Morton and his coworkers [BM80, MMS92, MS85] developed a theory of "optimal" Petrov-Galerkin methods. The basic idea is quite elegant: as seen in Section 2.2.1, for symmetric problems the Ritz-Galerkin technique is optimal with respect to the energy norm, so one tries to find test functions that yield a symmetric (or nearly symmetric) discrete problem. That is, one looks for a surjective mapping $\Phi : \; S_h \to T_h$ such that

$$B_s(v, w) := a(v, \Phi(w))$$

is a symmetric bilinear form. For one-dimensional problems this method works well, but it is difficult to generalize it to higher-dimensional problems, so it will not be discussed further.

Instead of trial and test functions that are linear within each mesh subinterval, O'Riordan [O'R84] proposes the use of *hinged elements*; these are only piecewise linear in each mesh subinterval, thus enabling better approximation of layers. One constructs them by introducing in each subinterval an additional mesh point whose position depends on a local Reynolds number. Recently, in the context of enriching the finite element space by bubble functions, a method using two additional mesh points in each subinterval is proposed in [BHMS03]. This can be considered as an extension of [O'R84] to handle the whole range of convection-diffusion to reaction-diffusion equations.

In recent years many other finite element methods of upwind type such as

- *streamline diffusion method (SDFEM)*
- *variational multiscale method (VMS)*
- *differentiated residual method (DRM)*
- *continuous interior penalty approach (CIP)*
- *Galerkin least squares techniques (GLS)*
- *local projection stabilization (LPS)*
- *discontinuous Galerkin methods (dGFEM)*
- *combined finite volume – finite element approaches (CFVFE)*

have been developed. To give the reader some impression of how higher-order finite element methods can be designed and analysed, the first three methods of this list will be considered in the next subsections; the others are deferred to Parts II and III.

2.2.3 Stabilized Higher-Order Methods

Consider as in Section 2.2.2 the singularly perturbed boundary value problem

$$Lu := -\varepsilon u'' + b(x)u' + c(x)u = f(x) \text{ on } (0,1), \quad u(0) = u(1) = 0, \tag{2.71a}$$

under the assumption that

$$c(x) - b'(x)/2 \geq \omega > 0 \quad \text{for all } x \in [0,1]. \tag{2.71b}$$

Our aim is to create a method that is more stable than the Galerkin approach and can be used for finite elements of any order. The improved stability property will be expressed in terms of a norm stronger than the standard energy norm.

The first idea is to add weighted residuals to the usual Galerkin finite element method. The method is called the streamline-diffusion finite element method (SDFEM); the reason for its name will become clear in the multidimensional case – see the interpretation following Remark III.3.28. Multiply the differential equation (2.71) by bv', integrate over each subinterval (x_{i-1}, x_i) for $i = 1, \dots, N$, and add this weighted sum to the standard Galerkin method; one gets the following discrete problem:

Find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = f_h(v_h) \quad \text{for all } v_h \in V_h, \tag{2.72}$$

where

$$a_h(v, w) := \varepsilon(v', w') + (bv' + cv, w) + \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \delta_i(-\varepsilon v'' + bv' + cv)bw'\,dx,$$

$$f_h(w) := (f, w) + \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \delta_i f\,bw'\,dx.$$

Here, $(\cdot, \cdot)$ denotes the inner product in $L_2(0, 1)$, δ_i is a user chosen parameter, called the SD parameter, which is usually constant on I_i. Note that since $v \in V_h$, in general v'' in $a_h(v, w)$ is defined only piecewise. Nevertheless, for a smooth solution $u \in H^2(0, 1)$ of (2.71) we have

$$a_h(u, v_h) = f_h(v_h) \quad \text{for all } v_h \in V_h. \tag{2.73}$$

A finite element method (2.72) that satisfies (2.73) for a sufficiently smooth solution of (2.71) is said to be *consistent*. This is *not* the same as consistency of a finite difference scheme, which was discussed in Section 2.1.1. Furthermore, finite element consistency implies *Galerkin orthogonality*, viz.,

$$a_h(u - u_h, v_h) = 0 \quad \text{for all } v_h \in V_h.$$

As regards coercivity of the discrete bilinear form $a_h(\cdot, \cdot)$, one has

$$\begin{aligned} a_h(v_h, v_h) &= \varepsilon |v_h|_1^2 + \int_0^1 (c - b'/2)\, v_h^2\,dx \\ &\quad + \sum_{i=1}^{N} \|\sqrt{\delta_i} bv'\|_{0,I_i}^2 + \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \delta_i\,(-\varepsilon v_h'' + cv_h)\, bv_h'\,dx \\ &\geq |||v_h|||_{SD}^2 + \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \delta_i\,(-\varepsilon v_h'' + cv_h)\, bv_h'\,dx, \end{aligned}$$

where $I_i = (x_{i-1}, x_i)$ and $\|.\|_{0,I_i}$ denote the i^{th} subinterval and the $L_2(I_i)$ norm. Furthermore, the streamline diffusion norm $|||\cdot|||_{SD}$ has been introduced:

$$|||v_h|||_{SD} := \left(\varepsilon |v_h|_1^2 + \omega \|v_h\|_0^2 + \sum_{i=1}^{N} \|\sqrt{\delta_i} b v'\|_{0,I_i}^2 \right)^{1/2} .$$

Let $h_i = x_i - x_{i-1}$ be the length of I_i. Using the inverse inequality

$$\|v_h''\|_{0,I_i} \le c_{inv}\, h_i^{-1} \|v_h'\|_{0,I_i}$$

and imposing the requirement on the SD parameter that

$$0 < \delta_i \le \frac{1}{2} \min \left\{ \frac{h_i^2}{\varepsilon c_{inv}^2}, \frac{\omega}{\|c\|_\infty^2} \right\}, \tag{2.74}$$

we estimate

$$\begin{aligned} &\left| \int_{x_{i-1}}^{x_i} \delta_i \left(-\varepsilon v_h'' + c v_h \right) b v_h' \, dx \right| \\ &\qquad \le \left(\sqrt{\frac{\varepsilon}{2}} \frac{h_i}{c_{inv}} \|v_h''\|_{0,I_i} + \sqrt{\frac{\omega}{2}} \|v_h\|_{0,I_i} \right) \|\sqrt{\delta_i} b v_h'\|_{0,I_i} \\ &\qquad \le \frac{\varepsilon}{2} \|v_h'\|_{0,I_i} + \frac{\omega}{2} \|v_h\|_{0,I_i}^2 + \frac{1}{2} \|\sqrt{\delta_i} b v_h'\|_{0,I_i}^2 . \end{aligned}$$

In the case of piecewise linear elements one has $v_h''|_{I_i} = 0$ for $i = 1, \dots, N$ and this inequality is still valid when (2.74) is replaced by the weaker assumption

$$0 < \delta_i \le \frac{\omega}{\|c\|_\infty^2} . \tag{2.75}$$

The above computation proves the following lemma:

Lemma 2.51. *Assume that (2.74) is satisfied. Then the SDFEM discrete bilinear form is coercive, viz.,*

$$a_h(v_h, v_h) \ge \frac{1}{2} |||v_h|||_{SD}^2 \qquad \textit{for all } v_h \in V_h.$$

For piecewise linear elements the assumption (2.74) can be replaced by (2.75).

Remark 2.52. Lemma 2.51 implies stability of the SDFEM with respect to the norm $|||\cdot|||_{SD}$. Now all $v_h \in V_h$ satisfy

$$|||v_h|||_{SD} \ge \min\{1, \omega\} \, \|v_h\|_\varepsilon .$$

Thus the stability of the SDFEM in the norm $|||\cdot|||_{SD}$ is stronger than the stability of the standard Galerkin method in the norm $\|\cdot\|_\varepsilon$. Furthermore, the quantity

$$\sum_{i=1}^{N} \|\sqrt{\delta_i} b u_h'\|_{0,I_i}^2$$

is bounded for the solution u_h of the SDFEM but in general this is not the case for the solution of the Galerkin method. ♣

Take V_h to be the space of piecewise linear functions on an equidistant mesh ($h_i = h$ for $i = 1, \dots, N$). Assume that b, c, f, and $\delta_i = \delta$ for $i = 1, \dots, N$ are all constant. Then the SDFEM reduces to the scheme

$$-(\varepsilon + b^2\delta)D^+D^-u_i + bD^0u_i + cu_i = f,$$

i.e., the fitted scheme (2.15) with $\sigma(q) = 1+b^2\delta/\varepsilon$, $q = bh/(2\varepsilon)$. Recall that for $\sigma(q) = q \coth q$ one gets the Il'in-Allen-Southwell scheme, which corresponds to choosing the SD parameter to be

$$\delta(q) = \frac{h}{2b}\left(\coth q - \frac{1}{q}\right).$$

Since

$$\coth q - \frac{1}{q} = \frac{q}{3} + \mathcal{O}(q^3) \quad \text{as } q \to 0 \quad \text{and} \quad \coth q - \frac{1}{q} = 1 + \mathcal{O}\left(\frac{1}{q}\right) \quad \text{as } q \to \infty,$$

the asymptotic limits $h \to 0$ for fixed ε, and $\varepsilon \to 0$ for fixed h, motivate the following choices of δ:

$$\delta(q) = \begin{cases} h^2/(12\varepsilon) & \text{if } 0 < q \ll 1, \\ h/(2b) & \text{if } q \gg 1. \end{cases} \tag{2.76}$$

The choice $\delta(q) = h/(2b)$ for $q \in (0, \infty)$ generates the simple upwind scheme.

We now study the convergence properties of the SDFEM in the case where $V_h \subset H_0^1(0,1)$ is the finite element space of piecewise polynomials of degree $k \geq 1$. For the nodal interpolant $u^I \in V_h$, one has the estimates

$$|u^I - u|_l \leq Ch^{k+1-l}|u|_{k+1} \quad \text{for } l = 0, \dots, k+1.$$

Theorem 2.53. *Let the SD parameter be specified by*

$$\delta_i = \begin{cases} C_0\, h_i^2/\varepsilon & \textit{if } \; h_i < \varepsilon, \\ C_0\, h_i & \textit{if } \; \varepsilon < h_i, \end{cases} \tag{2.77}$$

where the constant C_0 is small enough to satisfy (2.74) if $k \geq 2$ and (2.75) if $k = 1$. Then using piecewise polynomials of degree k, the solution u_h of the SDFEM satisfies the error estimate

$$|||u - u_h|||_{SD} \leq C(\varepsilon^{1/2}h^k + h^{k+1/2})\,|u|_{k+1}.$$

Proof. The coercivity of a_h (Lemma 2.51) and Galerkin orthogonality yield

$$\frac{1}{2}|||u^I - u_h|||_{SD}^2 \leq a_h(u^I - u_h, u^I - u_h) = a_h(u^I - u, u^I - u_h).$$

Each term in $a_h(u^I - u, u^I - u_h)$ will be estimated separately. Set $w_h = u^I - u_h$. First,

$$\left|\varepsilon((u^I - u)', w_h')\right| \le C\varepsilon^{1/2} h^k |u|_{k+1}\, |||w_h|||_{SD},$$
$$\left|(c(u^I - u), w_h)\right| \le C h^{k+1} |u|_{k+1}\, |||w_h|||_{SD}.$$

Then, using $\varepsilon\delta_i \le C_0\, h_i^2$ and $\delta_i \le C_0\, h_i$ we obtain

$$\begin{aligned}\left|\sum_{i=1}^N (-\varepsilon(u^I - u)'', \delta_i b w_h')_{I_i}\right| &\le C\sum_{i=1}^N \varepsilon^{1/2} h_i\, \|(u^I - u)''\|_{0,I_i} \|\sqrt{\delta_i} b w_h'\|_{0,I_i} \\ &\le C\varepsilon^{1/2} h^k |u|_{k+1}\, |||w_h|||_{SD},\end{aligned}$$

$$\left|\sum_{i=1}^N (b(u^I - u)' + c(u^I - u), \delta_i b w_h')_{I_i}\right| \le C(h^{k+1/2} + h^{k+3/2})\, |u|_{k+1}\, |||w_h|||_{SD}$$

It remains to estimate the convection term. The standard estimate would be

$$|(b(u^I - u)', w_h)| \le C h^k \|w_h\|_0 \le C h^k |u|_{k+1} |||w_h|||_{SD}$$

but thanks to the additional term $\sum_{i=1}^N \|\sqrt{\delta_i} b v'\|_{0,I_i}^2$ in the norm $|||\cdot|||_{SD}$, this estimate can be improved. To this end, one integrates by parts to get

$$|(b(u^I - u)', w_h)| \le |((u^I - u), b w_h')| + |((u^I - u), b' w_h)|$$

Here the second term is estimated in a standard way:

$$|((u^I - u), b' w_h)| \le C h^{k+1} |u|_{k+1} \|w_h\|_0 \le C h^{k+1} |u|_{k+1} |||w_h|||_{SD}.$$

The bound on the first term depends on $\varepsilon \le h_i$ or $\varepsilon > h_i$:

$$\begin{aligned}\left|\sum_{i=1}^N ((u^I - u), b w_h')_{I_i}\right| &\le C\sum_{\varepsilon \le h_i} \delta_i^{-1/2} h_i^{k+1} |u|_{k+1,I_i} \|\sqrt{\delta_i} b w_h'\|_{0,I_i} \\ &\quad + C\sum_{\varepsilon > h_i} h_i^{k+1/2} |u|_{k+1,I_i} \varepsilon^{1/2} |w_h|_1 \\ &\le C h^{k+1/2} |u|_{k+1} |||w_h|||_{SD}.\end{aligned}$$

Collecting all these estimates completes the proof of the theorem. □

The Cea lemma, Theorem 2.44, gives a quasi-optimal error estimate whose constant multiplier depends on the data of the problem. It says that the error is, up to a constant factor, less than or equal to the approximation error. Such an error estimate is highly desirable since it reduces the question of constructing a good solution to the corresponding task in approximation theory. In Section 2.2.2 the error $u - u_h$ has been measured in the energy norm $\|\cdot\|_\varepsilon = (\varepsilon|\cdot|_1^2 + |\cdot|_0^2)^{1/2}$, which forms part of the SD norm. But recalling (2.66), we have no uniform quasi-optimal error estimate in the norm $\|\cdot\|_\varepsilon$. Before considering the question of finding an appropriate norm in which a uniform quasi-optimal error estimate can be given, we demonstrate why the standard

H^1 norm $\|\cdot\|_1$ and the energy norm $\|\cdot\|_\varepsilon$ seem unsuited to our singularly perturbed problem.

Under the hypothesis (2.71b), the operator $\mathcal{L}_\varepsilon : H_0^1(0,1) \to H^{-1}(0,1)$ defined by

$$\langle \mathcal{L}_\varepsilon v, w\rangle = a(v,w) \quad \text{for all } v,w \in H_0^1(0,1)$$

is for each $\varepsilon > 0$ an isomorphism from $H_0^1(0,1)$ onto $H^{-1}(0,1)$. Let us consider two norms $\|\cdot\|_S$ and $\|\cdot\|_T$ on $H_0^1(0,1)$ that are equivalent for fixed ε and are such that the continuity and inf-sup conditions

$$|a(v,w)| \le \beta \|v\|_S \, \|w\|_T \qquad \text{for all } v,w \in H_0^1(0,1), \tag{2.78}$$

$$\inf_{v\in H_0^1(0,1)} \sup_{w\in H_0^1(0,1)} \frac{a(v,w)}{\|v\|_S \, \|w\|_T} \ge \alpha > 0, \tag{2.79}$$

hold true. From these inequalities one can deduce immediately that

$$\|\mathcal{L}_\varepsilon^{-1}\| := \sup_{f\in H^{-1}(0,1)} \frac{\|\mathcal{L}_\varepsilon^{-1} f\|_S}{\|f\|_{*,T}} = \sup_{v\in H_0^1(0,1)} \frac{\|v\|_S}{\|\mathcal{L}_\varepsilon v\|_{*,T}} \le \frac{1}{\alpha}\,,$$

$$\|\mathcal{L}_\varepsilon\| := \sup_{v\in H_0^1(0,1)} \frac{\|\mathcal{L}_\varepsilon v\|_{*,T}}{\|v\|_S} = \sup_{v\in H_0^1(0,1)} \sup_{w\in H_0^1(0,1)} \frac{\langle \mathcal{L}_\varepsilon v, w\rangle}{\|v\|_S \|w\|_T} \le \beta,$$

where $\|\cdot\|_{*,T}$ denotes the dual norm in $H^{-1}(0,1)$ defined by

$$\|f\|_{*,T} := \sup_{w\in H_0^1(0,1)} \frac{\langle f, w\rangle}{\|w\|_T}\,.$$

If α and β are independent of ε, then one can consider the norms $\|v\|_S$ and $\|w\|_T$ as natural for $\mathcal{L}_\varepsilon$ because for a given source term f and a perturbed source term $f + \delta f$ the relative perturbation in the solution is uniformly bounded by the relative perturbation of the source term. Indeed, if u and $u + \delta u$ denote the corresponding solutions, then

$$\frac{\|\delta u\|_S}{\|u\|_S} = \frac{\|\mathcal{L}_\varepsilon^{-1} \delta f\|_S}{\|u\|_S} \le \frac{\beta}{\alpha} \frac{\|\delta f\|_{*,T}}{\|\mathcal{L}_\varepsilon u\|_{*,T}} = \frac{\beta}{\alpha} \frac{\|\delta f\|_{*,T}}{\|f\|_{*,T}}\,.$$

If however $\|\cdot\|_S = \|\cdot\|_T = \|\cdot\|_1$, then (2.78) and (2.79) hold true only with constants α and β that depend on ε; this implies that

$$\|\mathcal{L}_\varepsilon^{-1}\| \le \frac{1}{\alpha} = \mathcal{O}\left(\frac{1}{\varepsilon}\right), \qquad \|\mathcal{L}_\varepsilon\| \le \beta = \mathcal{O}(1).$$

On the other hand, for $\|\cdot\|_S = \|\cdot\|_T = \|\cdot\|_\varepsilon$ one obtains

$$\|\mathcal{L}_\varepsilon^{-1}\| \le \frac{1}{\alpha} = \mathcal{O}(1), \qquad \|\mathcal{L}_\varepsilon\| \le \beta = \mathcal{O}\left(\frac{1}{\varepsilon}\right).$$

Suppose that we have appropriate norms $\|\cdot\|_S$ and $\|\cdot\|_T$ such that (2.78) and (2.79) hold with constants α and β that are independent of ε. Then one might hope that these inequalities yield a uniform quasi-optimal convergence result with respect to $\|v\|_S$, similarly to the Cea lemma, Theorem 2.44 – but this is not true. The reason is that the inf-sup condition (2.79) is weaker than the coercivity condition (2.52): imitating the proof of Theorem 2.44 one gets

$$\begin{aligned}\|u-u_h\|_S &\le \|u-v_h\|_S + \|v_h-u_h\|_S \\ &\le \|u-v_h\|_S + \frac{1}{\alpha} \sup_{w\in H_0^1(0,1)} \frac{a(v_h-u_h,w)}{\|w\|_T}\end{aligned}$$

but after using Galerkin orthogonality to replace $a(v_h-u_h,w)$ by $a(v_h-u,w)$, we are unable to take an infimum of the right-hand side over $H_0^1(0,1)$ – we can take the infimum only over the finite element space V_h where v_h lies. To surmount this obstacle, one needs an additional inf-sup condition on the discrete spaces S_h and T_h:

$$\inf_{v_h\in S_h} \sup_{w_h\in T_h} \frac{a(v_h,w_h)}{\|v_h\|_S\,\|w_h\|_T} \ge \alpha_1 > 0. \tag{2.80}$$

Then one can argue that

$$\begin{aligned}\|v_h-u_h\|_S &\le \frac{1}{\alpha_1} \sup_{w_h\in T_h} \frac{a(v_h-u_h,w_h)}{\|w_h\|_T} = \frac{1}{\alpha_1} \sup_{w_h\in T_h} \frac{a(v_h-u,w_h)}{\|w_h\|_T} \\ &\le \frac{\beta}{\alpha_1}\|v_h-u\|_S\end{aligned}$$

and use a triangle inequality to get the uniform quasi-optimal estimate

$$\|u-u_h\|_S \le \left(1+\frac{\beta}{\alpha_1}\right) \inf_{v_h\in S_h} \|u-v_h\|_S.$$

An investigation of norms $\|\cdot\|_S$ and $\|\cdot\|_T$ such that (2.78)–(2.80) are satisfied has been carried out by Sangalli [San05, San08].

Following [San05], we consider the simple model problem in which $b=1$ and $c=0$. Thus the bilinear form $a(\cdot,\cdot)$ becomes

$$a(v,w) := \varepsilon(v',w') + (w',v) \quad \text{for all } v,w\in H_0^1(0,1).$$

Let $L_0^2(0,1)$ denote the subset of $L^2(0,1)$ comprising functions of zero mean value. Let $\Pi_0 : L^2(0,1) \to L_0^2(0,1)$ be the L^2 projection onto $L_0^2(0,1)$ such that $(\Pi_0 w - w, v) = 0$ for all $v\in L_0^2(0,1)$ and $w = \Pi_0 w + \overline{w}$ where $\overline{w}$ denotes the mean value of w. The convection term (v',w) can be estimated via

$$|(v',w)| = |((\Pi_0 v)',w)| = |-(\Pi_0 v, w')| \le \|\Pi_0 v\|_0\,|w|_1$$

or equivalently, integrating by parts,

$$|(v', w)| = |-(v, w')| = |-(v, (\Pi_0 w)')| = |(v', \Pi_0 w)| \le |v|_1 \, \|\Pi_0 w\|_0,$$

which results in two continuity estimates of the form (2.78):

$$|a(v,w)| \le (\varepsilon|v|_1 + \|\Pi_0 v\|_0)\,|w|_1,$$
$$|a(v,w)| \le |v|_1\,(\varepsilon|w|_1 + \|\Pi_0 v\|_0).$$

Thus we shall consider $\varepsilon|\cdot|_1 + \|\Pi_0(\cdot)\|_0$ and $|\cdot|_1$ – or vice versa – as candidates for $\|\cdot\|_S$ and $\|\cdot\|_T$. The coercivity of $a(\cdot,\cdot)$ gives

$$\varepsilon|v|_1 \le \sup_{w\in H_0^1(0,1)} \frac{a(v,w)}{|w|_1} \qquad \text{for all } v \in H_0^1(0,1).$$

In order to show also that

$$\|\Pi_0 v\|_0 \le C \sup_{w\in H_0^1(0,1)} \frac{a(v,w)}{|w|_1} \qquad \text{for all } v \in H_0^1(0,1),$$

one uses the norm relationships $\|\Pi_0 v\|_0 \le C\,\|v'\|_{-1}$ and

$$\begin{aligned}\|v'\|_{-1} &= \sup_{w\in H_0^1(0,1)} \frac{(v',w)}{|w|_1} = \sup_{w\in H_0^1(0,1)} \frac{a(v,w) - \varepsilon(v',w')}{|w|_1}\\ &\le \sup_{w\in H_0^1(0,1)} \frac{a(v,w)}{|w|_1} + \varepsilon|v|_1 \le 2 \sup_{w\in H_0^1(0,1)} \frac{a(v,w)}{|w|_1}\,.\end{aligned}$$

Hence

$$\alpha(\varepsilon|v|_1 + \|\Pi_0 v\|_0) \le \sup_{w\in H_0^1(0,1)} \frac{a(v,w)}{|w|_1}$$

where α is independent of ε. A duality argument delivers the other estimate

$$\alpha\,|v|_1 \le \sup_{w\in H_0^1(0,1)} \frac{a(v,w)}{\varepsilon|w|_1 + \|\Pi_0 w\|_0}\,.$$

Thus, in agreement with our earlier discussion, the norms

$$v \mapsto \varepsilon|v|_1 + \|\Pi_0 v\|_0 \quad \text{and} \quad v \mapsto |v|_1$$

are suitable for this model problem.

In [San05] Sangalli proved a discrete inf-sup condition of type (2.80) from which uniform quasi-optimality with respect to the two norms follows.

Lemma 2.54. *Consider the bilinear form $a_h(\cdot,\cdot)$ of the SDFEM (2.72) with $b = 1$ and $c = 0$. Let V_h be the space of piecewise linear functions on an equidistant mesh. Then there is a constant α_1, which is independent of ε, such that*

$$\alpha_1(\varepsilon|v_h|_1 + \|\Pi_0 v_h\|_0) \le \sup_{w_h\in V_h} \frac{a_h(v_h,w_h)}{|w_h|_1} \qquad \forall v_h \in V_h,$$

$$\alpha\,|v_h|_1 \le \sup_{w_h\in V_h} \frac{a_h(v_h,w_h)}{\varepsilon|w_h|_1 + \|\Pi_0 w_h\|_0} \qquad \forall v_h \in V_h.$$

Proof. See [San05, Lemma 3.1]. □

Lemma 2.54 is the basis for using interpolation theory to construct a family of norms in which the SDFEM yields uniform quasi-optimal estimates; see [San05]. Note that the analysis given in [San05] is restricted to the model problem ($b = 1$ and $c = 0$) in one space dimension.

Next, following [CX08, CX05], we show that a variant of the SDFEM for continuous piecewise linear finite elements on an arbitrary family of meshes yields a solution u_h that is quasi-optimal with respect to the L^∞ norm, viz.,

$$\|u - u_h\|_\infty \le C \inf_{v_h \in V_h} \|u - v_h\|_\infty.$$

To concentrate on the main ideas, consider the simple model problem

$$-\varepsilon u'' + bu' = f \quad \text{on } (0,1), \qquad u(0) = u(1) = 0,$$

where b is a positive constant and f a given function. For a positive integer N, let $\mathcal{T}_N = \{x_i : 0 = x_0 < x_1 < \cdots < x_N = 1\}$ be an arbitrary grid with $h_i = x_i - x_{i-1}$ the local mesh size and $\{\varphi_i\}$ the standard piecewise linear hat functions that satisfy $\varphi_i(x_j) = \delta_{ij}$ for $i, j = 0, 1, \ldots, N$. Let the finite element space be

$$V_h := \operatorname{span}\{\varphi_1, \ldots, \varphi_{N-1}\} \subset H_0^1(0,1).$$

The SDFEM (2.72) can be written in the form

$$\text{Find } u_h \in V_h \text{ such that} \quad a_h(u_h, v_h) = f_h(v_h) \quad \text{for all } v_h \in V_h$$

where

$$a_h(v,w) := \varepsilon(v', w') + (bv', w) + \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \delta_i(-\varepsilon v'' + bv')bw'\,dx,$$

$$f_h(w) := (f, w) + \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \delta_i f\,bw'\,dx.$$

Unlike the usual choice of a piecewise-constant SD parameter, here we take

$$\delta_i := \frac{3h_i}{b} \min\{1, q_i\}\varphi_{i-1}(x)\varphi_i(x), \qquad q_i = \frac{bh_i}{2\varepsilon}\,. \tag{2.81}$$

Nevertheless the maximum of δ_i has the asymptotic behaviour (2.77) in the diffusion-dominated and convection-dominated regimes.

Let $A = (a_h(\varphi_j, \varphi_i))$, for $i, j = 1, \ldots, N-1$, be the coefficient matrix of the corresponding algebraic system. For $u_i = u_h(x_i)$ a direct calculation gives

$$\begin{aligned} -\left(\frac{\varepsilon + \overline{\delta}_i b^2}{h_i} + \frac{b}{2}\right) u_{i-1} + \left(\frac{\varepsilon + \overline{\delta}_i b^2}{h_i} + \frac{\varepsilon + \overline{\delta}_{i+1} b^2}{h_{i+1}}\right) u_i \\ - \left(\frac{\varepsilon + \overline{\delta}_{i+1} b^2}{h_{i+1}} - \frac{b}{2}\right) u_{i+1} = f_h(\varphi_i) \end{aligned} \tag{2.82}$$

where

$$\overline{\delta}_i := \frac{1}{h_i} \int_{x_{i-1}}^{x_i} \delta_i(x)\, dx = \frac{h_i}{2b} \min\{1, q_i\}.$$

Observe that

$$\frac{\varepsilon + \overline{\delta}_{i+1} b^2}{b h_{i+1}} = \frac{1 + q_{i+1} \min(1, q_{i+1})}{2q_{i+1}} = \begin{cases} \dfrac{1 + q_{i+1}^2}{2q_{i+1}} > 1 & \text{for } 0 < q_{i+1} < 1, \\[2ex] \dfrac{1 + q_{i+1}}{2q_{i+1}} > \dfrac{1}{2} & \text{for } q_{i+1} \geq 1, \end{cases}$$

so the matrix A of (2.82) is an M-matrix. The following uniform stability result is established in [CX05] by studying the properties of the discrete Green's function (compare Section 1.1.2 for the continuous analogue):

Lemma 2.55. *Define δ_i by* (2.81)*. Then the SDFEM is uniformly $(l_\infty, w^{-1,\infty})$ stable, i.e.,*

$$\|v_h\|_{\infty,d} \leq \frac{2}{b} \max_{j=1,\dots,N-1} \left| \sum_{k=j}^{N-1} (Av_h)_k \right| \qquad \forall v_h \in V_h,$$

where the right-hand side defines the discrete analogue of the norm $W^{-1,\infty}$.

Now consider the error $e_h = u^I - u_h \in V_h$ where u^I is the nodal interpolant. The consistency property $a_h(u, v_h) = f_h(v_h)$ for all $v_h \in V_h$ implies that (provided the solution u is sufficiently smooth) the error e_h is the solution of the problem

Find $e_h \in V_h$ such that $\quad a_h(e_h, v_h) = a_h(u^I - u, v_h) \quad$ for all $v_h \in V_h$.

Using $(u^I - u)(x_i) = 0$ for $i = 0, \dots, N$ and integration by parts, one sees that

$$(Ae_h)_k = a_h(e_h, \varphi_k) = a_h(u^I - u, \varphi_k) = r_k - r_{k+1}$$

where

$$r_k := \frac{b}{h_k} \Big[- \int_{x_{k-1}}^{x_k} (u^I - u)(x)\, dx + \int_{x_{k-1}}^{x_k} \delta_k(x) \varepsilon u''(x)\, dx \\ + \int_{x_{k-1}}^{x_k} b\delta_k(x)(u^I - u)'(x)\, dx. \Big]$$

Since the SD parameter δ_i vanishes at the mesh points, one can show by means of integration by parts that

$$\left|\frac{b}{h_k}\int_{x_{k-1}}^{x_k}(u^I-u)(x)\,dx\right| \le b\|u-u^I\|_\infty,$$

$$\begin{aligned}\left|\frac{b}{h_k}\int_{x_{k-1}}^{x_k}\delta_k(x)\varepsilon u''(x)\,dx\right| &= \frac{b}{h_k}\left|\int_{x_{k-1}}^{x_k}\delta_k''(x)\varepsilon(u-u^I)(x)\,dx\right| \\ &\le \frac{3b}{q_k}\min(1,q_k)\|u-u^I\|_\infty \le 3b\|u-u^I\|_\infty,\end{aligned}$$

$$\begin{aligned}\left|\frac{b}{h_k}\int_{x_{k-1}}^{x_k}b\delta_k(x)(u^I-u)'(x)\,dx\right| &= \frac{b^2}{h_k}\left|\int_{x_{k-1}}^{x_k}\delta_k'(x)(u^I-u)(x)\,dx\right| \\ &\le 3b\|u-u^I\|_\infty.\end{aligned}$$

Gathering all these bounds gives

$$|r_k| \le 7b\|u-u^I\|_\infty. \tag{2.83}$$

The discretization error can now be estimated using the interpolation error.

Lemma 2.56. *Let u_h be the solution of the SDFEM with δ_i given by* (2.81). *Then there is a positive constant C, independent of ε and the mesh, such that*

$$\|u-u_h\|_\infty \le C\|u-u^I\|_\infty.$$

Proof. By Lemma 2.55 and (2.83),

$$\begin{aligned}\|u^I-u_h\|_\infty &\le \frac{2}{b}\max_{j=1,\dots,N-1}\left|\sum_{k=j}^{N-1}(Ae_h)_k\right| = \frac{2}{b}\max_{j=1,\dots,N-1}|r_j-r_N| \\ &\le 28\|u-u^I\|_\infty\end{aligned}$$

and the desired estimate follows from the triangle inequality. □

Theorem 2.57. *Let u_h be the solution of the SDFEM with δ_i given by* (2.81). *Then there is a positive constant C, independent of ε and the mesh, such that*

$$\|u-u_h\|_\infty \le C\inf_{v_h\in V_h}\|u-v_h\|_\infty.$$

That is, the SDFEM is quasi-optimal in the L_∞ norm.

Proof. Let $P_h : H_0^1(0,1) \to V_h$ denote the solution operator of the SDFEM, i.e., $P_h u := u_h$. From Lemma 2.56 we infer that

$$\|u-u_h\|_\infty \le C\|u-u^I\|_\infty \le C(\|u\|_\infty + \|u^I\|_\infty) \le C\|u\|_\infty.$$

Thus the operator P_h is L_∞ stable since

$$\|P_h u\|_\infty = \|u_h\|_\infty \le \|u\|_\infty + \|u-u_h\|_\infty \le C\|u\|_\infty.$$

But $P_h^2 = P_h$, so for any $v_h \in V_h$ one has

$$\|u-u_h\|_\infty = \|(I-P_h)(u-v_h)\|_\infty \le C\|u-v_h\|_\infty.$$

The proof is then finished by taking the infimum over all $v_h \in V_h$. □

Remark 2.58. The quasi-optimality result of Lemma 2.56 reduces the question of L_∞-norm convergence of the SDFEM to a problem in approximation. Thus if layer-adapted meshes are used, convergence can be established in the L_∞ norm uniformly with respect to ε. Of course a detailed knowledge of the analytical structure of the solution u is needed in order to create a layer-adapted mesh. ♣

Remark 2.59. A quasi-optimality result in an L_p-type norm (where $1 \le p \le \infty$ is arbitrary) for a Petrov-Galerkin finite element method is given in [SB84]. This result could also be used to get a uniform convergence result on a suitable layer-adapted mesh. Moreover, [SB84] contains an asymptotically exact error estimator; such estimators will be the main topic of Section III.3.6. ♣

2.2.4 Variational Multiscale and Differentiated Residual Methods

The variational multiscale method (VMS) [HFMQ98, Hug95, HS07] was introduced to provide a framework for a better understanding of fine-to-coarse scale effects and as a platform for the development of new numerical methods.

We derive the method for the two-point boundary value problem

$$-\varepsilon u'' + b(x)u' + c(x)u = f(x) \quad \text{in } (0,1), \quad u(0) = u(1) = 0, \tag{2.84}$$

with sufficiently smooth functions b, c and f, where the parameter ε satisfies $0 < \varepsilon \ll 1$. Assume that

$$c(x) - \frac{1}{2}b'(x) \ge \omega > 0 \quad \text{for } x \in [0,1], \tag{2.85}$$

which guarantees the unique solvability of the problem.

The weak formulation of (2.84) is given by:

Find $u \in V := H_0^1(0,1)$ such that for all $v \in V$ one has

$$a(u,v) := \varepsilon(u',v') + (bu' + cu, v) = (f,v)\,. \tag{2.86}$$

The basic idea of the VMS approach is to split the solution space V into resolvable and unresolvable scales. This is done by choosing a finite element space V_h that represents the resolvable scales and a projection operator $P : V \to V_h$ such that

$$V = V_h \oplus V^\diamond, \qquad \text{so } u = Pu + (I-P)u = u_h + u^\diamond.$$

Now the weak formulation (2.86) can be restated as:

Find $u_h \in V_h$ and $u^\diamond \in V^\diamond$ such that

$$a(u_h + u^\diamond, v_h) = (f, v_h) \quad \forall v_h \in V_h, \tag{2.87a}$$
$$a(u_h + u^\diamond, v^\diamond) = (f, v^\diamond) \quad \forall v^\diamond \in V^\diamond. \tag{2.87b}$$