. \tag{15}$$

Let us consider any element $\hat{K} \subset \hat{\Delta}_j$. Then $\{\hat{v}_k\}_{k=1}^\infty$ is bounded in $\|\cdot\|_{1,\hat{K}}$ and hence it contains a subsequence which converges weakly in $H^1(\hat{K})$ and, consequently, strongly in $L^2(\hat{K})$ due to Rellich's theorem (cf. e.g. [23, p. 17]). In view of (15), $\{\hat{v}_k\}_{k=1}^\infty$ is a Cauchy sequence with respect to the seminorm $|\cdot|_{1,\hat{K}}$ and since $H^1(\hat{K})$ is complete, we finally deduce that the mentioned subsequence converges strongly in $H^1(\hat{K})$.

Thus, we see that there exists a subsequence of $\{\hat{v}_k\}_{k=1}^\infty$ which we again denote by $\{\hat{v}_k\}_{k=1}^\infty$ such that

$$\lim_{k\to\infty} \|\hat{v} - \hat{v}_k\|_{1,\hat{\Delta}_j,*} = 0,$$

where $\hat{v} \in L^2(\hat{\Delta}_j)$ is a function satisfying $\hat{v}|_{\hat{K}} \in H^1(\hat{K})$ for any $\hat{K} \subset \hat{\Delta}_j$. For any edge $\hat{E}_i$, $i = 1, \ldots, n$, it follows from trace theorems that

$$\|[\hat{v}]_{\hat{E}_i} - [\hat{v}_k]_{\hat{E}_i}\|_{0,\hat{E}_i} \le C\|\hat{v} - \hat{v}_k\|_{1,\hat{\Delta}_j,*}.$$

Since $\varphi([\hat{v}_k]_{\hat{E}_i} \circ \psi_{\hat{E}_i}) = 0$ for any $k \ge 1$, we deduce from (10) that

$$\varphi\Big([\hat{v}]_{\hat{E}_i} \circ \psi_{\hat{E}_i}\Big) = 0, \quad i = 1, \ldots, n. \tag{16}$$

From (15), we readily get

$$\|\hat{v}\|_{0,\hat{\Delta}_j} = 1, \quad |\hat{v}|_{1,\hat{\Delta}_j,*} = 0, \quad \int_{\hat{\Delta}_j} \hat{v}\,\mathrm{d}\hat{x} = 0, \tag{17}$$

which implies that $\hat{v}|_{\hat{K}} \in P_0(\hat{K})$ for any $\hat{K} \subset \hat{\Delta}_j$. Thus, it follows from (16) and (9) that $\hat{v} \in P_0(\hat{\Delta}_j)$, which is in contradiction with (17). □

Now we can establish error estimates for the operators r_i.

Theorem 1. *Let φ be any function satisfying (8) and (9) and let $\bar{W}_h$ be the corresponding set introduced in (12). Then there exists a constant C depending only on φ and σ from (3) such that, for any $i \in \{1, \ldots, N_h\}$, and any element $K \subset \Delta_i$, the operator r_i satisfies*

$$|v - r_i v|_{m,K} \le C h_K^{1-m} \left(\sum_{\tilde{K} \subset \Delta_i} |v|^2_{1,\tilde{K}} \right)^{1/2} \quad \forall\ v \in \bar{W}_h,\ m = 0, 1.$$

Proof. Let us consider any $i \in \{1, \ldots, N_h\}$, $K \subset \Delta_i$ and $v \in \bar{W}_h$ and let us set $\hat{v} = v \circ F_i$. Let $j \in \{1, \ldots, L\}$ be such that $\Delta_i = F_i(\hat{\Delta}_j)$. It was shown in [3, pp. 1898–1900] that

$$|v - r_i v|_{m,K} \le C h_K^{1-m} \inf_{\hat{q} \in P_0(\hat{\Delta}_j)} \|\hat{v} + \hat{q}\|_{1,\hat{\Delta}_j,*}, \quad m = 0, 1,$$

where C depends only on σ. Defining the linear transformations $\psi_{\hat{E}_i}$ and the jumps $[\cdot]_{\hat{E}_i}$ in the definition of $W(\hat{\Delta}_j)$ from Lemma 1 in a suitable way, we have $\hat{v} \in W(\hat{\Delta}_j)$. Thus, applying (13), we get

$$|v - r_i v|_{m,K} \le C h_K^{1-m} |\hat{v}|_{1,\hat{\Delta}_j,*}, \quad m = 0, 1. \tag{18}$$

Clearly, for each edge $\hat{E}_i$, there are only two possibilities for defining $\psi_{\hat{E}_i}$ and two possibilities for defining $[\cdot]_{\hat{E}_i}$. Since the number of edges $\hat{E}_i$ inside $\hat{\Delta}_j$ is bounded by a constant depending only on σ, we infer that, for any space $W(\hat{\Delta}_j)$, the constant C in (13) can be bounded by a constant depending only on φ and σ. Therefore, (18) also holds with a constant C depending only on φ and σ. According to [6, Section 15],

$$|\hat{v}|_{1,\hat{K}} \leq C|v|_{1,F_i(\hat{K})} \quad \forall\, \hat{K} \subset \hat{\Delta}_j$$

with C depending only on σ and the theorem follows. □

Using Theorem 1, one can prove the following approximation properties of the operator $\bar{R}_h$:

Theorem 2. *Let φ be any function satisfying (8) and (9) and let $\bar{W}_h$ be the corresponding set introduced in (12). Then there exists a constant C depending only on φ and σ from (3) such that*

$$|v - \bar{R}_h v|_{m,h} \leq Ch^{1-m}|v|_{1,h} \quad \forall\, v \in \bar{W}_h,\;\; m = 0, 1. \tag{19}$$

Moreover,

$$\|v - \bar{R}_h v\|_{0,\partial\Omega} \leq Ch^{1/2}|v|_{1,h} \quad \forall\, v \in \bar{W}_h. \tag{20}$$

Proof. The first step is to show that there exists a constant C depending only on φ and σ such that, for any element $K \in \mathcal{T}_h^*$,

$$|v - \bar{R}_h v|_{m,K} \leq Ch_K^{1-m}\left(\sum_{\substack{\tilde{K} \in \mathcal{T}_h^*,\\ \tilde{K} \cap K \neq \emptyset}} |v|_{1,\tilde{K}}^2\right)^{1/2} \quad \forall\, v \in \bar{W}_h,\;\; m = 0, 1. \tag{21}$$

The proof is the same as in [3, pp. 1909–1910] where the estimate (21) was proved for $v \in H^1(\Omega)$ (it suffices to use Theorem 1 instead of an analogous result for $v \in H^1(\Omega)$). Since the number of elements $\tilde{K}$ intersecting any element K is bounded by a constant depending only on σ, we obtain (19).

Let us mention how to prove (20). Let $E \subset \partial\Omega$ be any boundary edge and let $K \in \mathcal{T}_h^*$ be the adjacent element. We denote by $\hat{K}$ the standard reference triangle and by F_K the regular affine mapping which maps $\hat{K}$ onto K (cf. [6]). Finally, we consider any $v \in \bar{W}_h$ and set $\hat{v} = v \circ F_K$. Then, using a standard scaling argument and trace theorems, we get

$$\|v\|_{0,E} \leq h_K^{1/2}\|\hat{v}\|_{0,\partial\hat{K}} \leq Ch_K^{1/2}\|\hat{v}\|_{1,\hat{K}}. \tag{22}$$

This and results of [6, Section 15] imply that

$$\|v\|_{0,E} \leq C\left(h_K^{-1/2}\|v\|_{0,K} + h_K^{1/2}|v|_{1,K}\right),$$

where C depends only on σ. Replacing v by $v - \bar{R}_h v$, applying (21) and summing up the squares of the inequalities over all boundary edges, we derive (20). □

Before formulating approximation properties of the operator R_h, we first establish a simple auxiliary result.

Lemma 2. *Let $\hat{K}$ be the standard reference triangle and let $\hat{E}_1$ be any of its edges. Let $\psi_{\hat{E}_1}$ be a linear transformation of the reference edge $\hat{E}$ onto $\hat{E}_1$ and let φ be any function satisfying (8) and (9). Finally, let*

$$H(\hat{K}) = \left\{\hat{v} \in H^1(\hat{K});\ \varphi(\hat{v} \circ \psi_{\hat{E}_1}) = 0\right\}.$$

Then there exists a constant C such that

$$\|\hat{v}\|_{1,\hat{K}} \leq C|\hat{v}|_{1,\hat{K}} \quad \forall\ \hat{v} \in H(\hat{K}). \tag{23}$$

Proof. Let us assume that (23) does not hold. Then there exists a sequence $\left\{\hat{v}_k\right\}_{k=1}^{\infty} \subset H(\hat{K})$ such that

$$\|\hat{v}_k\|_{1,\hat{K}} = 1, \quad |\hat{v}_k|_{1,\hat{K}} < \frac{1}{k} \quad \forall\ k \geq 1. \tag{24}$$

In the same way as in the proof of Lemma 1, we deduce that this sequence contains a subsequence, again denoted by $\left\{\hat{v}_k\right\}_{k=1}^{\infty}$, such that

$$\lim_{k\to\infty} \|\hat{v} - \hat{v}_k\|_{1,\hat{K}} = 0$$

for some $\hat{v} \in H(\hat{K})$. Obviously, $|\hat{v}|_{1,\hat{K}} = 0$ and hence $\hat{v} \in P_0(\hat{K})$. Using (9), we infer that $\hat{v} = 0$, which contradicts the first part of (24). □

Now we are in a position to prove the desired approximation properties of the operator R_h.

Theorem 3. *Let φ be any function satisfying (8) and (9) and let W_h be the corresponding set from Section 3. Then there exists a constant C depending only on φ and σ from (3) such that*

$$|v - R_h v|_{m,h} \leq Ch^{1-m}|v|_{1,h} \quad \forall\ v \in W_h,\ m = 0, 1. \tag{25}$$

Proof. In view of Theorem 2, it suffices to establish an estimate of $\left|\bar{R}_h v - R_h v\right|_{m,h}$. Consider any $v \in W_h,\ K \in \mathcal{T}_h^*$ and $m \in \{0,\ 1\}$. Then

$$|\bar{R}_h v - R_h v|_{m,K} \leq \sum_{\substack{i=1,\\ a_i \in K}}^{N_h^D} |(r_i v)(a_i)||\varphi_i|_{m,\,K}.$$

Choose any $i \in \{1,\ \dots,\ N_h^D\}$ and let $K_1 \subset \Delta_i$ be any element of $\mathcal{T}_h^*$ adjacent to Γ^D. Let F_{K_1} be a regular affine mapping which maps the standard reference triangle $\hat{K}$ onto K_1 and set $\hat{v} = v \circ F_{K_1}$. Using the techniques of [3, p. 1912], we readily derive that

$$|(r_i v)(a_i)| \leq C\left(\sum_{\tilde{K} \subset \Delta_i} |v|_{1,\tilde{K}}^2\right)^{1/2} + C\|\hat{v}\|_{1,\hat{K}},$$

where C depends on φ and σ only. Denoting by $\hat{E}_1$ an edge of $\hat{K}$ satisfying $F_{K_1}(\hat{E}_1) \subset \Gamma^D$, there is a linear transformation $\psi_{\hat{E}_1}$ of the reference edge $\hat{E}$ onto $\hat{E}_1$ such that $\hat{v}$ belongs to the set $H(\hat{K})$ from Lemma 2. Therefore, it follows from (23) that $\|\hat{v}\|_{1,\hat{K}} \leq C|\hat{v}|_{1,\hat{K}}$ with a constant C depending only on φ. According to [6, Section 15], there exists a constant C depending only on σ such that $|\hat{v}|_{1,\hat{K}} \leq C|v|_{1,K_1}$ and $|\varphi_i|_{m,K} \leq C\,h_K^{1-m}$ for any $i \in \{1,\ \dots,\ N_h^D\}$. Hence, we deduce that

$$|\bar{R}_h v - R_h v|_{m,K} \leq Ch^{1-m}\left(\sum_{\tilde{K}\in\mathcal{T}_h^*,\ \tilde{K}\cap K\neq\emptyset} |v|_{1,\tilde{K}}^2\right)^{1/2},$$

where C depends on φ and σ only. Now, in the same way as in the proof of Theorem 2, we obtain

$$|\bar{R}_h v - R_h v|_{m,h} \leq Ch^{1-m}|v|_{1,h},$$

where C again depends on φ and σ only. Combining this inequality with (19), we obtain the theorem. □

5. PROOF OF THE DISCRETE FRIEDRICHS INEQUALITY

Using Theorem 3, it is very easy to prove the main result of this paper:

Theorem 4. *Let φ be any function satisfying (8) and (9) and let W_h be the corresponding set from Section 3. Then there exists a constant C depending only on Ω, Γ^D, φ and σ from (3) such that*

$$\|v\|_{0,\Omega} \leq C|v|_{1,h} \quad \forall\, v \in W_h. \tag{26}$$

Proof. Consider any $v \in W_h$. Then

$$\|v\|_{0,\Omega} \leq \|v - R_h v\|_{0,\Omega} + \|R_h v\|_{0,\Omega}.$$

Since $R_h v \in V$, we can apply the Friedrichs inequality (2), which gives

$$\|v\|_{0,\Omega} \leq \|v - R_h v\|_{0,\Omega} + C|R_h v|_{1,\Omega} \leq \big(1 + C)\|v - R_h v\|_{1,h} + C|v|_{1,h}.$$

Applying (25), we obtain the theorem. □

Consequences of this general discrete Friedrichs inequality were already discussed at the end of Section 3. We recall that Theorem 4 particularly implies that the discrete Friedrichs inequality holds with a constant independent of h for all usual nonconforming finite element spaces, e.g., those ones of [5], [8], [9], [16], [21], [22], and [25].

Similarly as above, we can also establish a discrete analogue of the Friedrichs inequality (cf. e.g. [23, p. 20])

$$\|v\|_{0,\Omega} \leq C(|v|_{1,\Omega} + \|v\|_{0,\Gamma^D}) \quad \forall\, v \in H^1(\Omega). \tag{27}$$

Theorem 5. *Let φ be any function satisfying (8) and (9) and let $\bar{W}_h$ be the corresponding set from Section 4. Then there exists a constant C depending only on Ω, Γ^D, φ and σ from (3) such that*

$$\|v\|_{0,\Omega} \leq C(|v|_{1,h} + \|v\|_{0,\Gamma^D}) \quad \forall\, v \in \bar{W}_h. \tag{28}$$

Proof. Consider any $v \in \bar{W}_h$. Then, using (27), we derive

$$\begin{aligned}\|v\|_{0,\Omega} &\leq \|v - \bar{R}_h v\|_{0,\Omega} + \|\bar{R}_h v\|_{0,\Omega} \leq \|v - \bar{R}_h v\|_{0,\Omega} + C(|\bar{R}_h v|_{1,\Omega} + \|\bar{R}_h v\|_{0,\Gamma^D}) \\ &\leq (1 + C)\|v - \bar{R}_h v\|_{1,h} + C\|v - \bar{R}_h v\|_{0,\Gamma^D} + C(|v|_{1,h} + \|v\|_{0,\Gamma^D}).\end{aligned}$$

Applying Theorem 2, we obtain (28). □

Theorem 5 has analogous consequences as Theorem 4. So, particularly, the inequality (28) holds with C independent of h for all usual nonconforming finite element spaces, e.g., for those ones mentioned below Theorem 4.

Remark 1. Consider any $E \in \mathcal{E}_h^D$ and let $K \in \mathcal{T}_h^*$ be the element adjacent to E. Like in the preceding section, we denote by $\hat{K}$ the standard reference triangle and by F_K the regular affine mapping which maps $\hat{K}$ onto K. For any $v \in W_h$, we set $\hat{v} = v \circ F_K$. Then it follows from (22) and (23) that

$$\|v\|_{0,E} \leq Ch^{1/2}|\hat{v}|_{1,\hat{K}} \quad \forall\ v \in W_h,$$

where C depends only on φ. According to [6, Section 15], any $v \in W_h$ satisfies $|\hat{v}|_{1,\hat{K}} \leq C|v|_{1,K}$ with C depending only on σ. Thus we deduce that

$$\|v\|_{0,\Gamma^D} \leq Ch^{1/2}|v|_{1,h} \quad \forall\ v \in W_h,$$

where C depends only on φ and σ. Therefore, (26) can also be derived as a consequence of (28).

6. REMARKS ON DISCRETE KORN'S INEQUALITY

For problems from linear elasticity or fluid mechanics with surface forces prescribed on the part $\partial\Omega\backslash\Gamma^D$ of the boundary (cf. e.g. [24] and [20], respectively), the Korn inequality (cf. [24])

$$|\boldsymbol{v}|_{1,\Omega} \leq C\|\nabla\boldsymbol{v}+\nabla\boldsymbol{v}^{\mathrm{T}}\|_{0,\Omega} \quad \forall\ \boldsymbol{v} \in \boldsymbol{V} \equiv [V]^d \tag{29}$$

has to be used in addition to the Friedrichs inequality (2). Clearly, to investigate discretizations obtained by approximating the space $\boldsymbol{V}$ by a nonconforming finite element space $\boldsymbol{V}_h$, the discrete Korn inequality

$$|\boldsymbol{v}|_{1,h}^2 \leq C \sum_{K\in\mathcal{T}_h} \|\nabla\boldsymbol{v} + \nabla\boldsymbol{v}^{\mathrm{T}}\|_{0,K}^2 \quad \forall\ \boldsymbol{v} \in \boldsymbol{V}_h \tag{30}$$

with C independent of h is needed. Since there has been a confusion in the last years concerning the validity of (30) for some low order spaces and since it is convenient to bring the inequalities (30) and (26) into a relation, we mention here some results on the validity of (30) in the two-dimensional case. We shall see that the situation is quite different from that one in the preceding sections.

It is well known since the beginning of the 90s that (30) generally does not hold for the linear triangular Crouzeix–Raviart element introduced in [9]. For this element we can even find a triangulation $\mathcal{T}_h$ such that the right-hand side of (30) vanishes for some $\boldsymbol{v} \in \boldsymbol{V}_h$ (cf. [12]). Recently, a counterexample was given in [19] which shows that a uniform validity of (30) also cannot be expected of typical nonconforming first order quadrilateral elements. So, for example, (30) does not hold uniformly for quadrilateral elements of [25] and [5] and for their various modifications which can be found in the literature.

However, under the assumptions of Section 2, it was shown in [19] that (30) holds uniformly whenever the space $\boldsymbol{V}_h$ satisfies the patch test of order 2. Similarly as in the preceding section, the validity of the inequality was established for the space

$$\boldsymbol{W}_h = \Big\{ \boldsymbol{v} \in L^2(\Omega)^2;\ \boldsymbol{v}|_K \in H^1(K)^2 \quad \forall\, K \in \mathcal{T}_h, \\ \int_E [\boldsymbol{v}]_E\, q \,\mathrm{d}\sigma = \mathbf{0} \quad \forall\, q \in P_1(E),\ \ E \in \mathcal{E}_h^i \cup \mathcal{E}_h^D \Big\}$$

which contains any nonconforming finite element space $\boldsymbol{V}_h$ defined over the triangulation $\mathcal{T}_h$, approximating the space $\boldsymbol{V}$ and satisfying the patch test of order 2. Thus, the discrete Korn inequality is also uniform with respect to the polynomial degrees of functions from $\boldsymbol{V}_h$.

The basic idea of the proof of (30) for $\boldsymbol{W}_h$ originates from [11] and completely differs from the technique used in the preceding section. The starting point is the identity

$$|\nabla \boldsymbol{v}|^2 = \frac{1}{2}(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^{\mathrm{T}}) \cdot (\nabla \boldsymbol{v} - \operatorname{curl} \boldsymbol{z}) + \nabla \boldsymbol{v} \cdot \operatorname{curl} \boldsymbol{z} + \frac{1}{2}(\operatorname{rot} \boldsymbol{v} - \operatorname{div} \boldsymbol{z}) \operatorname{rot} \boldsymbol{v}, \tag{31}$$

where

$$\operatorname{rot} \boldsymbol{v} = -\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}, \qquad \operatorname{curl} \boldsymbol{z} = \begin{pmatrix} \dfrac{\partial z_1}{\partial x_2} & -\dfrac{\partial z_1}{\partial x_1} \\ \dfrac{\partial z_2}{\partial x_2} & -\dfrac{\partial z_2}{\partial x_1} \end{pmatrix}$$

and the operator dot denotes the inner product of tensors, i.e., $A \cdot B = \sum_{i,j=1}^2 a_{ij}\, b_{ij}$ for $A = (a_{ij})_{i,j=1}^2$ and $B = (b_{ij})_{i,j=1}^2$. For any $\boldsymbol{v} \in \boldsymbol{W}_h$, the function $\boldsymbol{z} \in H^1(\Omega)^2$ in (31) is chosen in such a way that it continuously depends on $\boldsymbol{v}$ and, after integrating (31) over elements K and summing up, the last two terms in (31) vanish. We refer to [19] for details.

Observe that, in contrast to the proof of the discrete Friedrichs inequality, the 'continuous' Korn inequality (29) is not used in the proof of (30). On the contrary, (29) follows from (30) for $\boldsymbol{W}_h$ since $\boldsymbol{V} \subset \boldsymbol{W}_h$.

Finally, let us mention that combining the discrete Korn inequality for $\boldsymbol{W}_h$ with the discrete Friedrichs inequality proved in the preceding section, we deduce that the Korn inequality

$$\|\boldsymbol{v}\|_{1,h}^2 \le C \sum_{K \in \mathcal{T}_h} \|\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^{\mathrm{T}}\|_{0,K}^2 \quad \forall\ \boldsymbol{v} \in \boldsymbol{W}_h$$

holds with C independent of h. Again, this inequality is important for proving convergence of discrete solutions in the $L^2(\Omega)$ norm.

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