Algebraic flux correction for convection–diffusion problems

Petr Knobloch

Charles University, Prague

joint work with

Gabriel R. Barrenechea (Glasgow)

Volker John (Berlin)

ENUMATH 2017

Voss, 25–29 September, 2017

Outline

– algebraic flux correction scheme for a steady-state convection–diffusion–reaction equation

– theoretical analysis: error estimate, discrete maximum principle

– examples of limiters

– linearity preservation

– approximation of boundary layers

– difficulties caused by non-vanishing right-hand side

– numerical results

Steady-state convection–diffusion–reaction equation

$$-\varepsilon\,\Delta u + \mathbf{b}\cdot\nabla u + c\,u = f \quad \text{in } \Omega\,, \qquad u = u_b \quad \text{on } \partial\Omega$$

with constant $\varepsilon > 0$ and

$$\nabla\cdot\mathbf{b} = 0\,, \qquad c \geq \sigma_0 \geq 0 \qquad \text{in } \Omega\,.$$

FE discretization

Find $u_h \in W_h$ such that $u_h(x_i) = u_b(x_i)$, $i = M+1,\dots,N$, and

$$a_h(u_h, v_h) = (f, v_h) \qquad \forall\, v_h \in V_h\,,$$

where

$$W_h = \{v_h \in C(\overline{\Omega})\,;\; v|_K \in P_1(K)\;\forall K \in \mathscr{T}_h\}\,, \qquad V_h = W_h \cap H_0^1(\Omega)\,,$$

$$a_h(u_h, v_h) = \varepsilon\,(\nabla u_h, \nabla v_h) + (\mathbf{b}\cdot\nabla u_h, v_h) + \sum_{i=1}^{M} (c, \varphi_i)\, u_i\, v_i\,.$$

Algebraic problem

$$\sum_{j=1}^{N} a_{ij} u_j = f_i \,, \qquad i = 1, \ldots, M \,,$$

$$u_i = u_i^b \,, \qquad i = M+1, \ldots, N \,.$$

Algebraic problem

$$\sum_{j=1}^{N} a_{ij} u_j = f_i \,, \qquad i = 1, \ldots, M \,,$$

$$u_i = u_i^b \,, \qquad i = M+1, \ldots, N \,.$$

Properties: $(a_{ij})_{i,j=1}^{M}$ is positive definite,

$$\sum_{j=1}^{N} a_{ij} \geq 0 \quad \forall\, i = 1, \ldots, M$$

Algebraic problem

$$\sum_{j=1}^{N} a_{ij}\,u_j = f_i\,, \qquad i = 1,\dots,M\,,$$
$$u_i = u_i^b\,, \qquad i = M+1,\dots,N\,.$$

Properties: $(a_{ij})_{i,j=1}^{M}$ is positive definite,

$$\sum_{j=1}^{N} a_{ij} \geq 0 \quad \forall\, i = 1,\dots,M$$

Algebraic flux correction   Kuzmin et al. (2001–now)

Aim: manipulate the algebraic system in such a way that the solution satisfies DMP and layers are not smeared.

Algebraic problem

$$\sum_{j=1}^{N} a_{ij}\, u_j = f_i\,, \qquad i = 1,\dots,M\,,$$
$$u_i = u_i^b\,, \qquad i = M+1,\dots,N\,.$$

Properties: $(a_{ij})_{i,j=1}^{M}$ is positive definite,

$$\sum_{j=1}^{N} a_{ij} \geq 0 \quad \forall\, i = 1,\dots,M$$

$(a_{ij})_{i,j=1}^{N}$ ... FE matrix for homogeneous natural b.c.

$(d_{ij})_{i,j=1}^{N}$ ... symmetric artificial diffusion matrix:

$$d_{ij} = -\max\{a_{ij}, 0, a_{ji}\} \qquad \forall\, i \neq j\,, \qquad\qquad d_{ii} = -\sum_{j \neq i} d_{ij}\,.$$

Algebraic flux correction scheme

$$\sum_{j=1}^{N} a_{ij}\,u_j + \sum_{j=1}^{N} (1-\alpha_{ij})\,d_{ij}\,(u_j - u_i) = f_i\,, \qquad i = 1,\dots,M\,,$$

$$u_i = u_i^b\,, \qquad i = M+1,\dots,N\,,$$

where $\alpha_{ij} = \alpha_{ij}(u_1,\dots,u_N) \in [0,1]$,

$$\alpha_{ij} = \alpha_{ji}\,, \qquad i,j = 1,\dots,N\,.$$

and $\alpha_{ij}(U)(u_i - u_j)$ are continuous functions of $U = (u_1,\dots,u_N)$.

Algebraic flux correction scheme

$$\sum_{j=1}^{N} a_{ij}\, u_j + \sum_{j=1}^{N} (1-\alpha_{ij})\, d_{ij}\, (u_j - u_i) = f_i\,, \qquad i = 1,\ldots,M\,,$$

$$u_i = u_i^b\,, \qquad i = M+1,\ldots,N\,,$$

where $\alpha_{ij} = \alpha_{ij}(u_1,\ldots,u_N) \in [0,1]$,

$$\alpha_{ij} = \alpha_{ji}\,, \qquad i,j = 1,\ldots,N\,.$$

and $\alpha_{ij}(U)(u_i - u_j)$ are continuous functions of $U = (u_1,\ldots,u_N)$.

Theoretical analysis:

Barrenechea, John, K., IMAJNA (2015)
Barrenechea, John, K., SINUM (2016)
Barrenechea, John, K., M3AS (2017)

Variational form of the AFC scheme

Find $u_h \in W_h$ such that $u_h(x_i) = u_b(x_i)$, $i = M+1, \ldots, N$, and

$$a_h(u_h, v_h) + d_h(u_h; u_h, v_h) = (f, v_h) \qquad \forall\, v_h \in V_h$$

where

$$d_h(w_h; z, v) = \sum_{i,j=1}^{N} (1 - \alpha_{ij}(w_h))\, d_{ij}\, (z(x_j) - z(x_i))\, v(x_i)\,.$$

Variational form of the AFC scheme

Find $u_h \in W_h$ such that $u_h(x_i) = u_b(x_i)$, $i = M+1,\ldots,N$, and

$$a_h(u_h, v_h) + d_h(u_h; u_h, v_h) = (f, v_h) \qquad \forall\, v_h \in V_h$$

where

$$d_h(w_h; z, v) = \sum_{i,j=1}^{N} (1 - \alpha_{ij}(w_h))\, d_{ij}\, (z(x_j) - z(x_i))\, v(x_i)\,.$$

Error estimate (for $\sigma_0 > 0$)

Natural norm: $\|v_h\|_h = \left(\varepsilon\, |v|_{1,\Omega}^2 + \sigma_0\, \|v\|_{0,\Omega}^2 + d_h(u_h; v_h, v_h)\right)^{1/2}$

Variational form of the AFC scheme

Find $u_h \in W_h$ such that $u_h(x_i) = u_b(x_i)$, $i = M+1,\dots,N$, and

$$a_h(u_h, v_h) + d_h(u_h; u_h, v_h) = (f, v_h) \qquad \forall\, v_h \in V_h$$

where

$$d_h(w_h; z, v) = \sum_{i,j=1}^{N} (1 - \alpha_{ij}(w_h))\, d_{ij}\, (z(x_j) - z(x_i))\, v(x_i)\,.$$

Error estimate (for $\sigma_0 > 0$)

Natural norm: $\quad \|v_h\|_h = \left(\varepsilon\, |v|_{1,\Omega}^2 + \sigma_0\, \|v\|_{0,\Omega}^2 + d_h(u_h; v_h, v_h)\right)^{1/2}$

$$\Rightarrow \quad \|u - u_h\|_h \le C\, \|u - i_h u\|_h + \sup_{v_h \in V_h} \frac{a(u, v_h) - a_h(i_h u, v_h)}{\|v_h\|_h}$$

$$+\, d_h(u_h; i_h u, i_h u)^{1/2}$$

Variational form of the AFC scheme

Find $u_h \in W_h$ such that $u_h(x_i) = u_b(x_i)$, $i = M+1,\dots,N$, and

$$a_h(u_h,v_h) + d_h(u_h;u_h,v_h) = (f,v_h) \qquad \forall\, v_h \in V_h$$

where

$$d_h(w_h;z,v) = \sum_{i,j=1}^{N} (1-\alpha_{ij}(w_h))\, d_{ij}\, (z(x_j)-z(x_i))\, v(x_i)\,.$$

Error estimate (for $\sigma_0 > 0$)

Natural norm: $\quad \|v_h\|_h = \left(\varepsilon\, |v|_{1,\Omega}^2 + \sigma_0\, \|v\|_{0,\Omega}^2 + d_h(u_h;v_h,v_h)\right)^{1/2}$

$$\Rightarrow \quad \|u-u_h\|_h \le C\, \|u - i_h u\|_h + \sup_{v_h \in V_h} \frac{a(u,v_h) - a_h(i_h u, v_h)}{\|v_h\|_h}$$

$$+ d_h(u_h;i_h u, i_h u)^{1/2} \le C\, h\, \|u\|_{2,\Omega} + d_h(u_h;i_h u, i_h u)^{1/2}$$

Variational form of the AFC scheme

Find $u_h \in W_h$ such that $u_h(x_i) = u_b(x_i)$, $i = M+1,\ldots,N$, and

$$a_h(u_h,v_h) + d_h(u_h;u_h,v_h) = (f,v_h) \qquad \forall\, v_h \in V_h$$

where

$$d_h(w_h;z,v) = \sum_{i,j=1}^{N} (1-\alpha_{ij}(w_h))\, d_{ij}\, (z(x_j)-z(x_i))\, v(x_i)\,.$$

Error estimate (for $\sigma_0 > 0$)

Natural norm: $\quad \|v_h\|_h = \left(\varepsilon\, |v|^2_{1,\Omega} + \sigma_0\, \|v\|^2_{0,\Omega} + d_h(u_h;v_h,v_h)\right)^{1/2}$

$$\Rightarrow \quad \|u-u_h\|_h \le C\,\|u-i_hu\|_h + \sup_{v_h\in V_h} \frac{a(u,v_h)-a_h(i_hu,v_h)}{\|v_h\|_h}$$

$$+\, d_h(u_h;i_hu,i_hu)^{1/2} \le C\,h\,\|u\|_{2,\Omega} + d_h(u_h;i_hu,i_hu)^{1/2}$$

$$\le C\,h\,\|u\|_{2,\Omega} + C\,(\varepsilon + \|\mathbf{b}\|_{0,\infty,\Omega}\,h)^{1/2}\,|i_hu|_{1,\Omega}$$

Discrete maximum principle

Local index sets: $S_i \supset \{j \in \{1,\ldots,N\} \setminus \{i\};\, a_{ij} \neq 0$ or $a_{ji} > 0\}\,,$

$$i = 1,\ldots,M\,.$$

Assumption (A):

For any $i \in \{1,\ldots,M\}$ and any $U = (u_1,\ldots,u_N) \in \mathbb{R}^N$:

$u_i > u_j \quad \forall\, j \in S_i$ or $u_i < u_j \quad \forall\, j \in S_i$

$$\Rightarrow \qquad a_{ij} + (1 - \alpha_{ij}(U))\, d_{ij} \leq 0 \quad \forall\, j \in S_i\,.$$

Discrete maximum principle

Local index sets: $S_i \supset \{j \in \{1,\dots,N\} \setminus \{i\}\,;\, a_{ij} \neq 0$ or $a_{ji} > 0\}\,,$

$$i = 1,\dots,M\,.$$

Assumption (A):

For any $i \in \{1,\dots,M\}$ and any $U = (u_1,\dots,u_N) \in \mathbb{R}^N$:

$$u_i > u_j \quad \forall\, j \in S_i \quad \text{or} \quad u_i < u_j \quad \forall\, j \in S_i$$

$$\Rightarrow \qquad a_{ij} + (1 - \alpha_{ij}(U))\, d_{ij} \leq 0 \quad \forall\, j \in S_i\,.$$

Theorem Consider any $i \in \{1,\dots,M\}$. Then

$$f_i \leq 0 \;\; \Rightarrow \;\; u_i \leq \max_{j \in S_i} u_j^+\,, \qquad f_i \geq 0 \;\; \Rightarrow \;\; u_i \geq \min_{j \in S_i} u_j^-\,.$$

If $\sum_{j=1}^N a_{ij} = 0$, then

$$f_i \leq 0 \;\; \Rightarrow \;\; u_i \leq \max_{j \in S_i} u_j\,, \qquad f_i \geq 0 \;\; \Rightarrow \;\; u_i \geq \min_{j \in S_i} u_j\,.$$

Limiter 1

Zalesak (1979), Kuzmin (2007)

$$P_i^+ := \sum_{\substack{j=1 \\ a_{ji} \le a_{ij}}}^{N} f_{ij}^+ , \qquad Q_i^+ := -\sum_{j=1}^{N} f_{ij}^- , \qquad R_i^+ := \min\left\{1, \frac{Q_i^+}{P_i^+}\right\} ,$$

$$P_i^- := \sum_{\substack{j=1 \\ a_{ji} \le a_{ij}}}^{N} f_{ij}^- , \qquad Q_i^- := -\sum_{j=1}^{N} f_{ij}^+ , \qquad R_i^- := \min\left\{1, \frac{Q_i^-}{P_i^-}\right\} .$$

$$f_{ij} = d_{ij}\,(u_j - u_i)$$

$$\widetilde{\alpha}_{ij} := \begin{cases} R_i^+ & \text{if } f_{ij} > 0 , \\ 1 & \text{if } f_{ij} = 0 , \\ R_i^- & \text{if } f_{ij} < 0 , \end{cases} \qquad \alpha_{ij} = \alpha_{ji} := \begin{cases} \widetilde{\alpha}_{ij} & \text{if } a_{ji} \le a_{ij} , \\ \widetilde{\alpha}_{ji} & \text{else.} \end{cases}$$

Limiter 1

Zalesak (1979), Kuzmin (2007)

$$P_i^+ := \sum_{\substack{j=1 \\ a_{ji} \le a_{ij}}}^{N} f_{ij}^+ \,, \qquad Q_i^+ := -\sum_{j=1}^{N} f_{ij}^- \,, \qquad R_i^+ := \min\left\{1, \frac{Q_i^+}{P_i^+}\right\},$$

$$P_i^- := \sum_{\substack{j=1 \\ a_{ji} \le a_{ij}}}^{N} f_{ij}^- \,, \qquad Q_i^- := -\sum_{j=1}^{N} f_{ij}^+ \,, \qquad R_i^- := \min\left\{1, \frac{Q_i^-}{P_i^-}\right\}.$$

$$f_{ij} = d_{ij}\,(u_j - u_i)$$

$$\widetilde{\alpha}_{ij} := \begin{cases} R_i^+ & \text{if } f_{ij} > 0\,, \\ 1 & \text{if } f_{ij} = 0\,, \\ R_i^- & \text{if } f_{ij} < 0\,, \end{cases} \qquad \alpha_{ij} = \alpha_{ji} := \begin{cases} \widetilde{\alpha}_{ij} & \text{if } a_{ji} \le a_{ij}\,, \\ \widetilde{\alpha}_{ji} & \text{else.} \end{cases}$$

Assumption (A) satisfied if $\min\{a_{ij}, a_{ji}\} \le 0 \quad \forall\, i, j \in \{1, \ldots, N\}$

Limiter 1

Zalesak (1979), Kuzmin (2007)

$$P_i^+ := \sum_{\substack{j=1 \\ a_{ji} \le a_{ij}}}^{N} f_{ij}^+ , \qquad Q_i^+ := -\sum_{j=1}^{N} f_{ij}^- , \qquad R_i^+ := \min\left\{1, \frac{Q_i^+}{P_i^+}\right\} ,$$

$$P_i^- := \sum_{\substack{j=1 \\ a_{ji} \le a_{ij}}}^{N} f_{ij}^- , \qquad Q_i^- := -\sum_{j=1}^{N} f_{ij}^+ , \qquad R_i^- := \min\left\{1, \frac{Q_i^-}{P_i^-}\right\} .$$

$$f_{ij} = d_{ij}\,(u_j - u_i)$$

$$\widetilde{\alpha}_{ij} := \begin{cases} R_i^+ & \text{if } f_{ij} > 0 , \\ 1 & \text{if } f_{ij} = 0 , \\ R_i^- & \text{if } f_{ij} < 0 , \end{cases} \qquad \alpha_{ij} = \alpha_{ji} := \begin{cases} \widetilde{\alpha}_{ij} & \text{if } a_{ji} \le a_{ij} , \\ \widetilde{\alpha}_{ji} & \text{else.} \end{cases}$$

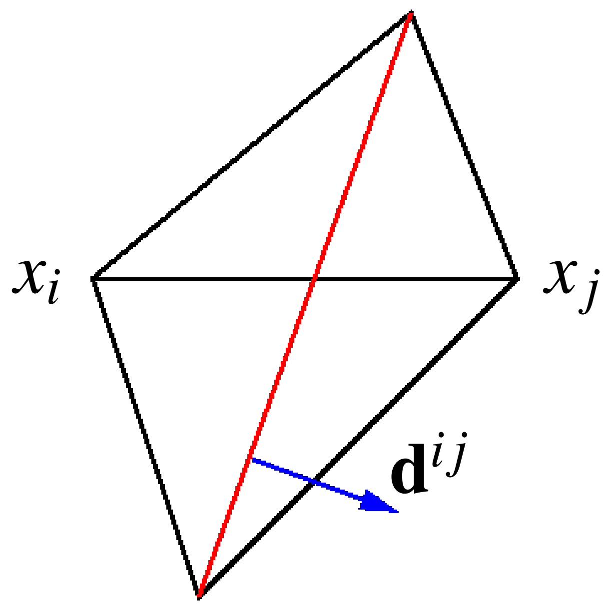
Assumption (A) satisfied if $\min\{a_{ij}, a_{ji}\} \le 0 \quad \forall\, i, j \in \{1, \dots, N\}$

$\Rightarrow$ DMP guranteed for Delaunay meshes,

... and often holds on non-Delaunay meshes!!!

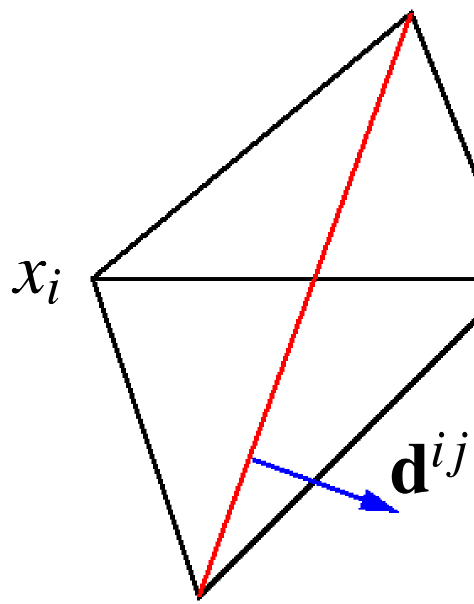
Condition $a_{ji} \leq a_{ij}$ for constant $\mathbf{b}$

$a_{ji} < a_{ij} \;\Leftrightarrow\; \mathbf{b} \cdot \mathbf{d}^{ij} > 0$ with $\mathbf{d}^{ij} = \int_\Omega \varphi_i \nabla \varphi_j \,\mathrm{d}x$



Condition $a_{ji} \leq a_{ij}$ for constant $\mathbf{b}$

$a_{ji} < a_{ij} \;\Leftrightarrow\; \mathbf{b}\cdot\mathbf{d}^{ij} > 0$ with $\mathbf{d}^{ij} = \int_\Omega \varphi_i \nabla\varphi_j \,\mathrm{d}x$



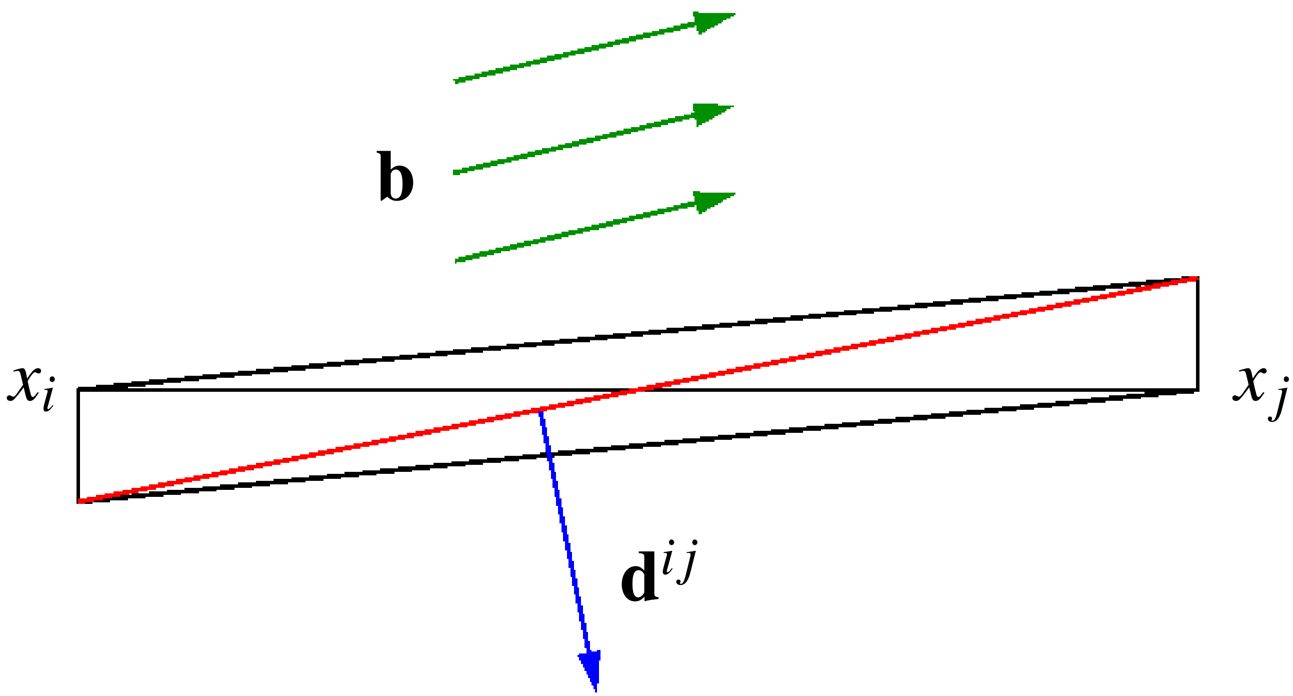
$\Rightarrow$ if $\mathbf{b} \parallel x_i x_j$ or red line $\perp x_i x_j$, then

$a_{ji} < a_{ij} \;\Leftrightarrow\; x_i$ is the upwind vertex.

Condition $a_{ji} \leq a_{ij}$ for constant $\mathbf{b}$

$a_{ji} < a_{ij} \;\Leftrightarrow\; \mathbf{b} \cdot \mathbf{d}^{ij} > 0$ with $\mathbf{d}^{ij} = \int_\Omega \varphi_i \nabla \varphi_j \,\mathrm{d}x$

BUT:



$a_{ji} > a_{ij}$, but x_i is the upwind vertex!

Limiter 2

Kuzmin (2012), Barrenechea, John, K. (2016)

$$u_i^{\max} := \max_{j\in S_i\cup\{i\}} u_j\,, \qquad u_i^{\min} := \min_{j\in S_i\cup\{i\}} u_j\,, \qquad q_i := \gamma_i \sum_{j\in S_i} d_{ij}\,,$$

$$P_i^+ := \sum_{j\in S_i} f_{ij}^+\,, \qquad Q_i^+ := q_i\,(u_i - u_i^{\max})\,, \qquad R_i^+ := \min\left\{1, \frac{Q_i^+}{P_i^+}\right\}\,,$$

$$P_i^- := \sum_{j\in S_i} f_{ij}^-\,, \qquad Q_i^- := q_i\,(u_i - u_i^{\min})\,, \qquad R_i^- := \min\left\{1, \frac{Q_i^-}{P_i^-}\right\}\,,$$

$$\widetilde{\alpha}_{ij} := \begin{cases} R_i^+ & \text{if } f_{ij} > 0\,, \\ 1 & \text{if } f_{ij} = 0\,, \\ R_i^- & \text{if } f_{ij} < 0\,, \end{cases} \qquad \alpha_{ij} = \alpha_{ji} := \min\{\widetilde{\alpha}_{ij}, \widetilde{\alpha}_{ji}\}.$$

Limiter 2 Kuzmin (2012), Barrenechea, John, K. (2016)

$$u_i^{\max} := \max_{j \in S_i \cup \{i\}} u_j \,, \qquad u_i^{\min} := \min_{j \in S_i \cup \{i\}} u_j \,, \qquad q_i := \gamma_i \sum_{j \in S_i} d_{ij} \,,$$

$$P_i^+ := \sum_{j \in S_i} f_{ij}^+ \,, \qquad Q_i^+ := q_i \, (u_i - u_i^{\max}) \,, \qquad R_i^+ := \min \left\{ 1, \frac{Q_i^+}{P_i^+} \right\} ,$$

$$P_i^- := \sum_{j \in S_i} f_{ij}^- \,, \qquad Q_i^- := q_i \, (u_i - u_i^{\min}) \,, \qquad R_i^- := \min \left\{ 1, \frac{Q_i^-}{P_i^-} \right\} ,$$

$$\widetilde{\alpha}_{ij} := \begin{cases} R_i^+ & \text{if } f_{ij} > 0 \,, \\ 1 & \text{if } f_{ij} = 0 \,, \\ R_i^- & \text{if } f_{ij} < 0 \,, \end{cases} \qquad \alpha_{ij} = \alpha_{ji} := \min\{\widetilde{\alpha}_{ij}, \widetilde{\alpha}_{ji}\}.$$

Assumption (A) always satisfied

$\Rightarrow$ DMP guranteed for arbitrary meshes!

Linearity preservation: $\alpha_{ij}(u) = 1$ for $u \in P_1(\overline{\Omega})$

Linearity preservation: $\alpha_{ij}(u) = 1$ **for** $u \in P_1(\overline{\Omega})$

Limiter 1: only for special meshes (e.g., Friedrichs–Keller)
and $\mathbf{b} = \text{const}$.

Linearity preservation: $\alpha_{ij}(u) = 1$ **for** $u \in P_1(\overline{\Omega})$

Limiter 1: only for special meshes (e.g., Friedrichs–Keller) and $\mathbf{b} =$ const.

Limiter 2: for arbitrary meshes if

$$\gamma_i = \frac{\max\limits_{x_j \in \partial\Delta_i} |x_i - x_j|}{\operatorname{dist}(x_i, \partial\Delta_i^{\mathrm{conv}})}, \qquad i = 1, \ldots, M\,.$$

Linearity preservation: $\alpha_{ij}(u) = 1$ **for** $u \in P_1(\overline{\Omega})$

Limiter 1: only for special meshes (e.g., Friedrichs–Keller) and $\mathbf{b} =$ const.

Limiter 2: for arbitrary meshes if

$$\gamma_i = \frac{\max\limits_{x_j \in \partial\Delta_i} |x_i - x_j|}{\operatorname{dist}(x_i, \partial\Delta_i^{\text{conv}})}, \qquad i = 1, \dots, M\,.$$

Improved error estimate on Delaunay meshes

Lipschitz-continuity $\quad\Rightarrow\quad d_h(u_h; i_h u, i_h u)$

$$\leq C \, \|\mathbf{b}\|_{0,\infty,\Omega} \, h \, |u - u_h|_{1,\Omega} \, |i_h u|_{1,\Omega} + C \, \|\mathbf{b}\|_{0,\infty,\Omega} \, h^2 \, |u|_{2,\Omega} \, |i_h u|_{1,\Omega}$$

Linearity preservation: $\alpha_{ij}(u) = 1$ **for** $u \in P_1(\overline{\Omega})$

Limiter 1: only for special meshes (e.g., Friedrichs–Keller)
and $\mathbf{b} = $ const.

Limiter 2: for arbitrary meshes if

$$\gamma_i = \frac{\max\limits_{x_j \in \partial\Delta_i} |x_i - x_j|}{\operatorname{dist}(x_i, \partial\Delta_i^{\text{conv}})}\,, \qquad i = 1, \dots, M\,.$$

Improved error estimate on Delaunay meshes

Lipschitz-continuity $\quad \Rightarrow \quad d_h(u_h; i_h u, i_h u)$

$$\leq C\,\|\mathbf{b}\|_{0,\infty,\Omega}\, h\, |u - u_h|_{1,\Omega}\, |i_h u|_{1,\Omega} + C\,\|\mathbf{b}\|_{0,\infty,\Omega}\, h^2\, |u|_{2,\Omega}\, |i_h u|_{1,\Omega}$$

$$\Rightarrow \quad \|u - u_h\|_h \leq C\, h\, \|u\|_{2,\Omega} + C\, \frac{h}{\sqrt{\varepsilon}}\, |i_h u|_{1,\Omega}$$

Limiter 3

$$u_i^{\max} := \max_{j\in S_i\cup\{i\}} u_j\,, \qquad u_i^{\min} := \min_{j\in S_i\cup\{i\}} u_j\,, \qquad q_i := \gamma_i \sum_{a_{ji}\le a_{ij}} d_{ij}\,,$$

$$P_i^+ := \sum_{a_{ji}\le a_{ij}} f_{ij}^+\,, \qquad Q_i^+ := q_i\,(u_i - u_i^{\max})\,, \qquad R_i^+ := \min\left\{1, \frac{Q_i^+}{P_i^+}\right\},$$

$$P_i^- := \sum_{a_{ji}\le a_{ij}} f_{ij}^-\,, \qquad Q_i^- := q_i\,(u_i - u_i^{\min})\,, \qquad R_i^- := \min\left\{1, \frac{Q_i^-}{P_i^-}\right\},$$

$$f_{ij} = d_{ij}\,(u_j - u_i)$$

$$\widetilde{\alpha}_{ij} := \begin{cases} R_i^+ & \text{if } f_{ij} > 0\,, \\ 1 & \text{if } f_{ij} = 0\,, \\ R_i^- & \text{if } f_{ij} < 0\,, \end{cases} \qquad \alpha_{ij} = \alpha_{ji} := \begin{cases} \widetilde{\alpha}_{ij} & \text{if } a_{ji} \le a_{ij}\,, \\ \widetilde{\alpha}_{ji} & \text{else.} \end{cases}$$

Limiter 3

$$u_i^{\max} := \max_{j \in S_i \cup \{i\}} u_j\,, \qquad u_i^{\min} := \min_{j \in S_i \cup \{i\}} u_j\,, \qquad q_i := \gamma_i \sum_{a_{ji} \le a_{ij}} d_{ij}\,,$$

$$P_i^+ := \sum_{a_{ji} \le a_{ij}} f_{ij}^+\,, \qquad Q_i^+ := q_i\,(u_i - u_i^{\max})\,, \qquad R_i^+ := \min\left\{1, \frac{Q_i^+}{P_i^+}\right\},$$

$$P_i^- := \sum_{a_{ji} \le a_{ij}} f_{ij}^-\,, \qquad Q_i^- := q_i\,(u_i - u_i^{\min})\,, \qquad R_i^- := \min\left\{1, \frac{Q_i^-}{P_i^-}\right\},$$

$$f_{ij} = d_{ij}\,(u_j - u_i)$$

$$\tilde{\alpha}_{ij} := \begin{cases} R_i^+ & \text{if } f_{ij} > 0\,, \\ 1 & \text{if } f_{ij} = 0\,, \\ R_i^- & \text{if } f_{ij} < 0\,, \end{cases} \qquad \alpha_{ij} = \alpha_{ji} := \begin{cases} \tilde{\alpha}_{ij} & \text{if } a_{ji} \le a_{ij}\,, \\ \tilde{\alpha}_{ji} & \text{else.} \end{cases}$$

Assumption (A) satisfied if $\min\{a_{ij}, a_{ji}\} \le 0 \quad \forall\, i, j \in \{1, \dots, N\}$

Limiter 3

$$u_i^{\max} := \max_{j\in S_i\cup\{i\}} u_j\,, \qquad u_i^{\min} := \min_{j\in S_i\cup\{i\}} u_j\,, \qquad q_i := \gamma_i \sum_{a_{ji}\le a_{ij}} d_{ij}\,,$$

$$P_i^+ := \sum_{a_{ji}\le a_{ij}} f_{ij}^+\,, \qquad Q_i^+ := q_i\,(u_i - u_i^{\max})\,, \qquad R_i^+ := \min\left\{1, \frac{Q_i^+}{P_i^+}\right\},$$

$$P_i^- := \sum_{a_{ji}\le a_{ij}} f_{ij}^-\,, \qquad Q_i^- := q_i\,(u_i - u_i^{\min})\,, \qquad R_i^- := \min\left\{1, \frac{Q_i^-}{P_i^-}\right\},$$

$$f_{ij} = d_{ij}\,(u_j - u_i)$$

$$\widetilde{\alpha}_{ij} := \begin{cases} R_i^+ & \text{if } f_{ij} > 0\,, \\ 1 & \text{if } f_{ij} = 0\,, \\ R_i^- & \text{if } f_{ij} < 0\,, \end{cases} \qquad \alpha_{ij} = \alpha_{ji} := \begin{cases} \widetilde{\alpha}_{ij} & \text{if } a_{ji} \le a_{ij}\,, \\ \widetilde{\alpha}_{ji} & \text{else.} \end{cases}$$

Limiter 4

$\alpha_{ij} = \alpha_{ji}$ defined as for Limiter 3 if $\min\{a_{ij}, a_{ji}\} \le 0$ and as for Limiter 2 otherwise.

Example 1 (polynomial solution)

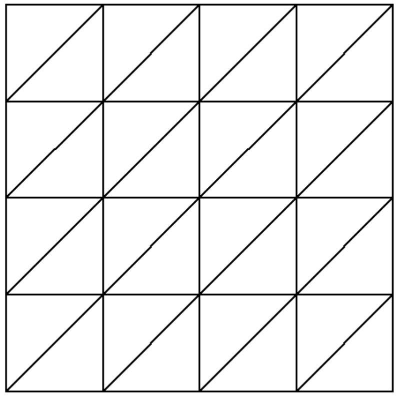
$$\Omega = (0,1)^2\,, \qquad \mathbf{b} = (3,2)\,, \qquad c = 1\,, \qquad u_b = 0\,.$$

The right-hand side f is chosen such that, for given ε,

$$u(x,y) = 100\,x^2\,(1-x)^2\,y\,(1-y)\,(1-2y)$$

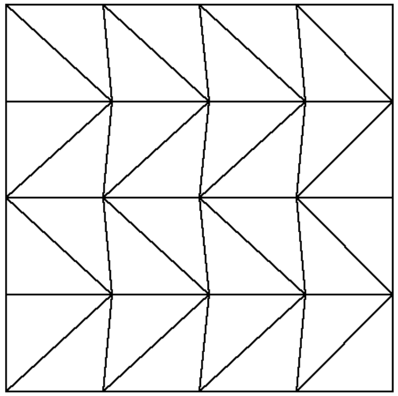
is the exact solution.

Example 1, Limiter 1, $\varepsilon = 10^{-8}$



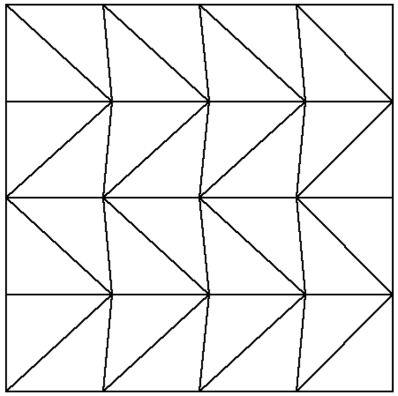
l	$\|e_h\|_{0,\Omega}$	ord.	$\|e_h\|_{1,\Omega}$	ord.	$d_h^{1/2}(u_h)$	ord.
3	5.457e−3	1.85	2.287e−1	1.10	1.163e−2	2.11
4	1.408e−3	1.95	1.074e−1	1.09	2.683e−3	2.12
5	3.493e−4	2.01	5.113e−2	1.07	6.410e−4	2.07
6	8.652e−5	2.01	2.546e−2	1.01	1.633e−4	1.97
7	2.152e−5	2.01	1.321e−2	0.95	4.099e−5	1.99
8	5.357e−6	2.01	6.822e−3	0.95	1.018e−5	2.01

Example 1, Limiter 1, $\varepsilon = 10$
(non-Delaunay grid)



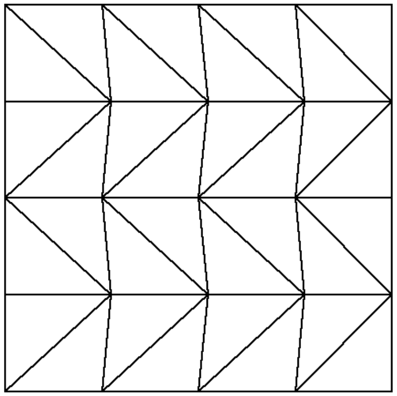
l	$\|e_h\|_{0,\Omega}$	ord.	$\|e_h\|_{1,\Omega}$	ord.	$d_h^{1/2}(u_h)$	ord.
3	1.248e−2	0.48	2.229e−1	0.79	7.211e−1	0.77
4	1.123e−2	0.15	1.558e−1	0.52	5.135e−1	0.49
5	1.090e−2	0.04	1.333e−1	0.22	4.452e−1	0.21
6	1.080e−2	0.01	1.269e−1	0.07	4.259e−1	0.06
7	1.077e−2	0.00	1.252e−1	0.02	4.207e−1	0.02
8	1.076e−2	0.00	1.248e−1	0.00	4.193e−1	0.00

Example 1, Limiter 4, $\varepsilon = 10$
(non-Delaunay grid)



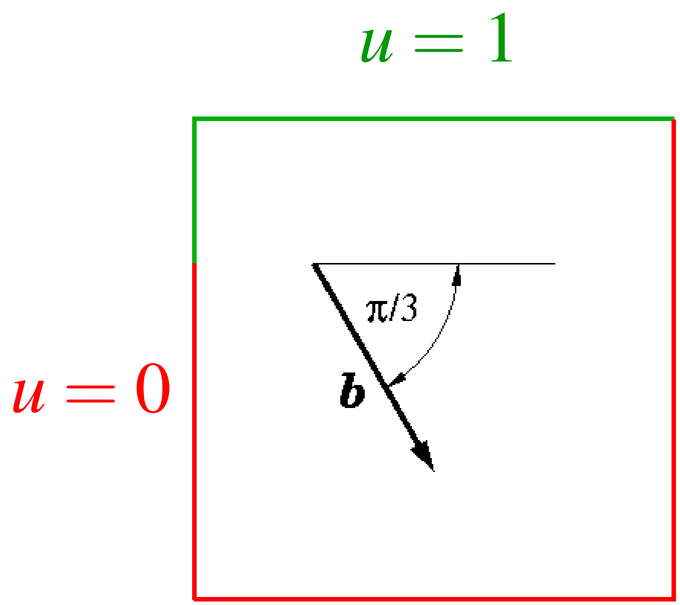
l	$\|e_h\|_{0,\Omega}$	ord.	$\|e_h\|_{1,\Omega}$	ord.	$d_h^{1/2}(u_h)$	ord.
3	2.259e−3	2.01	1.850e−1	0.99	5.850e−1	0.99
4	5.588e−4	2.02	9.278e−2	1.00	2.934e−1	1.00
5	1.391e−4	2.01	4.643e−2	1.00	1.468e−1	1.00
6	3.472e−5	2.00	2.322e−2	1.00	7.343e−2	1.00
7	8.675e−6	2.00	1.161e−2	1.00	3.672e−2	1.00

Example 1, Limiter 4, $\varepsilon = 10$, $\gamma_i := \gamma_i/10$ (non-Delaunay grid)



l	$\|e_h\|_{0,\Omega}$	ord.	$\lvert e_h\rvert_{1,\Omega}$	ord.	$d_h^{1/2}(u_h)$	ord.
3	1.938e−2	0.40	2.395e−1	0.73	7.741e−1	0.71
4	1.746e−2	0.15	1.749e−1	0.45	5.739e−1	0.43
5	1.679e−2	0.06	1.530e−1	0.19	5.066e−1	0.18
6	1.652e−2	0.02	1.463e−1	0.06	4.859e−1	0.06
7	1.640e−2	0.01	1.442e−1	0.02	4.794e−1	0.02

Example 2 (interior layer and exponential boundary layers)



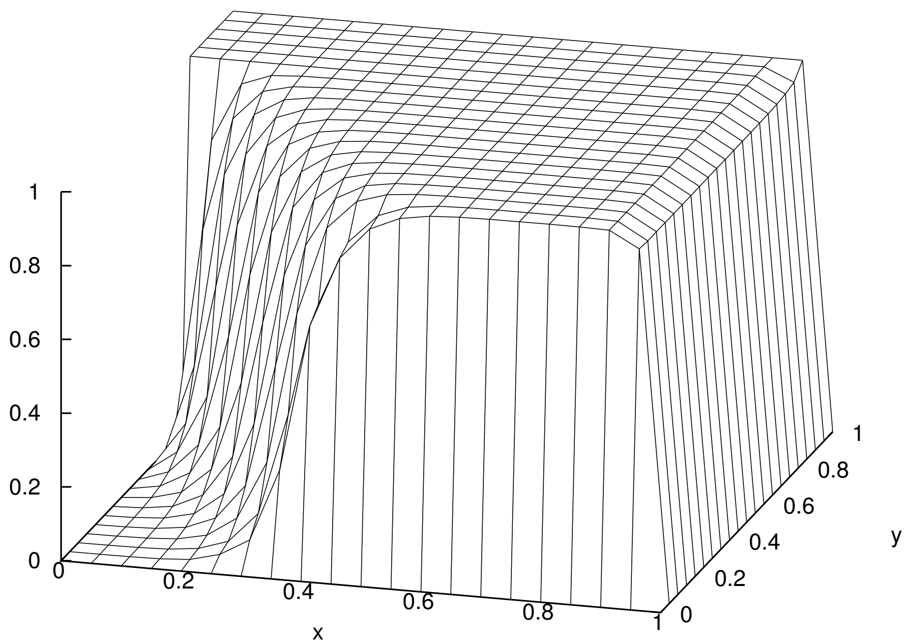
$$\varepsilon = 10^{-8}$$

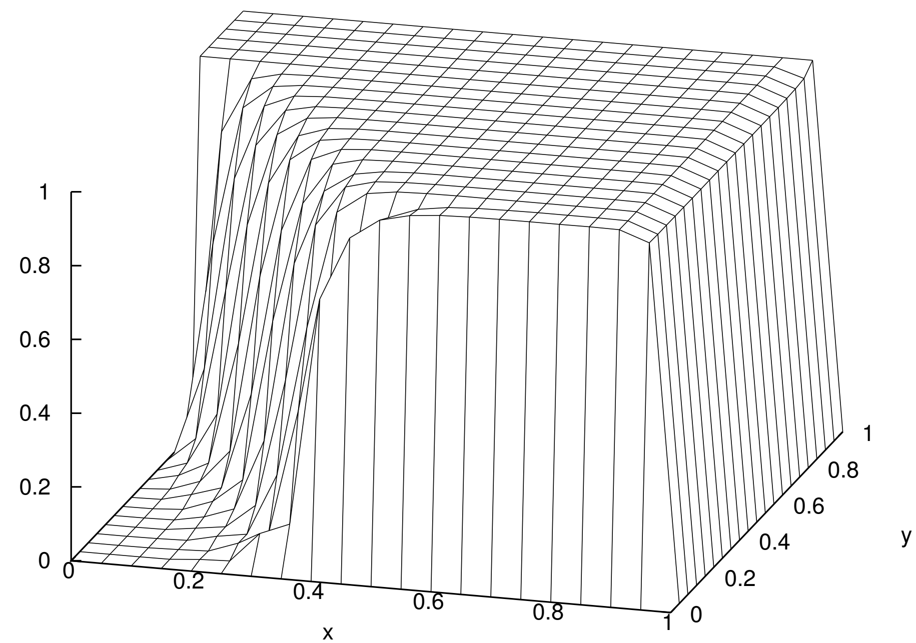
$$|\mathbf{b}| = 1$$

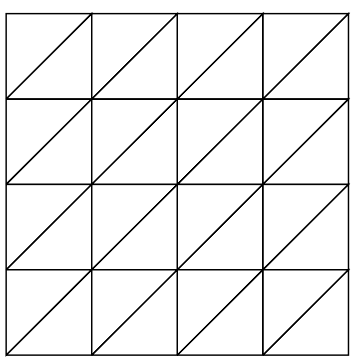
$$c = 0$$

$$f = 0$$

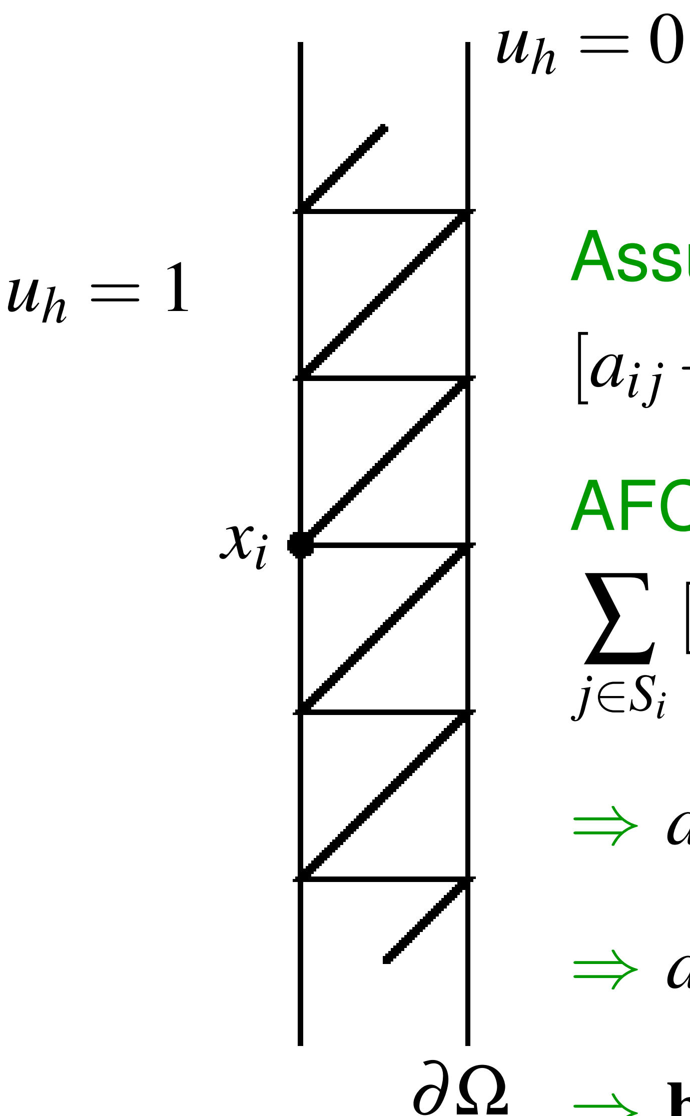
Example 2, Limiter 4, 21x21 mesh, $\gamma_i = 1$ vs. $\gamma_i = 2$







Approximation of boundary layers



Assumption (A) & continuity $\Rightarrow$

$$[a_{ij} + (1 - \alpha_{ij}(U))\, d_{ij}]\,(u_j - u_i) \geq 0 \quad \forall\, j \in S_i$$

AFC scheme:

$$\sum_{j \in S_i} [a_{ij} + (1 - \alpha_{ij}(U))\, d_{ij}]\,(u_j - u_i) = 0$$

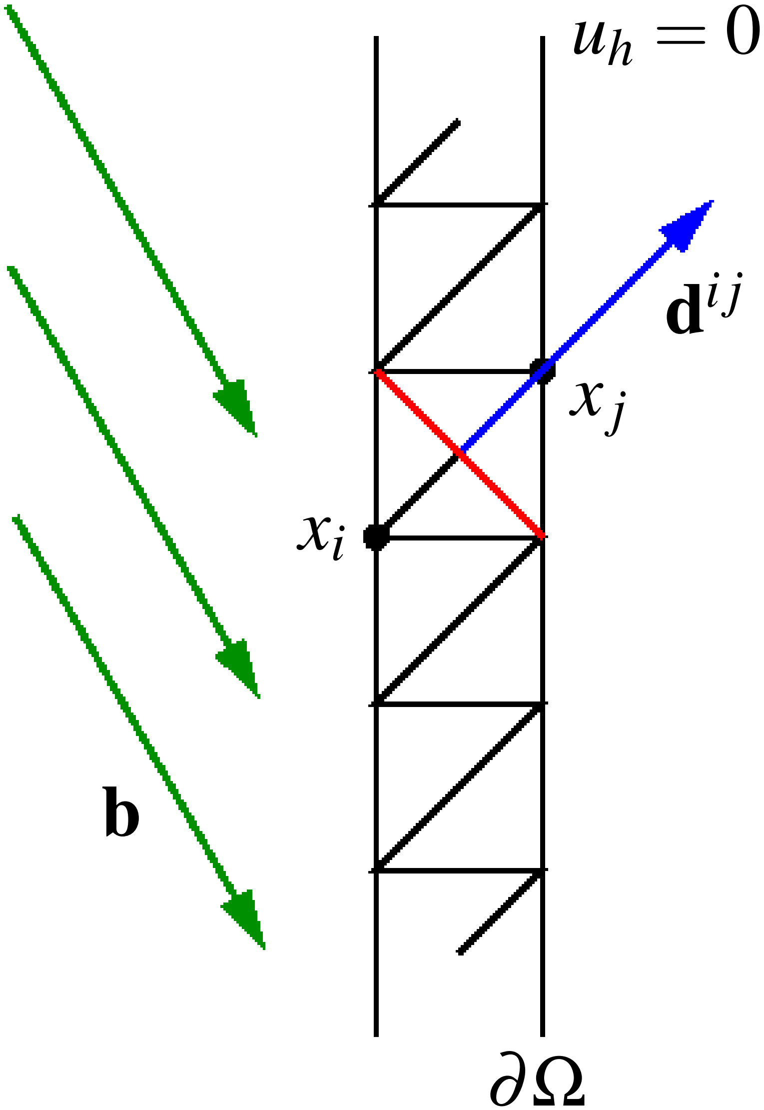
$$\Rightarrow\ a_{ij} + (1 - \alpha_{ij}(U))\, d_{ij} = 0 \quad \forall\, j \in S_i^{\partial\Omega}$$

$\Rightarrow\ a_{ij} \geq 0 \quad \forall\, j \in S_i^{\partial\Omega}$ also sufficient!

$$\Rightarrow\ \mathbf{b} \cdot \mathbf{d}^{ij} \geq 0 \quad \forall\, j \in S_i^{\partial\Omega}$$

since $a_{ij} = \varepsilon\,(\nabla\varphi_j, \nabla\varphi_i) + \mathbf{b} \cdot \mathbf{d}^{ij}$

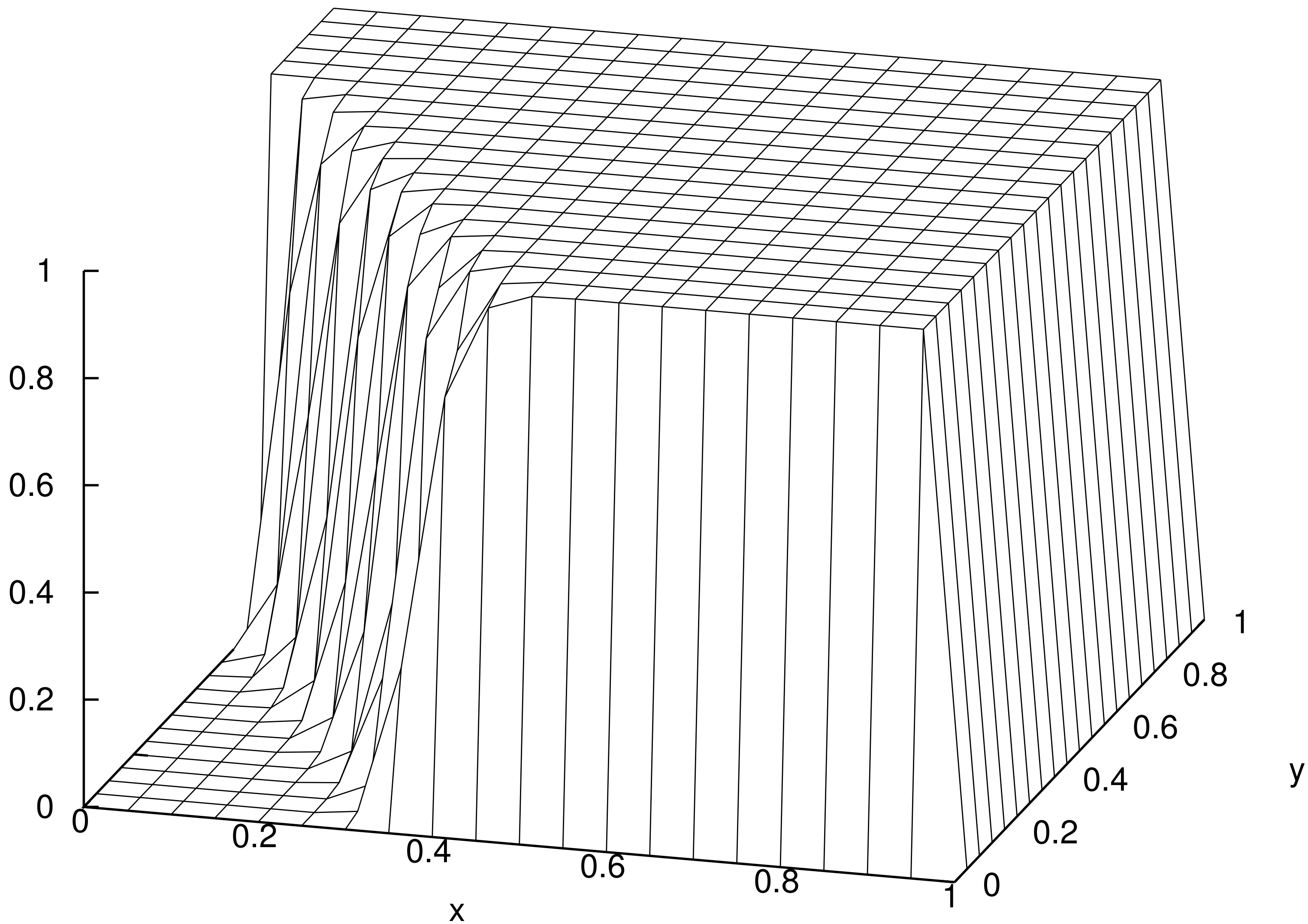
Approximation of boundary layers

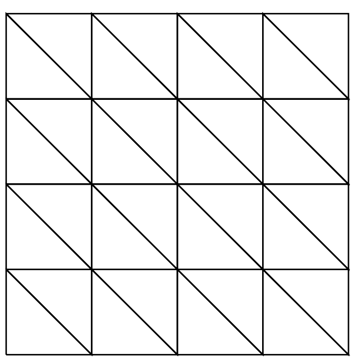


$\mathbf{b} \cdot \mathbf{d}^{ij} < 0$

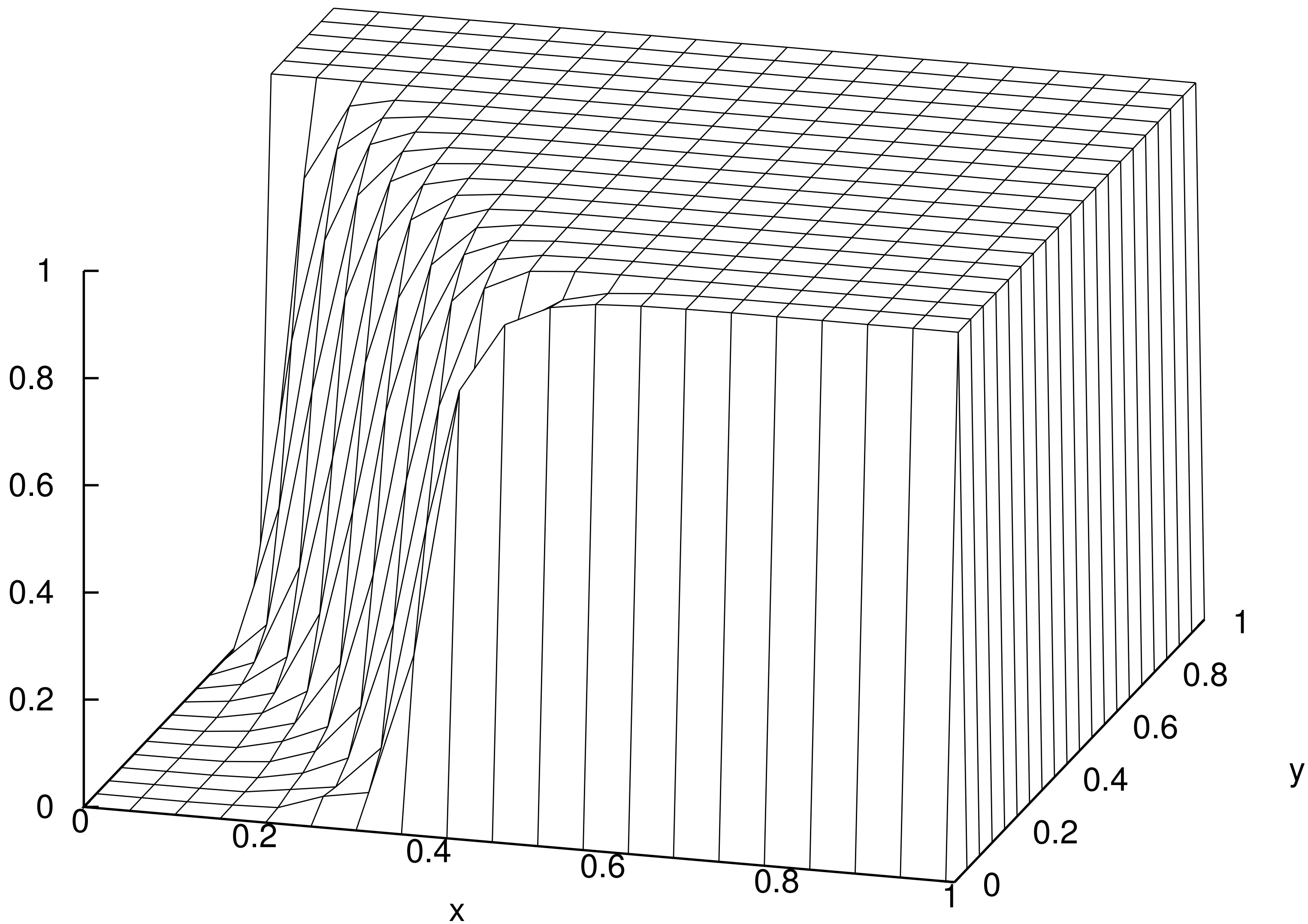
sharp approximation of the boundary layer not possible with any limiter for meshes of this type!!!

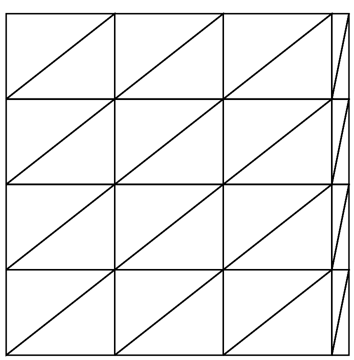
Example 2, Limiter 4, 21x21 mesh





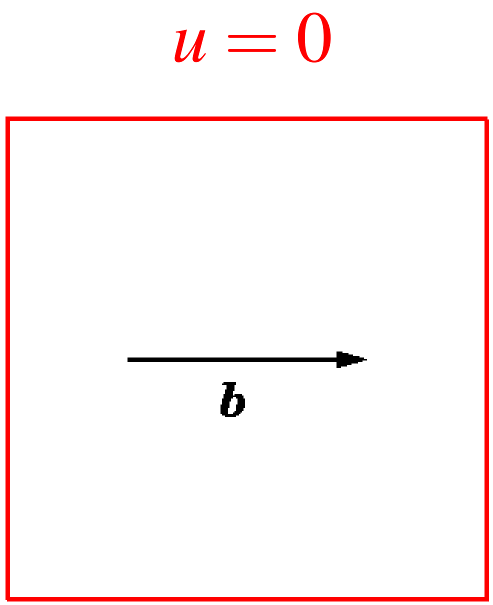
Example 2, Limiter 4, 21x21 mesh





Geometry of the mesh important,
not only the directions of edges!

Example 3 (exponential and parabolic boundary layers)



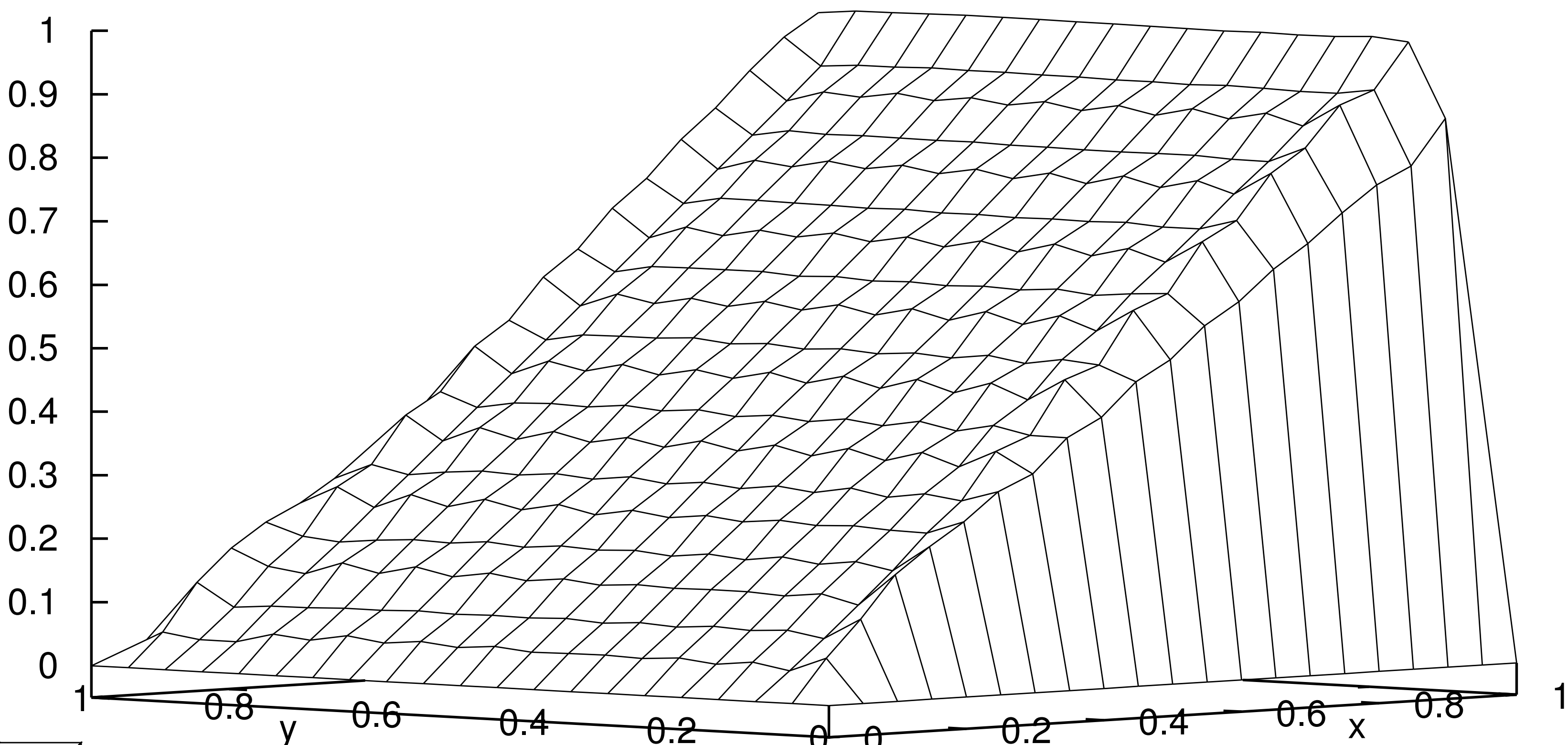
$$\varepsilon = 10^{-8}$$

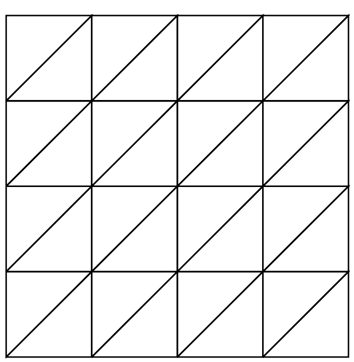
$$|\mathbf{b}| = 1$$

$$c = 0$$

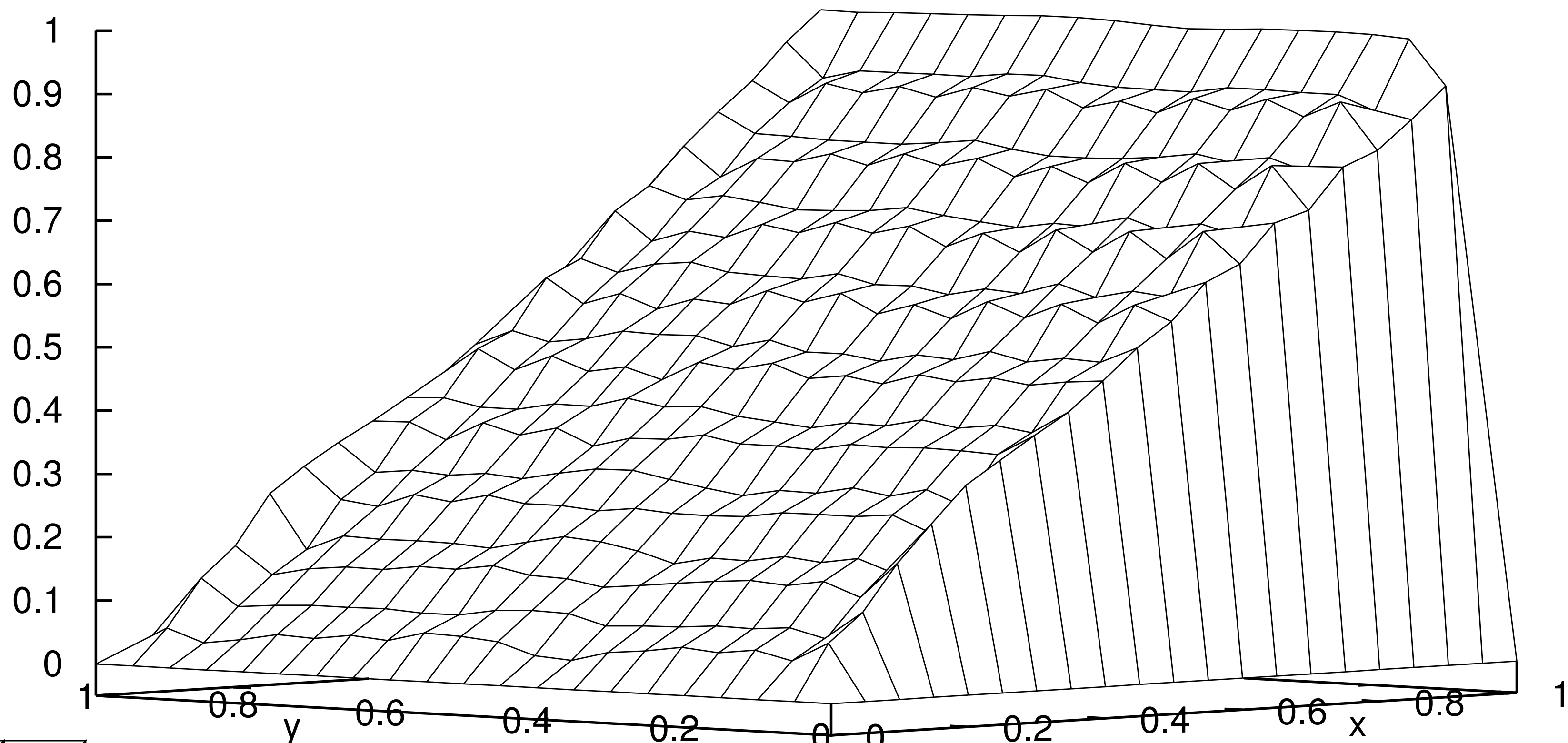
$$f = 1$$

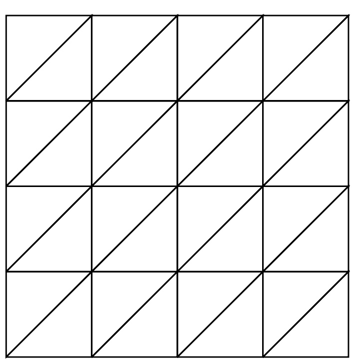
Example 3, Limiter 4, 21x21 mesh, $\gamma_i = 1$



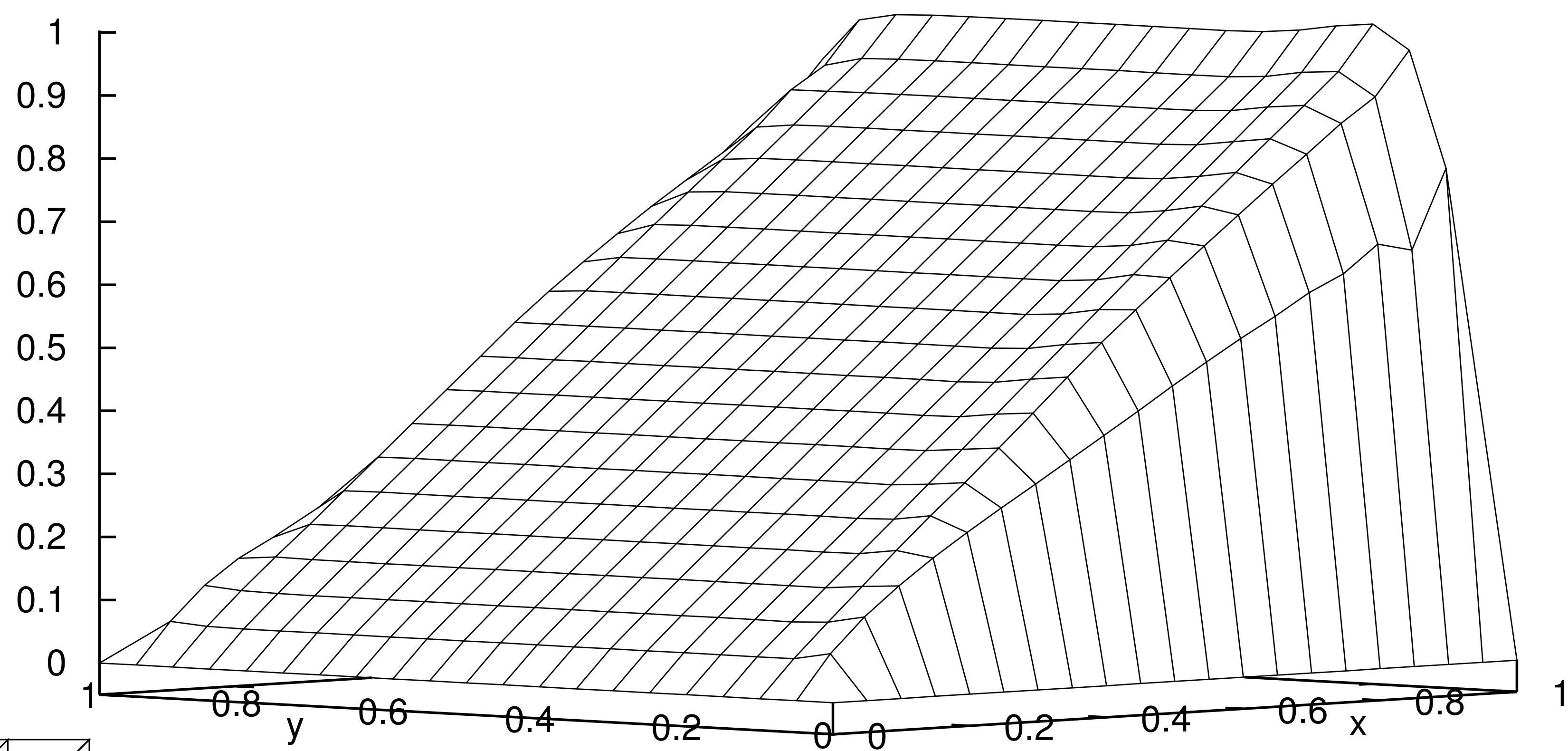


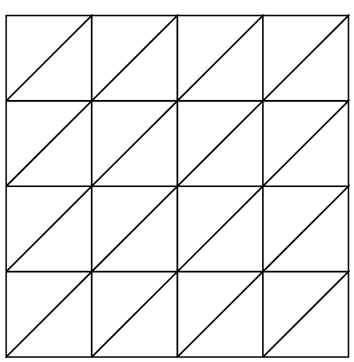
Example 3, Limiter 4, 21x21 mesh, $\gamma_i = 2$





Example 3, Limiter 4, 21x21 mesh, $\gamma_i = 0.5$





Two-step method

The problem

$$\mathscr{L}\,u := -\varepsilon\,\Delta u + \mathbf{b}\cdot\nabla u + c\,u = f \quad \text{in } \Omega\,, \qquad u = u_b \quad \text{on } \partial\Omega$$

is decomposed into two problems:

$$\mathscr{L}\,u_1 := f \quad \text{in } \Omega\,, \qquad u_1 = u_b \ \text{ on } \Gamma_-\,, \qquad \frac{\partial u_1}{\partial \mathbf{n}} = 0 \ \text{ on } \Gamma_+ \cup \Gamma_0$$

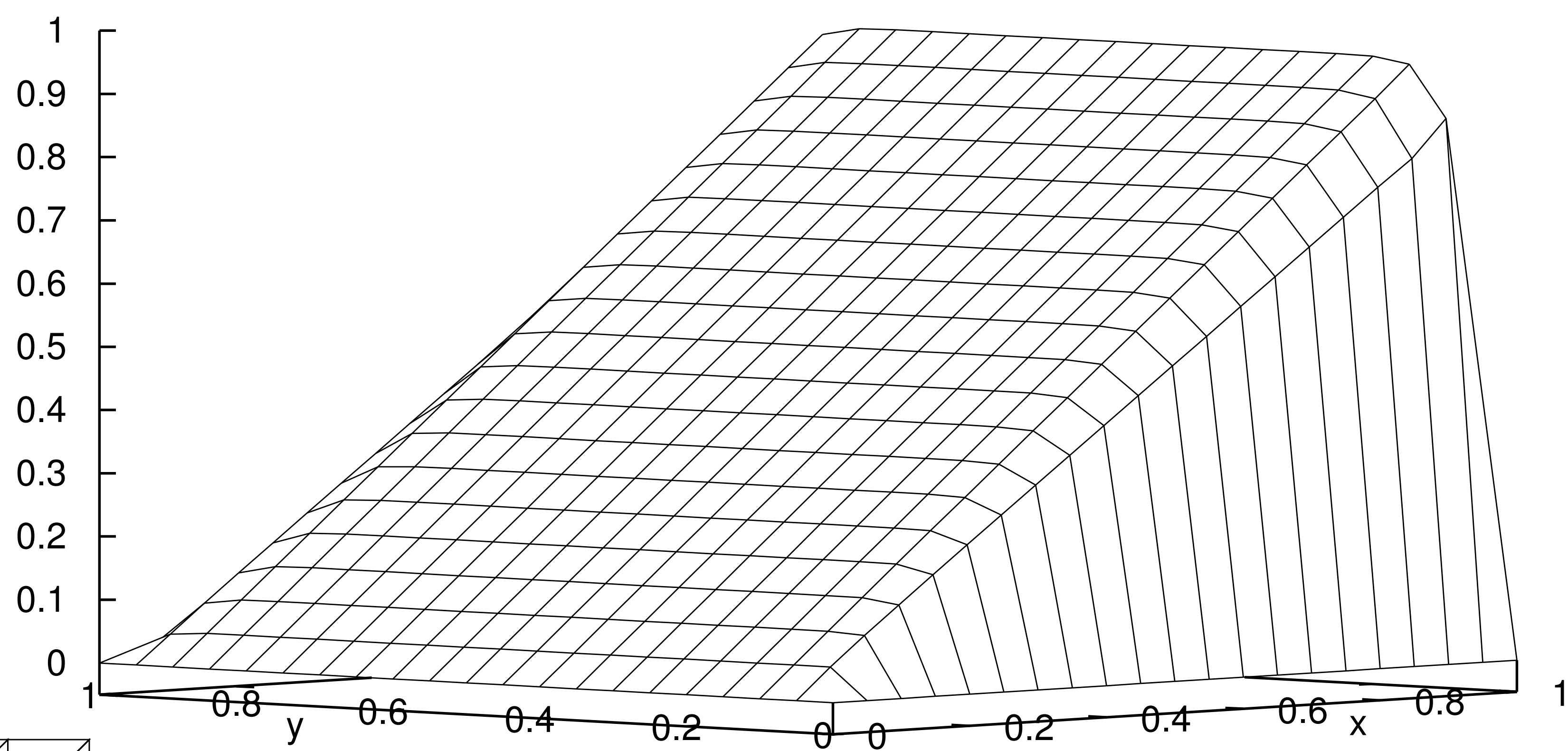
and

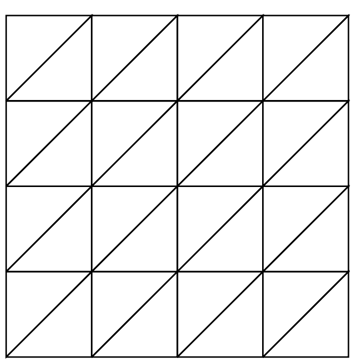
$$\mathscr{L}\,u_2 := 0 \quad \text{in } \Omega\,, \qquad u_2 = u_b - u_1 \quad \text{on } \partial\Omega\,.$$

Then $u = u_1 + u_2$.

The AFC scheme is applied to each of these subproblems separately.

Example 3, two-step method, Limiter 4, 21x21 mesh, $\gamma_i = 2$





Example 4 (exponential boundary layers)

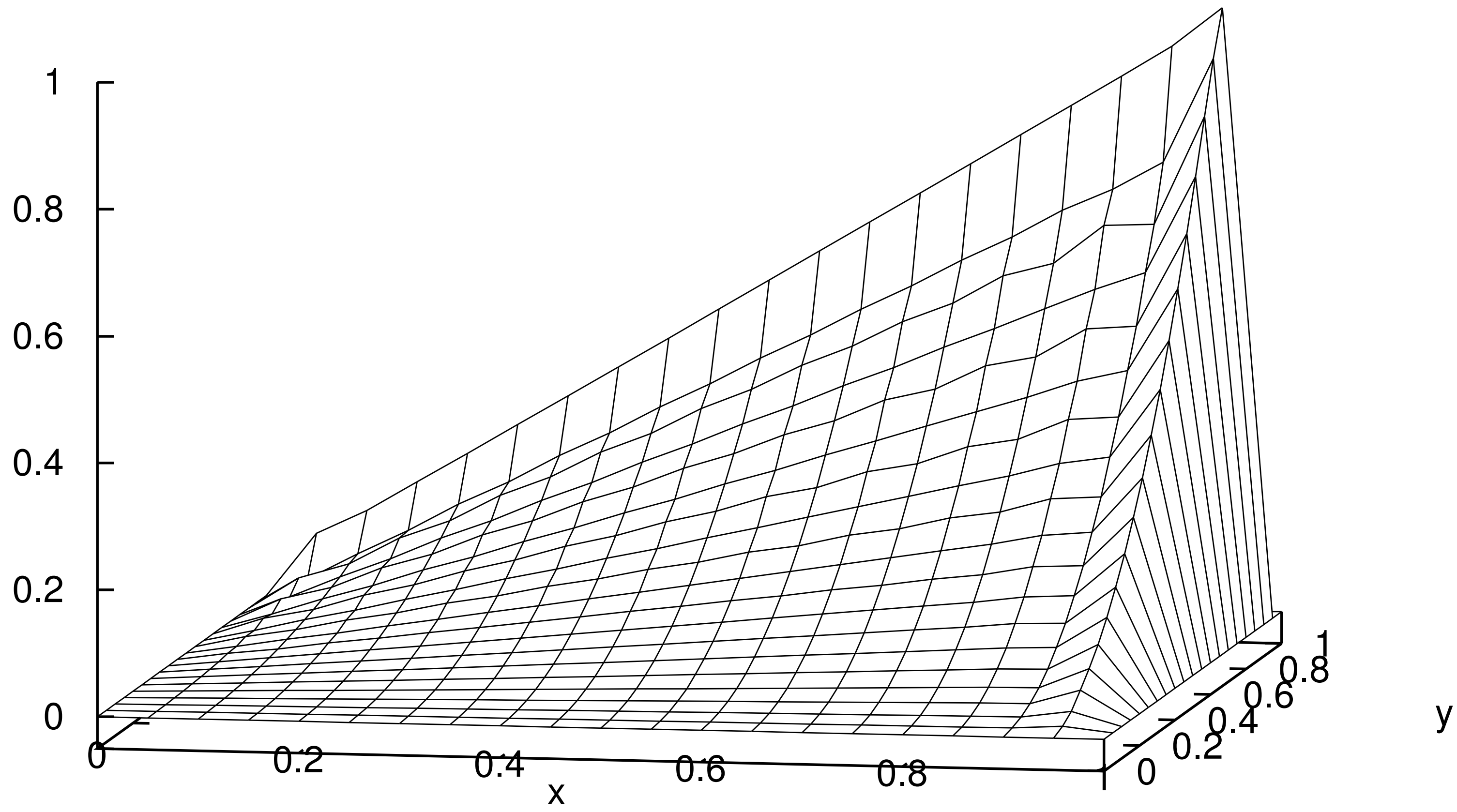
$$\Omega = (0,1)^2\,, \qquad \varepsilon = 10^{-8}\,, \qquad \mathbf{b} = (2,3)\,, \qquad c = 0\,.$$

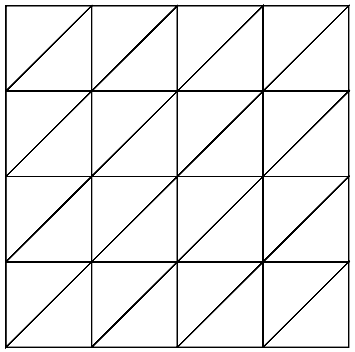
The functions f and u_b are chosen in such a way that

$$u(x,y) = xy^2 - y^2 \exp\left(\frac{2\,(x-1)}{\varepsilon}\right) - x\exp\left(\frac{3\,(y-1)}{\varepsilon}\right) + \exp\left(\frac{2\,(x-1)+3\,(y-1)}{\varepsilon}\right)$$

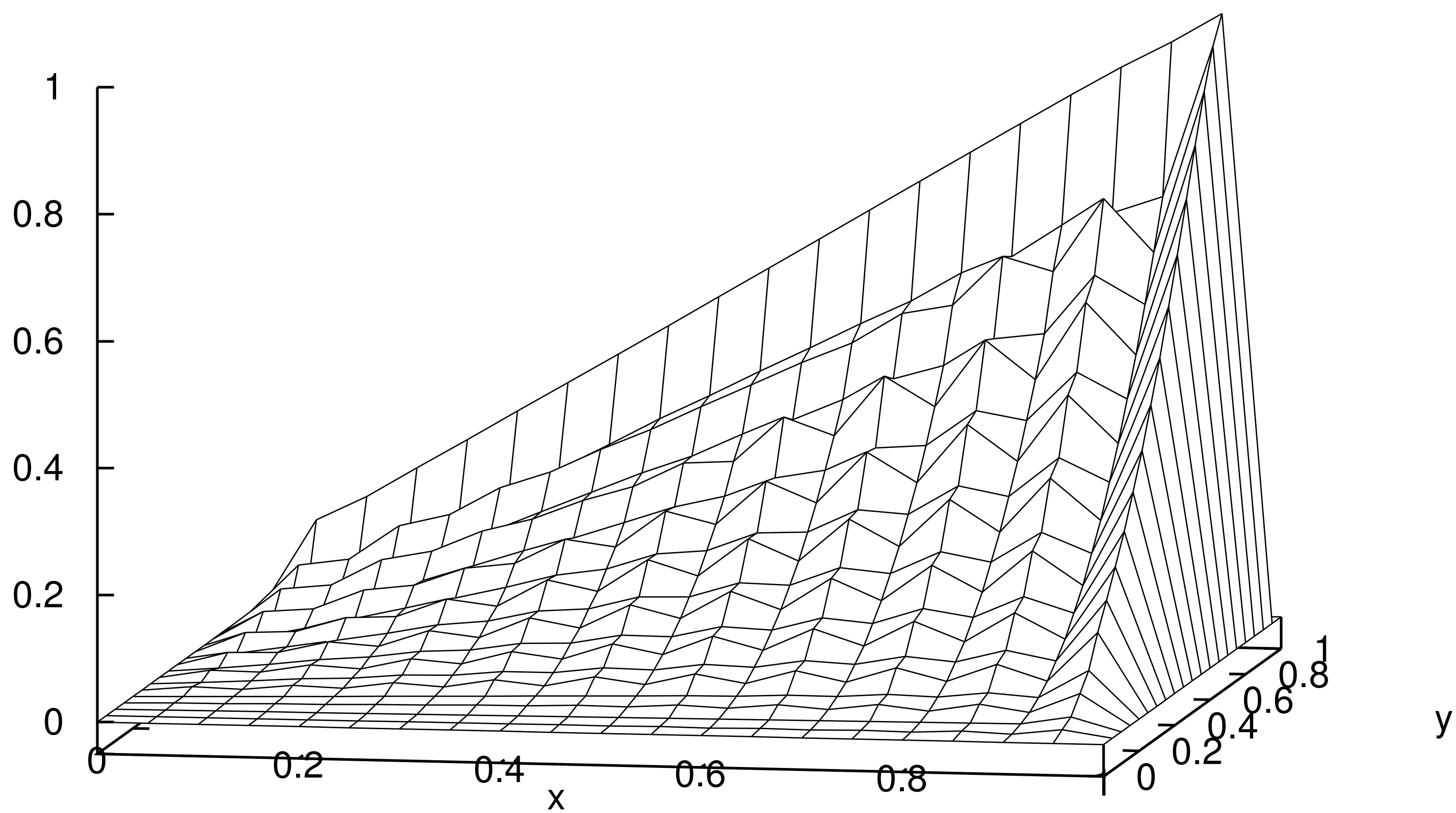
is the exact solution.

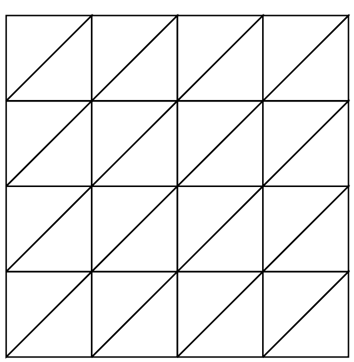
Example 4, Limiter 4, 21x21 mesh, $\gamma_i = 1$



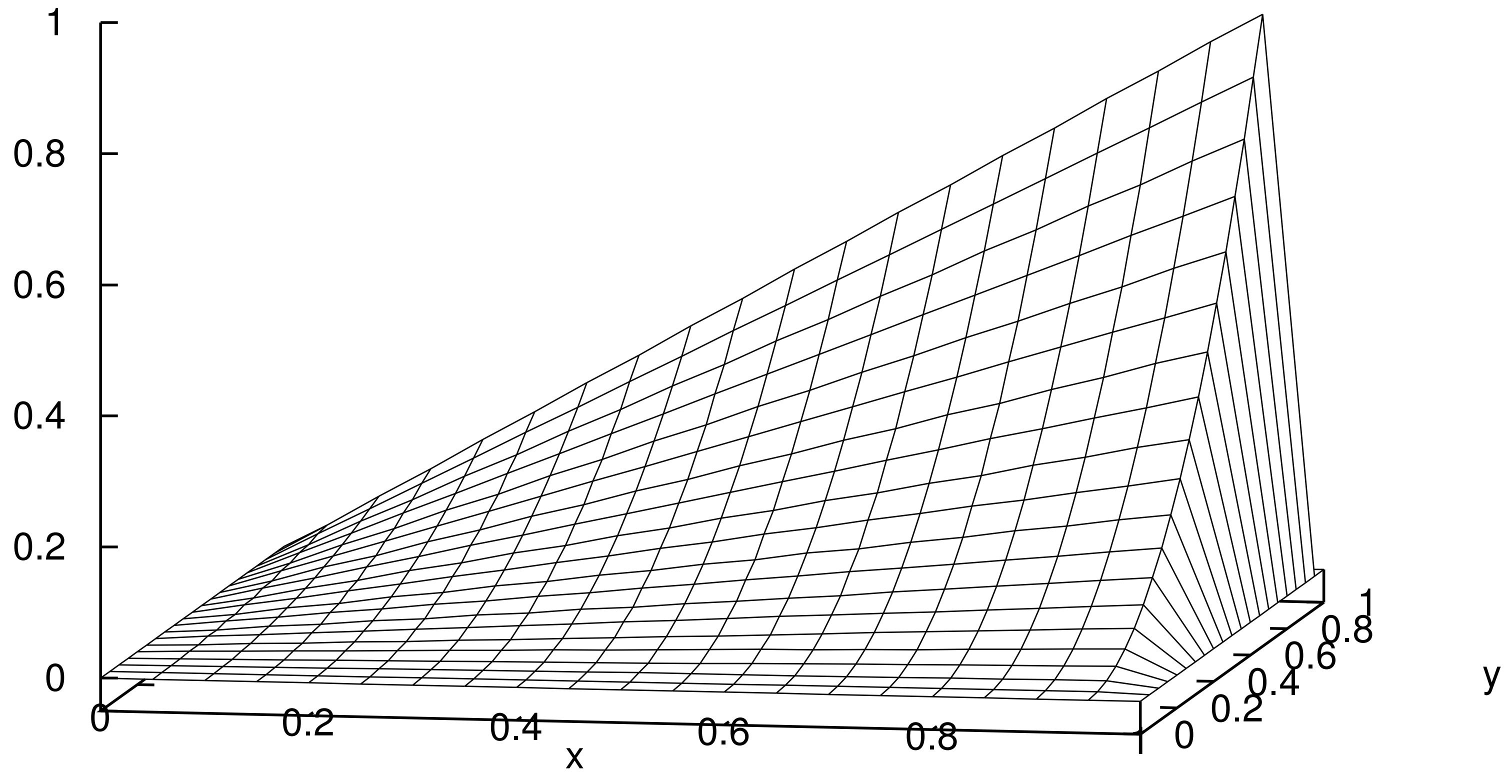


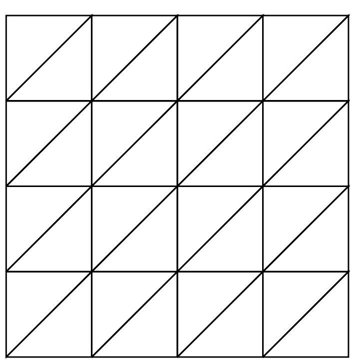
Example 4, Limiter 4, 21x21 mesh, $\gamma_i = 2$





Example 4, two-step method, Limiter 4, 21x21 mesh, $\gamma_i = 2$





Example 4, two-step method, Limiter 4, NxN mesh, $\gamma_i = 2$

N	$\|\|e_h\|\|^*_{0,\Omega}$	$\|e_h\|^*_{1,\Omega}$	$\|e_h\|^*_{0,\infty,\Omega}$	$\|e_h\|_{0,\infty,\Omega}$
6	9.29e−3	1.50e−1	3.01e−2	3.01e−2
11	1.93e−3	7.67e−2	9.25e−3	9.25e−3
21	4.48e−4	3.88e−2	2.12e−3	2.29e−3
41	1.12e−4	1.99e−2	5.43e−4	5.91e−4
81	2.80e−5	1.01e−2	1.37e−4	1.53e−4
ord.	2.00	0.98	1.99	1.95

Conclusions

– general theoretical analysis for algebraic flux correction schemes applied to convection–diffusion–reaction equations

– new limiter of upwind type leading to the discrete maximum principle and linearity preservation on arbitrary meshes

– AFC scheme alone cannot guarantee sharp approximations of boundary layers – the use of appropriate meshes is essential

– two-step method as a remedy for difficulties caused by non-vanishing right-hand side