Fluids under Pressure

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Preface

This book is based on lectures being presented at a thematic summer school named Fluids under Pressure, that was held in Prague in August 2016 in the series Prague-Sum events started back in 2011 . The aim of this monograph is to cover various roles of pressure in physics as well as in mathematical modeling and analysis of fluids flows problems. The pressure is a common denominator in all the chapters of the book. Besides of several theoretical problems concerning namely the well-posedness of the Navier-Stokes equations, some chapters are devoted to the role of pressure in finite-element and finite-volume methods and their CFD applications. All the chapters are written by world renown experts in the corresponding fields, which makes this volume an excellent summary of state of the art knowledge in this area.

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Finite Element Pressure Stabilizations for Incompressible Flow Problems

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3.1 Introduction

The behavior of incompressible flows is modeled by the incompressible Navier–Stokes equations, given here already in dimensionless form,

$$\begin{array}{rcll} \partial_t \boldsymbol{u} - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p & = & \boldsymbol{f} & \text{in } (0,T]\times\Omega, \\ \nabla\cdot\boldsymbol{u} & = & 0 & \text{in } (0,T]\times\Omega, \end{array} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, is the flow domain, T the final time, $\boldsymbol{u}$ the velocity field, p the pressure, ν the (dimensionless) kinematic viscosity, and $\boldsymbol{f}$ represents forces acting on the fluid. The first equation describes the conservation of linear momentum and the second equation, the so-called continuity equation, the conservation of mass. System (3.1) has to be equipped with an initial velocity condition and with boundary conditions on the boundary $\partial\Omega$.

There are three aspects that might lead to difficulties in the analysis and numerical simulation of the incompressible Navier–Stokes equations:

- It is a coupled system with two unknowns, where the pressure does not appear in the continuity equation. One obtains a so-called saddle point problem.
- The Navier–Stokes equations form a nonlinear system.

- In the case of (very) small viscosities, the first order term $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$ dominates in the momentum equation. This situation corresponds to turbulent flows. System (3.1) is convection-dominated and its numerical simulation requires special approaches, so-called turbulence models.

This review will discuss numerical methods for treating the coupling of velocity and pressure. To concentrate on this issue, it suffices to consider the (scaled) stationary Stokes equations with homogeneous Dirichlet boundary conditions

$$\begin{aligned} -\nu\Delta\boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u} &= 0 \quad \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} \quad \text{on } \partial\Omega. \end{aligned} \tag{3.2}$$

System (3.2) is a linear saddle point problem. The theory of linear saddle point problems was developed in the early 1970s in the seminal papers [6, 24]. In this theory, the weak or variational form of (3.2) is studied. It turns out that this form is well posed, i.e., there exists a unique solution that depends continuously on the right-hand side, if the spaces V for the velocity and Q for the pressure are chosen appropriately.

Applying a Galerkin finite element method to discretize the variational form of the Stokes equations, i.e., solely replacing the infinite-dimensional spaces V and Q with finite-dimensional spaces V^h and Q^h, leads to a finite-dimensional linear saddle point problem, whose algebraic form is

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{0} \end{pmatrix}.$$

From the theory of linear saddle point problems, it follows that the Galerkin finite element method is only well posed for appropriate choices of the finite element spaces. Concretely, the spaces have to satisfy a discrete inf-sup condition

$$\inf_{q^h\in Q^h\setminus\{0\}} \sup_{\boldsymbol{v}^h\in V^h\setminus\{\boldsymbol{0}\}} \frac{\left(\nabla\cdot\boldsymbol{v}^h, q^h\right)}{\|\nabla\boldsymbol{v}^h\|_{L^2(\Omega)}\|q^h\|_{L^2(\Omega)}} \geq \beta_{\text{is}}^h > 0. \tag{3.3}$$

For obtaining optimal order convergence, β_{is}^h has to be independent of the mesh width h.

In practice, it turns out that the inequality (3.3) requires the use of different finite element spaces for velocity and pressure. However, it was proved that lowest order spaces, using continuous linear or d-linear functions for the finite element velocity and piecewise constant functions for the discrete pressure, do not satisfy (3.3). Thus, implementing finite element methods that respect (3.3) requires some effort. Another issue in practice is that many standard preconditioners for iterative solvers of linear systems of equations cannot be applied to linear saddle point problems due to the zeros in the main diagonal of the system matrix.

In view of these drawbacks, numerical methods were developed in order to circumvent the discrete inf-sup condition (3.3). The main idea of these so-called pressure stabilizations consists in introducing a pressure term in the finite element continuity equation to remove the saddle point character of the discrete problem, leading to an algebraic system of the form

$$\begin{pmatrix} A & D \\ B & -C \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{g} \end{pmatrix}. \tag{3.4}$$

Several methods were proposed in the 1980s, the first one by Brezzi and Pitkäranta in [26] and a number of residual-based pressure stabilizations in [55, 54, 39]. At the end of the 1990s and during the 2000s, new approaches were developed, which often contain terms where only the pressure appears, e.g., in [34, 13, 38, 29]. In recent years, variants of stabilized methods were proposed that allow an easier implementation as previous variants, e.g., in [12, 7, 31], or a finite element error analysis was presented with less regularity assumptions on the solution of the continuous problem in [83].

Altogether, there are many different proposals for pressure stabilizations. However, to the best of our knowledge, there is no up-to-date comprehensive survey of this topic in the literature available. In addition, it was pointed out as an open problem in [57] that *Systematic assessments of the proposed stabilized methods are missing that clarify their advantages and drawbacks and give finally proposals which ones should be preferred in simulations.* The present paper aims to close these gaps to some extent. However, there will be also some limitations of this survey. It is restricted to conforming finite element methods and to the discussion of the a priori error analysis.

Throughout the paper, standard notation for Lebesgue and Sobolev spaces is used. The inner product of $L^2(\Omega)^d$, $d \in \{1, 2, 3\}$, will be denoted by $(\cdot, \cdot)$. All constants C, C_1, etc. do neither depend on the viscosity coefficient ν nor on the mesh width h. The notation C indicates a general constant that can have different values at different places.

The paper is organized as follows. Section 3.2 introduces the considered finite element spaces and provides some properties which are used in the numerical analysis. Available convergence results for inf-sup stable discretizations are summarized in Section 3.3, to allow an easy comparison with the results for pressure-stabilized discretizations. The topic of Section 3.4 is the class of residual-based stabilizations. For some of them, the finite element analysis is presented in detail. Stabilizations that use only the pressure are described in Section 3.5. A detailed presentation of the analysis is provided for a local projection stabilization (LPS) scheme. Section 3.6 describes the connection of some stabilized discretizations to inf-sup stable methods that are enriched with bubble functions. Finally, numerical studies involving three residual-based stabilizations and one LPS method are presented in Section 3.7.

3.2 Weak Form of the Stokes Equations, Finite Element Spaces

Throughout the remaining part of this chapter, the following assumptions on the data of the Stokes problem (3.2) will be made. It will be assumed that Ω is a bounded domain with a polygonal resp. polyhedral Lipschitz-continuous boundary, the viscosity ν is a positive constant, and $\boldsymbol{f} \in L^2(\Omega)^d$.

A weak form of the Stokes equations (3.2) reads: Given $\boldsymbol{f} \in L^2(\Omega)^d$, find $(\boldsymbol{u}, p) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$ such that

$$\nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) - (\nabla \cdot \boldsymbol{v}, p) + (\nabla \cdot \boldsymbol{u}, q) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall\, (\boldsymbol{v}, q) \in H_0^1(\Omega)^d \times L_0^2(\Omega). \tag{3.5}$$

We shall use the notation $V = H_0^1(\Omega)^d$ and $Q = L_0^2(\Omega)$. The unique solvability of (3.5) is closely connected with the fact that the spaces V and Q satisfy the inf–sup condition

$$\inf_{q \in Q \setminus \{0\}} \sup_{\boldsymbol{v} \in V \setminus \{\boldsymbol{0}\}} \frac{(\nabla \cdot \boldsymbol{v}, q)}{\|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \|q\|_{L^2(\Omega)}} \geq \beta_{\mathrm{is}} > 0. \tag{3.6}$$

The inequality

$$\|\nabla \cdot \boldsymbol{v}\|_{L^2(\Omega)} \leq \|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \quad \forall\, \boldsymbol{v} \in V \tag{3.7}$$

will be used in the analysis. The space of weakly divergence-free functions is given by

$$V_{\mathrm{div}} = \{\boldsymbol{v} \in V \ : \ (\nabla \cdot \boldsymbol{v}, q) = 0 \ \forall\, q \in Q\}.$$

We assume that we are given a family $\{\mathcal{T}^h\}$ of triangulations of Ω consisting of simplices, quadrilaterals or hexahedra and possessing the usual compatibility properties. The set of interior faces (edges for $d = 2$) will be denoted by $\mathcal{E}^h$. We denote $h_K := \mathrm{diam}(K)$ and $h_E := \mathrm{diam}(E)$ for any $K \in \mathcal{T}^h$ and $E \in \mathcal{E}^h$ and assume that $h_K \leq h$ for all $K \in \mathcal{T}^h$. For each face $E \in \mathcal{E}^h$, we denote by $\boldsymbol{n}_E$ a fixed unit normal vector to E and by $[|q|]_E$ the jump of the function q across the face E such that $[|q|]_E > 0$ if q decreases in the direction of $\boldsymbol{n}_E$.

For each $\mathcal{T}^h$, we introduce finite element spaces $V^h \subset V$ and $Q^h \subset Q$ containing piecewise (mapped) polynomials of degree $k \geq 1$ and $l \geq 0$, respectively. We assume that the finite element spaces V^h and Q^h possess standard interpolation properties. More precisely, we denote by $I^h : V \cap H^{k+1}(\Omega)^d \to V_h$ and

$J^h : Q \cap H^{l+1}(\Omega) \to Q_h$ interpolation operators satisfying

$$\left(\sum_{K\in\mathcal{T}^h} h_K^{-2} \left\| \boldsymbol{v} - I^h \boldsymbol{v} \right\|_{L^2(K)}^2 \right)^{1/2} + \left\| \nabla(\boldsymbol{v} - I^h\boldsymbol{v}) \right\|_{L^2(\Omega)} + \left(\sum_{K\in\mathcal{T}^h} h_K^{2} \left\| \Delta(\boldsymbol{v} - I^h \boldsymbol{v}) \right\|_{L^2(K)}^2 \right)^{1/2} + \left(\sum_{E\in\mathcal{E}^h} h_E^{-1} \left\| \boldsymbol{v} - I^h \boldsymbol{v} \right\|_{L^2(E)}^2 \right)^{1/2} \le C\, h^k \, \|\boldsymbol{v}\|_{H^{k+1}(\Omega)}, \tag{3.8}$$

$$\left(\sum_{K\in\mathcal{T}^h} h_K^{-2} \left\| q - J^h q \right\|_{L^2(K)}^2 \right)^{1/2} + \left(\sum_{K\in\mathcal{T}^h} \left\| \nabla(q - J^h q) \right\|_{L^2(K)}^2 \right)^{1/2} + \left(\sum_{K\in\mathcal{T}^h} \sum_{E\subset\partial K} h_E^{-1} \left\| q - (J^h q)|_K \right\|_{L^2(E)}^2 \right)^{1/2} \le C\, h^l \, \|q\|_{H^{l+1}(\Omega)}, \tag{3.9}$$

for $\boldsymbol{v} \in V \cap H^{k+1}(\Omega)^d$ and $q \in Q \cap H^{l+1}(\Omega)$. The operator I^h may be the standard Lagrange interpolation. The definition of J^h depends on the construction of Q^h. For example, if $Q^h \subset H^1(\Omega)$, the operator J^h may be defined as the Lagrange interpolation projected into Q. If the functions in Q^h are discontinuous across faces, the operator J^h may be defined as the projection into a polynomial space on each element of the triangulation.

In addition, for $\boldsymbol{v} \in V$, we shall use a piecewise (multi)linear interpolant $\mathcal{I}^h\boldsymbol{v} \in V^h$ (e.g., the Clément or Scott–Zhang interpolant) satisfying

$$\left(\sum_{K\in\mathcal{T}^h} h_K^{-2} \left\| \boldsymbol{v} - \mathcal{I}^h \boldsymbol{v} \right\|_{L^2(K)}^2 \right)^{1/2} + \left\| \nabla \mathcal{I}^h\boldsymbol{v} \right\|_{L^2(\Omega)} + \left(\sum_{E\in\mathcal{E}^h} h_E^{-1} \left\| \boldsymbol{v} - \mathcal{I}^h \boldsymbol{v} \right\|_{L^2(E)}^2 \right)^{1/2} \le C \, \|\nabla\boldsymbol{v}\|_{L^2(\Omega)}. \tag{3.10}$$

Similarly, for $q \in Q \cap H^1(\Omega)$, we introduce an interpolant $\mathcal{J}^h q \in Q_h$ satisfying

$$\left(\sum_{K\in\mathcal{T}^h} h_K^{-2} \left\| q - \mathcal{J}^h q \right\|_{L^2(K)}^2 \right)^{1/2} + \left(\sum_{K\in\mathcal{T}^h} \left\| \nabla \mathcal{J}^h q \right\|_{L^2(K)}^2 \right)^{1/2} + \left(\sum_{K\in\mathcal{T}^h} \sum_{E\subset\partial K} h_E^{-1} \left\| q - (\mathcal{J}^h q)|_K \right\|_{L^2(E)}^2 \right)^{1/2} \le C \, \|\nabla q\|_{L^2(\Omega)}. \tag{3.11}$$

Finally, it is assumed that the following inverse inequality holds

$$\left\|\Delta \boldsymbol{v}^h\right\|_{L^2(K)} \le C_{\mathrm{inv}} h_K^{-1} \left\|\nabla \boldsymbol{v}^h\right\|_{L^2(K)} \qquad \forall\ \boldsymbol{v}^h \in V^h,\ K \in \mathcal{T}^h. \tag{3.12}$$

Note that C_{inv} depends on the polynomial degree. It was shown in [53] for some types of mesh cells that it increases with increasing polynomial degree. For example, it has the value $0, 48, 149.1$ for $P_1(K)$, $P_2(K)$, and $P_3(K)$, respectively, in the case that K is a right isoscale triangle.

3.3 Inf-Sup Stable Finite Element Discretizations

Inf-sup stable pairs of finite element spaces satisfy the discrete inf-sup condition (3.3). For the well-posedness of the discrete problem, the introduction of a pressure stabilization is not necessary. This section provides a survey on the most important results from the finite element convergence theory for inf-sup stable finite element discretizations to facilitate the comparison with the convergence results for stabilized discretizations presented in the subsequent sections.

Let the spaces V^h and Q^h satisfy the discrete inf-sup condition (3.3). Then, the conforming discretization of the Stokes problem reads as follows: Find $\left(\boldsymbol{u}^h, p^h\right) \in V^h \times Q^h$ such that

$$\nu\left(\nabla \boldsymbol{u}^h, \nabla \boldsymbol{v}^h\right) - \left(\nabla \cdot \boldsymbol{v}^h, p^h\right) + \left(\nabla \cdot \boldsymbol{u}^h, q^h\right) = \left(\boldsymbol{f}, \boldsymbol{v}^h\right) \quad \forall\ \left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h. \tag{3.13}$$

The natural norms for the analysis of the Stokes problem are the $L^2(\Omega)$ norm of the velocity gradient and the $L^2(\Omega)$ norm of the pressure. Since the error analysis for these norms utilizes typical tools and it is rather short, the proofs will be presented in detail. The presentation follows [56, Section 4.2.1].

A crucial role in the analysis plays the subspace of discretely divergence-free functions

$$V_{\mathrm{div}}^h = \left\{\boldsymbol{v}^h \in V^h \ :\ \left(\nabla \cdot \boldsymbol{v}^h, q^h\right) = 0\ \forall\, q^h \in Q^h\right\}.$$

The solution of (3.13) belongs to this subspace. Note that in general functions from this subspace are not weakly divergence-free, i.e., it holds $V_{\mathrm{div}}^h \not\subset V_{\mathrm{div}}$.

Theorem 3.3.1 (Error estimate for the $L^2(\Omega)$ norm of the velocity gradient)**.** *Let $(\boldsymbol{u}, p) \in V \times Q$ be the unique solution of the Stokes problem (3.5) and assume that the spaces V^h and Q^h satisfy (3.3). Then, the solution of the conforming discretization (3.13) satisfies the error estimate*

$$\left\|\nabla(\boldsymbol{u} - \boldsymbol{u}^h)\right\|_{L^2(\Omega)} \le 2 \inf_{\boldsymbol{v}^h \in V_{\mathrm{div}}^h} \left\|\nabla(\boldsymbol{u} - \boldsymbol{v}^h)\right\|_{L^2(\Omega)} + \frac{1}{\nu} \inf_{q^h \in Q^h} \left\|p - q^h\right\|_{L^2(\Omega)}. \tag{3.14}$$

Proof. The proof starts by formulating the error equation. Since $V^h \subset V$, functions from V^h can be used as test functions in (3.5). Subtracting (3.13) from (3.5) and setting $q = q^h = 0$ yields the so-called error equation

$$\nu \left(\nabla(\boldsymbol{u} - \boldsymbol{u}^h), \nabla \boldsymbol{v}^h\right) - \left(\nabla \cdot \boldsymbol{v}^h, p - p^h\right) = 0 \qquad \forall\, \boldsymbol{v}^h \in V^h. \tag{3.15}$$

Now, restricting the test functions to the space V^h_{div}, the second term on the left-hand side is modified such that an approximation term with respect to the pressure is obtained. One observes that $\left(\nabla \cdot \boldsymbol{v}^h, q^h\right) = 0$ for all $\boldsymbol{v}^h \in V^h_{\text{div}}$ and $q^h \in Q^h$, which leads to

$$\nu \left(\nabla(\boldsymbol{u} - \boldsymbol{u}^h), \nabla \boldsymbol{v}^h\right) - \left(\nabla \cdot \boldsymbol{v}^h, p - q^h\right) = 0 \qquad \forall\, \boldsymbol{v}^h \in V^h_{\text{div}},\, q^h \in Q^h. \tag{3.16}$$

Next, an approximation error for the velocity is introduced. To this end, the error is decomposed into

$$\boldsymbol{u} - \boldsymbol{u}^h = \left(\boldsymbol{u} - \boldsymbol{w}^h\right) - \left(\boldsymbol{u}^h - \boldsymbol{w}^h\right) = \boldsymbol{\eta} - \boldsymbol{\phi}^h,$$

where $\boldsymbol{w}^h$ denotes an arbitrary interpolant of $\boldsymbol{u}$ in V^h_{div}. Hence, $\boldsymbol{\eta}$ is an approximation error which depends only on the finite element space V^h_{div}. The goal consists in estimating $\boldsymbol{\phi}^h \in V^h_{\text{div}}$ by approximation errors as well. Therefore, this decomposition is inserted in (3.16) and the test function $\boldsymbol{v}^h = \boldsymbol{\phi}^h$ is chosen. It follows that

$$\nu \left\|\nabla \boldsymbol{\phi}^h\right\|^2_{L^2(\Omega)} = \nu \left(\nabla \boldsymbol{\phi}^h, \nabla \boldsymbol{\phi}^h\right) = \nu \left(\nabla \boldsymbol{\eta}, \nabla \boldsymbol{\phi}^h\right) - \left(\nabla \cdot \boldsymbol{\phi}^h, p - q^h\right) \quad \forall\, q^h \in Q^h. \tag{3.17}$$

The first term on the right-hand side is estimated with the Cauchy–Schwarz inequality

$$\nu \left|\left(\nabla \boldsymbol{\eta}, \nabla \boldsymbol{\phi}^h\right)\right| \le \nu \left\|\nabla \boldsymbol{\eta}\right\|_{L^2(\Omega)} \left\|\nabla \boldsymbol{\phi}^h\right\|_{L^2(\Omega)}.$$

For the second term, one uses in addition (3.7), which gives

$$\begin{aligned} \left|-\left(\nabla \cdot \boldsymbol{\phi}^h, p - q^h\right)\right| &\le \left\|p - q^h\right\|_{L^2(\Omega)} \left\|\nabla \cdot \boldsymbol{\phi}^h\right\|_{L^2(\Omega)} \\ &\le \left\|p - q^h\right\|_{L^2(\Omega)} \left\|\nabla \boldsymbol{\phi}^h\right\|_{L^2(\Omega)}. \end{aligned}$$

Inserting these estimates in (3.17) and dividing by $\nu \left\|\nabla \boldsymbol{\phi}^h\right\|_{L^2(\Omega)} \neq 0$ yields

$$\left\|\nabla \boldsymbol{\phi}^h\right\|_{L^2(\Omega)} \le \left\|\nabla \boldsymbol{\eta}\right\|_{L^2(\Omega)} + \frac{1}{\nu} \left\|p - q^h\right\|_{L^2(\Omega)}.$$

This estimate is trivially true if $\left\|\nabla \boldsymbol{\phi}^h\right\|_{L^2(\Omega)} = 0$.

With the triangle inequality, it follows that

$$\begin{aligned}\left\|\nabla(\boldsymbol{u}-\boldsymbol{u}^h)\right\|_{L^2(\Omega)} &\leq \left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(\Omega)} + \left\|\nabla\boldsymbol{\eta}\right\|_{L^2(\Omega)} \\ &\leq 2\left\|\nabla\boldsymbol{\eta}\right\|_{L^2(\Omega)} + \frac{1}{\nu}\left\|p-q^h\right\|_{L^2(\Omega)}\end{aligned}$$

for all $\boldsymbol{w}^h \in V^h_{\mathrm{div}}$ and for all $q^h \in Q^h$, such that (3.14) follows. □

Theorem 3.3.2 (Error estimate for the $L^2(\Omega)$ norm of the pressure)**.** *Let the assumption of Theorem 3.3.1 be satisfied. Then the following error estimate holds*

$$\begin{aligned}\left\|p-p^h\right\|_{L^2(\Omega)} &\leq \frac{2\nu}{\beta^h_{\mathrm{is}}}\inf_{\boldsymbol{v}^h\in V^h_{\mathrm{div}}}\left\|\nabla(\boldsymbol{u}-\boldsymbol{v}^h)\right\|_{L^2(\Omega)} \\ &\quad + \left(1+\frac{2}{\beta^h_{\mathrm{is}}}\right)\inf_{q^h\in Q^h}\left\|p-q^h\right\|_{L^2(\Omega)}. \end{aligned} \tag{3.18}$$

Proof. Let $q^h \in Q^h$ be arbitrary, then the triangle inequality implies

$$\left\|p-p^h\right\|_{L^2(\Omega)} \leq \left\|p-q^h\right\|_{L^2(\Omega)} + \left\|p^h-q^h\right\|_{L^2(\Omega)}.$$

Replacing the right-hand side of the momentum equation of the finite element Stokes problem (3.13) by the left-hand side of the the momentum equation of the continuous Stokes problem (3.5) for $\boldsymbol{v}^h \in V^h$ yields

$$\begin{aligned}-\left(\nabla\cdot\boldsymbol{v}^h, p^h-q^h\right) &= -\nu\left(\nabla\boldsymbol{u}^h,\nabla\boldsymbol{v}^h\right) + \left(\boldsymbol{f},\boldsymbol{v}^h\right) + \left(\nabla\cdot\boldsymbol{v}^h, q^h\right) \\ &= \nu\left(\nabla\left(\boldsymbol{u}-\boldsymbol{u}^h\right),\nabla\boldsymbol{v}^h\right) - \left(\nabla\cdot\boldsymbol{v}^h, p-q^h\right)\end{aligned}$$

for all $\left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h$. With the discrete inf-sup condition (3.3), the Cauchy–Schwarz inequality, and (3.7), it follows now that

$$\begin{aligned}&\left\|p^h-q^h\right\|_{L^2(\Omega)} \\ &\leq \frac{1}{\beta^h_{\mathrm{is}}}\sup_{\boldsymbol{v}^h\in V^h\setminus\{\boldsymbol{0}\}}\frac{-\left(\nabla\cdot\boldsymbol{v}^h,p^h-q^h\right)}{\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)}} \\ &= \frac{1}{\beta^h_{\mathrm{is}}}\sup_{\boldsymbol{v}^h\in V^h\setminus\{\boldsymbol{0}\}}\frac{\nu\left(\nabla\left(\boldsymbol{u}-\boldsymbol{u}^h\right),\nabla\boldsymbol{v}^h\right)-\left(\nabla\cdot\boldsymbol{v}^h,p-q^h\right)}{\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)}} \\ &\leq \frac{1}{\beta^h_{\mathrm{is}}}\sup_{\boldsymbol{v}^h\in V^h\setminus\{\boldsymbol{0}\}}\frac{\nu\left\|\nabla\left(\boldsymbol{u}-\boldsymbol{u}^h\right)\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)}+\left\|p-q^h\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)}}{\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)}} \\ &= \frac{1}{\beta^h_{\mathrm{is}}}\left(\nu\left\|\nabla\left(\boldsymbol{u}-\boldsymbol{u}^h\right)\right\|_{L^2(\Omega)}+\left\|p-q^h\right\|_{L^2(\Omega)}\right) \quad \forall\, q^h\in Q^h.\end{aligned}$$

Inserting the error bound (3.14) for the velocity yields the error estimate (3.18) for the pressure. □

The best approximation error in the subspace V^h_{div} can be estimated by the best approximation error in V^h

$$\inf_{\boldsymbol{v}^h \in V^h_{\text{div}}} \left\| \nabla \left(\boldsymbol{u} - \boldsymbol{v}^h \right) \right\|_{L^2(\Omega)} \leq \left(1 + \frac{1}{\beta^h_{\text{is}}} \right) \inf_{\boldsymbol{w}^h \in V^h} \left\| \nabla \left(\boldsymbol{u} - \boldsymbol{w}^h \right) \right\|_{L^2(\Omega)}, \tag{3.19}$$

e.g., see [56, Lemma 3.60]. With respect to the dependency on the discrete inf-sup constant, estimate (3.19) is a worst case estimate. For many pairs of finite element spaces, an alternative estimate using a quasi-local Fortin projection is possible which does not depend on the inverse of β^h_{is}, see [48]. Applying (3.19) to the error bounds (3.14) and (3.18) gives the following estimate.

Corollary 3.3.3 (Error estimate)**.** *Let the spaces V^h and Q^h satisfy* (3.3) *with β^h_{is} bounded from below by $\beta_0 > 0$ independent of h. Assume that the solution of* (3.5) *satisfies $(\boldsymbol{u}, p) \in H^{k+1}(\Omega)^d \times H^{l+1}(\Omega)$, then one has the error estimate*

$$\begin{aligned} &\nu \left\| \nabla (\boldsymbol{u} - \boldsymbol{u}^h) \right\|_{L^2(\Omega)} + \left\| p - p^h \right\|_{L^2(\Omega)} \\ &\qquad \leq C \left(\nu h^k \left\| \boldsymbol{u} \right\|_{H^{k+1}(\Omega)} + h^{l+1} \left\| p \right\|_{H^{l+1}(\Omega)} \right). \end{aligned} \tag{3.20}$$

Another norm of interest is the $L^2(\Omega)$ norm of the velocity because its square is proportional to the kinetic energy of the flow. Applying the Poincaré–Friedrichs inequality, one observes that the estimate from Corollary 3.3.3 also holds for $\nu \left\| \boldsymbol{u} - \boldsymbol{u}^h \right\|_{L^2(\Omega)}$. However, such an error estimate is suboptimal with respect to h. In what follows, an optimal estimate of the velocity error in the $L^2(\Omega)$ norm will be derived using the usual Aubin–Nitsche technique. To this end, a regularity assumption on the Stokes problem in the following sense will be needed.

Definition 3.3.4. The Stokes problem (3.2) is regular if, for any $\boldsymbol{f} \in L^2(\Omega)^d$, the solution of the weak formulation (3.5) satisfies $(\boldsymbol{u}, p) \in H^2(\Omega)^d \times H^1(\Omega)$ and it holds

$$\nu \left\| \boldsymbol{u} \right\|_{H^2(\Omega)} + \left\| p \right\|_{H^1(\Omega)} \leq C \left\| \boldsymbol{f} \right\|_{L^2(\Omega)}$$

with a constant C independent of $\boldsymbol{f}$ and ν.

Theorem 3.3.5 (L^2 estimate of the velocity error)**.** *Let the spaces V^h and Q^h satisfy* (3.3) *with β^h_{is} bounded from below by $\beta_0 > 0$ independent of h. Assume that the solution of* (3.5) *satisfies $(\boldsymbol{u}, p) \in H^{k+1}(\Omega)^d \times H^{l+1}(\Omega)$ and let the Stokes problem* (3.2) *be regular. Then there holds the error estimate*

$$\left\| \boldsymbol{u} - \boldsymbol{u}^h \right\|_{L^2(\Omega)} \leq C \left(h^{k+1} \left\| \boldsymbol{u} \right\|_{H^{k+1}(\Omega)} + \frac{h^{l+2}}{\nu} \left\| p \right\|_{H^{l+1}(\Omega)} \right). \tag{3.21}$$

Proof. Let $(\boldsymbol{z}, r) \in V \times Q$ be the solution of the problem

$$\nu(\nabla \boldsymbol{z}, \nabla \boldsymbol{v}) - (\nabla \cdot \boldsymbol{v}, r) + (\nabla \cdot \boldsymbol{z}, q) = \nu(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{v}) \qquad \forall\, (\boldsymbol{v}, q) \in V \times Q. \quad (3.22)$$

Then, according to the regularity assumption, one has $(\boldsymbol{z}, r) \in H^2(\Omega)^d \times H^1(\Omega)$ and

$$\nu \|\boldsymbol{z}\|_{H^2(\Omega)} + \|r\|_{H^1(\Omega)} \leq C\nu \left\|\boldsymbol{u} - \boldsymbol{u}^h\right\|_{L^2(\Omega)}. \quad (3.23)$$

Since $\boldsymbol{u} - \boldsymbol{u}^h \in V$, one can set $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{u}^h$ and $q = 0$ in (3.22), which gives

$$\nu \left\|\boldsymbol{u} - \boldsymbol{u}^h\right\|^2_{L^2(\Omega)} = \nu(\nabla \boldsymbol{z}, \nabla(\boldsymbol{u} - \boldsymbol{u}^h)) - (\nabla \cdot (\boldsymbol{u} - \boldsymbol{u}^h), r). \quad (3.24)$$

Let $\boldsymbol{z}^I \in V^h$ be the continuous piecewise (multi)linear Lagrange interpolant of $\boldsymbol{z}$ satisfying (3.8) with $k = 1$ and let $r^I = \mathcal{J}^h r \in Q^h$ be an interpolant of r satisfying (3.11). Then

$$\left\|\nabla(\boldsymbol{z} - \boldsymbol{z}^I)\right\|_{L^2(\Omega)} \leq Ch \|\boldsymbol{z}\|_{H^2(\Omega)} \leq Ch \left\|\boldsymbol{u} - \boldsymbol{u}^h\right\|_{L^2(\Omega)}, \quad (3.25)$$

$$\left\|r - r^I\right\|_{L^2(\Omega)} \leq Ch \|\nabla r\|_{L^2(\Omega)} \leq C\nu h \left\|\boldsymbol{u} - \boldsymbol{u}^h\right\|_{L^2(\Omega)}. \quad (3.26)$$

It follows from (3.24) that

$$\begin{aligned} \nu \left\|\boldsymbol{u} - \boldsymbol{u}^h\right\|^2_{L^2(\Omega)} &= \nu(\nabla(\boldsymbol{z} - \boldsymbol{z}^I), \nabla(\boldsymbol{u} - \boldsymbol{u}^h)) - (\nabla \cdot (\boldsymbol{u} - \boldsymbol{u}^h), r - r^I) \\ &\quad + \nu(\nabla \boldsymbol{z}^I, \nabla(\boldsymbol{u} - \boldsymbol{u}^h)) - (\nabla \cdot (\boldsymbol{u} - \boldsymbol{u}^h), r^I). \end{aligned} \quad (3.27)$$

Applying the Cauchy–Schwarz inequality and (3.25), (3.26), the first two terms in (3.27) can be estimated by

$$\begin{aligned} &\nu(\nabla(\boldsymbol{z} - \boldsymbol{z}^I), \nabla(\boldsymbol{u} - \boldsymbol{u}^h)) - (\nabla \cdot (\boldsymbol{u} - \boldsymbol{u}^h), r - r^I) \\ &\leq C\nu h \left\|\nabla(\boldsymbol{u} - \boldsymbol{u}^h)\right\|_{L^2(\Omega)} \left\|\boldsymbol{u} - \boldsymbol{u}^h\right\|_{L^2(\Omega)}. \end{aligned} \quad (3.28)$$

Setting $\boldsymbol{v}^h = \boldsymbol{z}^I$ in (3.15), using the fact that $\nabla \cdot \boldsymbol{z} = 0$ and applying the Cauchy–Schwarz inequality and (3.25), one derives

$$\begin{aligned} \nu(\nabla \boldsymbol{z}^I, \nabla(\boldsymbol{u} - \boldsymbol{u}^h)) &= \left(\nabla \cdot (\boldsymbol{z}^I - \boldsymbol{z}), p - p^h\right) \\ &\leq \left\|\nabla(\boldsymbol{z} - \boldsymbol{z}^I)\right\|_{L^2(\Omega)} \left\|p - p^h\right\|_{L^2(\Omega)} \\ &\leq Ch \left\|\boldsymbol{u} - \boldsymbol{u}^h\right\|_{L^2(\Omega)} \left\|p - p^h\right\|_{L^2(\Omega)}. \end{aligned}$$

Finally, the last term in (3.27) vanishes since, according to (3.5) and (3.13), $\boldsymbol{u}$ is weakly divergence-free and $\boldsymbol{u}^h$ is discretely divergence-free. Combining the above estimates gives

$$\nu \left\|\boldsymbol{u} - \boldsymbol{u}^h\right\|_{L^2(\Omega)} \leq Ch \left(\nu \|\nabla(\boldsymbol{u} - \boldsymbol{u}^h)\|_{L^2(\Omega)} + \|p - p^h\|_{L^2(\Omega)}\right)$$

and the statement of the theorem follows from Corollary 3.3.3. □

It should be noted that the velocity error bounds (3.20) and (3.21) improve substantially if an inf-sup stable pair of finite element spaces is used with $V_{\mathrm{div}}^h \subset V_{\mathrm{div}}$. Such pairs exist, e.g., the Scott–Vogelius pair P_2/P_1^{disc} applied on special meshes. Then, the pressure term in the error equation (3.16) vanishes and consequently the pressure terms vanish on the right-hand sides of the estimates (3.20) and (3.21). The consequences are that the velocity error bounds do not depend on the pressure and they do not depend explicitly on inverse powers of the viscosity. Even for spaces with $V_{\mathrm{div}}^h \not\subset V_{\mathrm{div}}$, an approach has been developed such that the velocity error bounds have these two properties, see [70, 71] or the recent survey paper [58]. To derive velocity error bounds with these two properties for pressure-stabilized methods, as presented in the following sections, is impossible.

For inf-sup stable pairs of finite element spaces, error estimates with respect to the norms of other Lebesgue spaces can be proved. In particular, estimates in $L^\infty(\Omega)$ were derived in [47, 33, 51, 46, 52] that are of the form

$$\begin{aligned} &\nu \left\|\nabla\left(\boldsymbol{u}-\boldsymbol{u}^h\right)\right\|_{L^\infty(\Omega)} + \left\|p-p^h\right\|_{L^\infty(\Omega)} \\ &\quad \leq C\left(\nu \inf_{\boldsymbol{v}^h\in V^h}\left\|\nabla\left(\boldsymbol{u}-\boldsymbol{v}^h\right)\right\|_{L^\infty(\Omega)} + \inf_{q^h\in V^h}\left\|p-q^h\right\|_{L^\infty(\Omega)}\right). \end{aligned} \tag{3.29}$$

In [46], even an estimate of the form

$$\begin{aligned} &\nu \left\|\nabla\left(\boldsymbol{u}-\boldsymbol{u}^h\right)\right\|_{L^r(\Omega)} + \left\|p-p^h\right\|_{L^r(\Omega)} \\ &\quad \leq C\left(\nu \inf_{\boldsymbol{v}^h\in V^h}\left\|\nabla\left(\boldsymbol{u}-\boldsymbol{v}^h\right)\right\|_{L^r(\Omega)} + \inf_{q^h\in V^h}\left\|p-q^h\right\|_{L^r(\Omega)}\right), \quad 2\leq r\leq\infty \end{aligned} \tag{3.30}$$

was shown. The current state of the art is that estimates of form (3.29) and (3.30) can be proved for convex polyhedral domains.

3.4 Residual-Based Stabilizations

For another review of residual-based stabilizations, it is referred to [42].

3.4.1 A Framework

A framework for the derivation of residual-based stabilizations was presented in [19]. Starting point is the regularization of the Galerkin finite element method (3.13) with respect to the norm of Q^h

$$\nu\left(\nabla\boldsymbol{u}^h,\nabla\boldsymbol{v}^h\right) - \left(\nabla\cdot\boldsymbol{v}^h,p^h\right) + \left(\nabla\cdot\boldsymbol{u}^h,q^h\right) + \delta\left(p^h,q^h\right) = (\boldsymbol{f},\boldsymbol{v}^h),$$

where $\delta > 0$ is a stabilization parameter. However, this stabilization acts like a penalty term which prevents the method from being optimally convergent for higher order finite element spaces. Thus, this stabilization should be replaced by a stabilization that is, on the one hand, similarly strong but, on the other hand, possesses a sufficiently small consistency error. Using [49, Cor. 2.1], it is known that there are positive constants C_1 and C_2 such that

$$C_1 \|q\|_{L^2(\Omega)} \le \|\nabla q\|_{H^{-1}(\Omega)} \le C_2 \|q\|_{L^2(\Omega)} \quad \forall\, q \in Q,$$

i.e., the $H^{-1}(\Omega)^d$ norm of ∇q is equivalent to the $L^2(\Omega)$ norm of q. Consequently, $\left\|\nabla q^h\right\|^2_{H^{-1}(\Omega)}$ has the same stabilization effect like $\left\|q^h\right\|^2_{L^2(\Omega)}$. The term $\left(\nabla p^h, \nabla q^h\right)_{-1}$ can be included in a stabilization term naturally by using the residual, where $(\cdot,\cdot)_{-1}$ is the inner product in $H^{-1}(\Omega)^d$, see [19] for a definition of this inner product.

For simplicity of presentation, only the case $Q^h \subset H^1(\Omega)$ is considered. The prototype of a residual-based stabilization from [19] has the form: Find $\left(\boldsymbol{u}^h, p^h\right) \in V^h \times Q^h$ such that

$$\begin{aligned} &\nu\left(\nabla \boldsymbol{u}^h, \nabla \boldsymbol{v}^h\right) - \left(\nabla \cdot \boldsymbol{v}^h, p^h\right) + \left(\nabla \cdot \boldsymbol{u}^h, q^h\right) \\ &\qquad + \delta\left(-\nu\Delta\boldsymbol{u}^h + \nabla p^h, \kappa\nu\Delta\boldsymbol{v}^h + \nabla q^h\right)_{-1} \\ &= \quad (\boldsymbol{f}, \boldsymbol{v}) + \delta\left(\boldsymbol{f}, \kappa\nu\Delta\boldsymbol{v}^h + \nabla q^h\right)_{-1} \quad \forall\ \left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h, \end{aligned} \tag{3.31}$$

with $\kappa \in \{-1, 0, 1\}$ and $\delta > 0$. There are still two issues in (3.31). First, $(\cdot,\cdot)_{-1}$ is not computable and second, $\Delta\boldsymbol{u}^h, \Delta\boldsymbol{v}^h$ are not defined. Thanks to the regularity assumption on Q^h, the functions $\nabla p^h, \nabla q^h$ are well defined.

A standard way to resolve these issues consists in approximating $(\cdot,\cdot)_{-1}$ by a weighted $L^2(\Omega)$ inner product, leading to the following problem: Find $\left(\boldsymbol{u}^h, p^h\right) \in V^h \times Q^h$ such that for all $\left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h$

$$\begin{aligned} &\nu\left(\nabla \boldsymbol{u}^h, \nabla \boldsymbol{v}^h\right) - \left(\nabla \cdot \boldsymbol{v}^h, p^h\right) + \left(\nabla \cdot \boldsymbol{u}^h, q^h\right) \\ &\qquad + \sum_{K\in\mathcal{T}^h} \delta h_K^2 \left(-\nu\Delta\boldsymbol{u}^h + \nabla p^h, \kappa\nu\Delta\boldsymbol{v}^h + \nabla q^h\right)_K \\ &= \quad (\boldsymbol{f}, \boldsymbol{v}) + \sum_{K\in\mathcal{T}^h} \delta h_K^2 \left(\boldsymbol{f}, \kappa\nu\Delta\boldsymbol{v}^h + \nabla q^h\right)_K. \end{aligned} \tag{3.32}$$

For $\kappa = 0$, one obtains the PSPG method, which is discussed in Section 3.4.2, for $\kappa = 1$ the symmetric GLS method, see Section 3.4.3, and for $\kappa = -1$ the non-symmetric GLS method presented in Section 3.4.4.

In [19], a new proposal for approximating the inner product in $H^{-1}(\Omega)^d$ was presented. This proposal is discussed briefly in Section 3.4.5.

Definition 3.4.1 (Absolutely and conditionally stable methods)**.** A stabilized discrete method is called absolutely stable if it is stable for all $\delta > 0$. Otherwise, if it is stable only for a restricted set of parameters, it is called conditionally stable.

3.4.2 The PSPG Method

The Pressure Stabilizing Petrov–Galerkin (PSPG) method was proposed for finite element spaces with continuous discrete pressures in [55]. In the case of piecewise polynomial but discontinuous finite element pressure spaces, an additional term is necessary, which was introduced in [54, 39].

The PSPG method has the form: Find $\left(\boldsymbol{u}^h, p^h\right) \in V^h \times Q^h$ such that

$$A_{\text{pspg}}\left(\left(\boldsymbol{u}^h, p^h\right), \left(\boldsymbol{v}^h, q^h\right)\right) = L_{\text{pspg}}\left(\left(\boldsymbol{v}^h, q^h\right)\right) \quad \forall\ \left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h, \qquad (3.33)$$

where the bilinear form $A_{\text{pspg}}\ :\ \left(\tilde{V} \times \tilde{Q}\right) \times \left(V \times \tilde{Q}\right) \to \mathbb{R}$ is given by

$$\begin{aligned} A_{\text{pspg}}\left((\boldsymbol{u}, p), (\boldsymbol{v}, q)\right) &= \nu\left(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}\right) - \left(\nabla \cdot \boldsymbol{v}, p\right) + \left(\nabla \cdot \boldsymbol{u}, q\right) \\ &\quad + \sum_{E \in \mathcal{E}^h} \gamma_E \left([|p|]_E, [|q|]_E\right)_E + \sum_{K \in \mathcal{T}^h} \left(-\nu \Delta \boldsymbol{u} + \nabla p, \delta_K \nabla q\right)_K \qquad (3.34) \end{aligned}$$

and the linear form $L_{\text{pspg}}\ :\ V \times \tilde{Q} \to \mathbb{R}$ by

$$L_{\text{pspg}}\left((\boldsymbol{v}, q)\right) = (\boldsymbol{f}, \boldsymbol{v}) + \sum_{K \in \mathcal{T}^h} \left(\boldsymbol{f}, \delta_K \nabla q\right)_K, \qquad (3.35)$$

with

$$\begin{aligned} \tilde{V} &= \left\{\boldsymbol{v} \in V\ :\ \boldsymbol{v}|_K \in H^2(K)^d \text{ for all } K \in \mathcal{T}^h\right\}, \qquad (3.36) \\ \tilde{Q} &= \left\{q \in Q\ :\ q|_K \in H^1(K) \text{ for all } K \in \mathcal{T}^h\right\} \qquad (3.37) \end{aligned}$$

and nonnegative stabilization parameters γ_E and δ_K. Their appropriate choices will be based on the study of the existence and uniqueness of a solution of (3.33), see Lemma 3.4.3, and on finite element error estimates, see Theorem 3.4.6. The volume integrals in the stabilization terms contain the so-called strong residual of the Stokes equations.

The definition of $\tilde{Q}$ ensures that the jumps of the pressure across the faces of the mesh cells are well defined. If $Q^h \subset H^1(\Omega)$, then the jumps of the pressure vanish almost everywhere on the faces. From the practical point of view, the case of piecewise polynomial and continuous discrete pressure functions is very important such that then even $Q^h \subset C(\overline{\Omega})$.

Lemma 3.4.2 (A norm in $V^h \times Q^h$ containing the stabilization terms)**.** *Let $\delta_K > 0$ for all $K \in \mathcal{T}^h$ and, in the case $Q^h \not\subset H^1(\Omega)$, let $\gamma_E > 0$ for all $E \in \mathcal{E}^h$. Then*

$$\begin{aligned} \left\|\left(\boldsymbol{v}^h, q^h\right)\right\|_{\text{pspg}} &= \left(\nu \left\|\nabla \boldsymbol{v}^h\right\|^2_{L^2(\Omega)} + \sum_{E \in \mathcal{E}^h} \gamma_E \left\|[|q^h|]_E\right\|^2_{L^2(E)} \right. \\ &\quad \left. + \sum_{K \in \mathcal{T}^h} \delta_K \left\|\nabla q^h\right\|^2_{L^2(K)}\right)^{1/2} \qquad (3.38) \end{aligned}$$

defines a norm in $V^h \times Q^h$.

Proof. Expression (3.38) is the square root of a sum of squares of seminorms. Thus, it is clearly a seminorm itself. It remains to prove that from $\left\|\left(\boldsymbol{v}^h, q^h\right)\right\|_{\text{pspg}} = 0$, it follows that $\boldsymbol{v}^h = \mathbf{0}$ and $q^h = 0$.

Let $\left\|\left(\boldsymbol{v}^h, q^h\right)\right\|_{\text{pspg}} = 0$, then all terms in (3.38) vanish. In particular, it holds $\left\|\nabla \boldsymbol{v}^h\right\|_{L^2(\Omega)} = 0$. Since this expression is a norm in V^h, it follows that $\boldsymbol{v}^h = \mathbf{0}$. With this result, one gets

$$0 = \sum_{E\in\mathcal{E}^h} \gamma_E \left\|\left[\left|q^h\right|\right]_E\right\|^2_{L^2(E)} + \sum_{K\in\mathcal{T}^h} \delta_K \left\|\nabla q^h\right\|^2_{L^2(K)}.$$

Because δ_K is assumed to be positive for all mesh cells, it follows that $\left\|\nabla q^h\right\|_{L^2(K)} = 0$ for all $K \in \mathcal{T}^h$. If $Q^h \subset H^1(\Omega)$, then $\left\|\left[\left|q^h\right|\right]_E\right\|_{L^2(E)} = 0$ for all faces. Otherwise, one gets this property from the assumption $\gamma_E > 0$ for all faces. Altogether, it follows that q^h is constant on Ω. The only globally constant function in Q^h is $q^h = 0$. Hence $\left\|\left(\boldsymbol{v}^h, q^h\right)\right\|_{\text{pspg}}$ defines a norm on $V^h \times Q^h$. □

Lemma 3.4.3 (Existence and uniqueness of a solution of (3.33))**.** *Let the assumptions of Lemma 3.4.2 be satisfied and let*

$$\delta_K \le \frac{h_K^2}{\nu C_{\text{inv}}^2}. \tag{3.39}$$

Then the PSPG problem (3.33) *possesses a unique solution.*

Proof. First, the coercivity of the bilinear form $A_{\text{pspg}}(\cdot,\cdot)$ with respect to the norm $\|\cdot\|_{\text{pspg}}$ will be shown for any $(\boldsymbol{v}^h, q^h) \in V^h \times Q^h$. One obtains with the Cauchy–Schwarz inequality, the inverse inequality (3.12), the Young inequality, and the condition (3.39) on the stabilization parameters

$$\begin{aligned}
&A_{\text{pspg}}\left(\left(\boldsymbol{v}^h, q^h\right), \left(\boldsymbol{v}^h, q^h\right)\right)\\
&\quad\ge\ \nu\left\|\nabla \boldsymbol{v}^h\right\|^2_{L^2(\Omega)} + \sum_{E\in\mathcal{E}^h} \gamma_E \left\|\left[\left|q^h\right|\right]_E\right\|^2_{L^2(E)} + \sum_{K\in\mathcal{T}^h} \delta_K \left\|\nabla q^h\right\|^2_{L^2(K)}\\
&\qquad - \sum_{K\in\mathcal{T}^h} \delta_K \nu \left\|\Delta \boldsymbol{v}^h\right\|_{L^2(K)} \left\|\nabla q^h\right\|_{L^2(K)}\\
&\quad\ge\ \left\|\left(\boldsymbol{v}^h, q^h\right)\right\|^2_{\text{pspg}} - \sum_{K\in\mathcal{T}^h} \delta_K h_K^{-1} C_{\text{inv}} \nu \left\|\nabla \boldsymbol{v}^h\right\|_{L^2(K)} \left\|\nabla q^h\right\|_{L^2(K)}\\
&\quad\ge\ \left\|\left(\boldsymbol{v}^h, q^h\right)\right\|^2_{\text{pspg}} - \frac{1}{2}\sum_{K\in\mathcal{T}^h} \frac{\delta_K C_{\text{inv}}^2 \nu^2}{h_K^2} \left\|\nabla \boldsymbol{v}^h\right\|^2_{L^2(K)} - \frac{1}{2}\sum_{K\in\mathcal{T}^h} \delta_K \left\|\nabla q^h\right\|^2_{L^2(K)}\\
&\quad\ge\ \frac{1}{2}\left\|\left(\boldsymbol{v}^h, q^h\right)\right\|^2_{\text{pspg}}.
\end{aligned} \tag{3.40}$$

The PSPG problem (3.33) is equivalent to a system of linear algebraic equations with a square matrix. The coercivity (3.40) implies that the homogeneous PSPG problem (for $\boldsymbol{f} = \mathbf{0}$) has only the trivial solution. Consequently, the matrix is non-singular, which proves the lemma. $\square$

Since the stabilization parameters have to satisfy (3.39), they depend on the local mesh size. Hence, the norm $\|\cdot\|_{\text{pspg}}$ is a mesh-dependent norm. Note that in the case that $\Delta \boldsymbol{v}^h|_K = \mathbf{0}$ for all mesh cells K, as it is given, e.g., for P_1 finite elements, the restriction (3.39) on the stabilization parameter is not necessary.

Lemma 3.4.4 (Stability estimate)**.** *Let the assumptions of Lemmas 3.4.2 and 3.4.3 be satisfied. Then the solution of the PSPG problem* (3.33) *satisfies the stability estimate*

$$\left\|\left(\boldsymbol{u}^h, p^h\right)\right\|_{\text{pspg}} \le \frac{C}{\nu^{1/2}} \|\boldsymbol{f}\|_{L^2(\Omega)} + 2\left(\sum_{K\in\mathcal{T}^h} \delta_K \|\boldsymbol{f}\|_{L^2(K)}^2\right)^{1/2}. \tag{3.41}$$

Proof. Using the Cauchy–Schwarz inequality, the Poincaré–Friedrichs inequality, and the Cauchy–Schwarz inequality for sums, one obtains

$$\begin{aligned}
& L_{\text{pspg}}\left(\left(\boldsymbol{v}^h, q^h\right)\right) \\
& \quad \le \|\boldsymbol{f}\|_{L^2(\Omega)} \left\|\boldsymbol{v}^h\right\|_{L^2(\Omega)} + \sum_{K\in\mathcal{T}^h} \delta_K \|\boldsymbol{f}\|_{L^2(K)} \left\|\nabla q^h\right\|_{L^2(K)} \\
& \quad \le C \|\boldsymbol{f}\|_{L^2(\Omega)} \left\|\nabla \boldsymbol{v}^h\right\|_{L^2(\Omega)} \\
& \qquad + \left(\sum_{K\in\mathcal{T}^h} \delta_K \|\boldsymbol{f}\|_{L^2(K)}^2\right)^{1/2} \left(\sum_{K\in\mathcal{T}^h} \delta_K \left\|\nabla q^h\right\|_{L^2(K)}^2\right)^{1/2} \\
& \quad \le \left(\frac{C}{\nu^{1/2}} \|\boldsymbol{f}\|_{L^2(\Omega)} + \left(\sum_{K\in\mathcal{T}^h} \delta_K \|\boldsymbol{f}\|_{L^2(K)}^2\right)^{1/2}\right) \left\|\left(\boldsymbol{v}^h, q^h\right)\right\|_{\text{pspg}},
\end{aligned}$$

for all $\left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h$. Inserting this estimate in (3.33) and setting $\left(\boldsymbol{v}^h, q^h\right) = \left(\boldsymbol{u}^h, p^h\right)$, the stability estimate follows using the coercivity (3.40). $\square$

Lemma 3.4.5 (Consistency and Galerkin orthogonality)**.** *Let the solution of* (3.5) *satisfy* $(\boldsymbol{u}, p) \in H^2(\Omega)^d \times H^1(\Omega)$ *and let* $(\boldsymbol{u}^h, p^h) \in V^h \times Q^h$ *be the solution of the PSPG method* (3.33)*. The PSPG method is consistent, i.e., it holds*

$$A_{\text{pspg}}\left((\boldsymbol{u}, p), \left(\boldsymbol{v}^h, q^h\right)\right) = L_{\text{pspg}}\left(\left(\boldsymbol{v}^h, q^h\right)\right) \quad \forall \left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h \tag{3.42}$$

and it satisfies the Galerkin orthogonality

$$A_{\text{pspg}}\left(\left(\boldsymbol{u} - \boldsymbol{u}^h, p - p^h\right), \left(\boldsymbol{v}^h, q^h\right)\right) = 0 \quad \forall \left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h. \tag{3.43}$$

Proof. The residual vanishes for $(\boldsymbol{u}, p)$ and with that the residual-based stabilization terms in A_{pspg} and L_{pspg} are equal. Moreover, the stabilization term with pressure jumps vanishes since $p \in H^1(\Omega)$. Thus, only the terms from the weak formulation (3.5) remain and since the finite element spaces are conforming, (3.42) holds.

The Galerkin orthogonality is obtained by subtracting (3.33) from (3.42). $\square$

To prove error estimates for the solution of (3.33), we shall need additional assumptions on the stabilization parameters. It will be assumed that there are positive constants δ_0, δ_1 and γ_0, γ_1 independent of ν and h such that

$$0 < \delta_0 \frac{h_K^2}{\nu} \le \delta_K \le \delta_1 \frac{h_K^2}{\nu} \qquad \forall\, K \in \mathcal{T}^h \tag{3.44}$$

and

$$0 < \gamma_0 \frac{h_E}{\nu} \le \gamma_E \le \gamma_1 \frac{h_E}{\nu} \qquad \forall\, E \in \mathcal{E}^h. \tag{3.45}$$

Theorem 3.4.6 (Error estimate)**.** *Let the solution of* (3.5) *satisfy* $(\boldsymbol{u}, p) \in H^{k+1}(\Omega)^d \times H^{l+1}(\Omega)$ *and let* $(\boldsymbol{u}^h, p^h) \in V^h \times Q^h$ *be the solution of the PSPG problem* (3.33)*. Assume that the stabilization parameters satisfy* (3.44) *and* (3.45) *with* $\delta_1 \le 1/C_{\mathrm{inv}}^2$*. Then the following error estimate holds*

$$\left\|\left(\boldsymbol{u} - \boldsymbol{u}^h, p - p^h\right)\right\|_{\mathrm{pspg}} \le C \left(\nu^{1/2} h^k \left\|\boldsymbol{u}\right\|_{H^{k+1}(\Omega)} + \frac{h^{l+1}}{\nu^{1/2}} \left\|p\right\|_{H^{l+1}(\Omega)} \right). \tag{3.46}$$

Proof. The triangle inequality gives

$$\begin{aligned} &\left\|\left(\boldsymbol{u} - \boldsymbol{u}^h, p - p^h\right)\right\|_{\mathrm{pspg}} \\ &\qquad \le \left\|\left(\boldsymbol{u} - I^h \boldsymbol{u}, p - J^h p\right)\right\|_{\mathrm{pspg}} + \left\|\left(\boldsymbol{u}^h - I^h \boldsymbol{u}, p^h - J^h p\right)\right\|_{\mathrm{pspg}}, \end{aligned} \tag{3.47}$$

where I^h and J^h are the interpolation operators satisfying (3.8) and (3.9). Both terms on the right-hand side of (3.47) are estimated separately.

One obtains with the interpolation estimates (3.8) and (3.9), and with the assumptions (3.44) and (3.45) on the stabilization parameters

$$\begin{aligned} &\left\|\left(\boldsymbol{u} - I^h \boldsymbol{u}, p - J^h p\right)\right\|_{\mathrm{pspg}}^2 \\ &\quad \le \ \nu \left\|\nabla\left(\boldsymbol{u} - I^h \boldsymbol{u}\right)\right\|_{L^2(\Omega)}^2 + \frac{\gamma_1 h}{\nu} \sum_{E \in \mathcal{E}^h} \left\|[|p - J^h p|]_E\right\|_{L^2(E)}^2 \\ &\qquad + \frac{\delta_1 h^2}{\nu} \sum_{K \in \mathcal{T}^h} \left\|\nabla\left(p - J^h p\right)\right\|_{L^2(K)}^2 \\ &\quad \le \ C \left(\nu h^{2k} \left\|\boldsymbol{u}\right\|_{H^{k+1}(\Omega)}^2 + \frac{h^{2(l+1)}}{\nu} \left\|p\right\|_{H^{l+1}(\Omega)}^2 \right). \end{aligned} \tag{3.48}$$

The estimate of the second term of (3.47) starts with the coercivity (3.40) and the Galerkin orthogonality (3.43)

$$\begin{aligned}
&\left\|\left(\boldsymbol{u}^h - I^h\boldsymbol{u}, p^h - J^h p\right)\right\|_{\mathrm{pspg}}^2 \\
&\quad\leq\ 2A_{\mathrm{pspg}}\left(\left(\boldsymbol{u}^h - I^h\boldsymbol{u}, p^h - J^h p\right), \left(\boldsymbol{u}^h - I^h\boldsymbol{u}, p^h - J^h p\right)\right) \\
&\quad=\ 2A_{\mathrm{pspg}}\left(\left(\boldsymbol{u} - I^h\boldsymbol{u}, p - J^h p\right), \left(\boldsymbol{u}^h - I^h\boldsymbol{u}, p^h - J^h p\right)\right).
\end{aligned} \qquad (3.49)$$

Now, each term of the right-hand side of (3.49) is estimated separately. The goal of these estimates is to obtain interpolation errors and to hide the other terms in the left-hand side of (3.49).

Using the Cauchy–Schwarz inequality, the Young inequality, and the interpolation estimate (3.8), one obtains for the viscous term

$$\begin{aligned}
&\nu\left(\nabla\left(\boldsymbol{u} - I^h\boldsymbol{u}\right), \nabla\left(\boldsymbol{u}^h - I^h\boldsymbol{u}\right)\right) \\
&\quad\leq\ \nu\left\|\nabla\left(\boldsymbol{u} - I^h\boldsymbol{u}\right)\right\|_{L^2(\Omega)}\left\|\nabla\left(\boldsymbol{u}^h - I^h\boldsymbol{u}\right)\right\|_{L^2(\Omega)} \\
&\quad\leq\ 4\nu\left\|\nabla\left(\boldsymbol{u} - I^h\boldsymbol{u}\right)\right\|_{L^2(\Omega)}^2 + \frac{\nu}{16}\left\|\nabla\left(\boldsymbol{u}^h - I^h\boldsymbol{u}\right)\right\|_{L^2(\Omega)}^2 \\
&\quad\leq\ C\nu h^{2k}\left\|\boldsymbol{u}\right\|_{H^{k+1}(\Omega)}^2 + \frac{\nu}{16}\left\|\nabla\left(\boldsymbol{u}^h - I^h\boldsymbol{u}\right)\right\|_{L^2(\Omega)}^2.
\end{aligned}$$

The last term can be absorbed in the left-hand side of (3.49). In a similar way, using (3.9), one gets

$$\left(\nabla\cdot\left(\boldsymbol{u}^h - I^h\boldsymbol{u}\right), p - J^h p\right) \leq C\frac{h^{2(l+1)}}{\nu}\left\|p\right\|_{H^{l+1}(\Omega)}^2 + \frac{\nu}{16}\left\|\nabla\left(\boldsymbol{u}^h - I^h\boldsymbol{u}\right)\right\|_{L^2(\Omega)}^2.$$

The estimate of the next term requires an integration by parts

$$\begin{aligned}
\left(\nabla\cdot\left(\boldsymbol{u} - I^h\boldsymbol{u}\right), p^h - J^h p\right) \ =\ &\sum_{E\in\mathcal{E}^h}\left(\left(\boldsymbol{u} - I^h\boldsymbol{u}\right)\cdot\boldsymbol{n}_E, [|p^h - J^h p|]_E\right)_E \\
&- \sum_{K\in\mathcal{T}^h}\left(\boldsymbol{u} - I^h\boldsymbol{u}, \nabla\left(p^h - J^h p\right)\right)_K.
\end{aligned} \qquad (3.50)$$

Both terms on the right-hand side of (3.50) are estimated more or less in the same way, e.g., one obtains for the last term with the Cauchy–Schwarz inequality, the Young inequality, the property (3.44) of the stabilization parameters, and the interpolation estimate (3.8)

$$
\begin{aligned}
\sum_{K\in\mathcal{T}^h} \left(\boldsymbol{u}-I^h\boldsymbol{u}, \nabla\left(p^h-J^hp\right)\right)_K &\le \sum_{K\in\mathcal{T}^h} \left\|\boldsymbol{u}-I^h\boldsymbol{u}\right\|_{L^2(K)} \left\|\nabla\left(p^h-J^hp\right)\right\|_{L^2(K)} \\
&\le 4\sum_{K\in\mathcal{T}^h} \frac{1}{\delta_K}\left\|\boldsymbol{u}-I^h\boldsymbol{u}\right\|_{L^2(K)}^2 + \frac{1}{16}\sum_{K\in\mathcal{T}^h}\delta_K\left\|\nabla\left(p^h-J^hp\right)\right\|_{L^2(K)}^2 \\
&\le \frac{4\nu}{\delta_0}\sum_{K\in\mathcal{T}^h} h_K^{-2}\left\|\boldsymbol{u}-I^h\boldsymbol{u}\right\|_{L^2(K)}^2 + \frac{1}{16}\sum_{K\in\mathcal{T}^h}\delta_K\left\|\nabla\left(p^h-J^hp\right)\right\|_{L^2(K)}^2 \\
&\le \frac{C\nu}{\delta_0}h^{2k}\left\|\boldsymbol{u}\right\|_{H^{k+1}(\Omega)}^2 + \frac{1}{16}\sum_{K\in\mathcal{T}^h}\delta_K\left\|\nabla\left(p^h-J^hp\right)\right\|_{L^2(K)}^2 .
\end{aligned}
$$

The estimate of the other term on the right-hand side of (3.50) uses (3.45). All stabilization terms are estimated with the same tools used so far. One gets

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}^h} \left(-\nu\Delta\left(\boldsymbol{u}-I^h\boldsymbol{u}\right), \delta_K\nabla\left(p^h-J^hp\right)\right)_K \\
&\quad\le C\nu h^{2k}\left\|\boldsymbol{u}\right\|_{H^{k+1}(\Omega)}^2 + \frac{1}{16}\sum_{K\in\mathcal{T}^h}\delta_K\left\|\nabla\left(p^h-J^hp\right)\right\|_{L^2(K)}^2 ,
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}^h} \left(\nabla\left(p-J^hp\right), \delta_K\nabla\left(p^h-J^hp\right)\right)_K \\
&\quad\le C\frac{h^{2(l+1)}}{\nu}\left\|p\right\|_{H^{l+1}(\Omega)}^2 + \frac{1}{16}\sum_{K\in\mathcal{T}^h}\delta_K\left\|\nabla\left(p^h-J^hp\right)\right\|_{L^2(K)}^2 .
\end{aligned}
$$

Finally, for the term with the pressure jumps, one gets with (3.9)

$$
\begin{aligned}
&\sum_{E\in\mathcal{E}^h} \gamma_E\left(\left[|p-J^hp|\right]_E, \left[|p^h-J^hp|\right]_E\right)_E \\
&\quad\le C\frac{h^{2(l+1)}}{\nu}\left\|p\right\|_{H^{l+1}(\Omega)}^2 + \frac{1}{16}\sum_{E\in\mathcal{E}^h}\gamma_E\left\|\left[|p^h-J^hp|\right]_E\right\|_{L^2(E)}^2 .
\end{aligned}
$$

Collecting all estimates proves the statement of the theorem. □

To derive an error estimate for the pressure in the L^2 norm, the following auxiliary problem (a kind of Stokes projection) will be considered: Find $(\boldsymbol{w}, r) \in V \times Q$ such that

$$
\begin{aligned}
(\nabla\boldsymbol{w}, \nabla\boldsymbol{v}) - (\nabla\cdot\boldsymbol{v}, r) &= 0 && \forall\, \boldsymbol{v}\in V, \\
-(\nabla\cdot\boldsymbol{w}, q) &= \left(p-p^h, q\right) && \forall\, q\in Q.
\end{aligned} \tag{3.51}
$$

It follows from the theory of linear saddle point problems that (3.51) possesses a unique solution.

Lemma 3.4.7 (Stability estimate for (3.51))**.** *For the unique solution of* (3.51) *there holds the stability estimate*

$$\|\nabla \boldsymbol{w}\|_{L^2(\Omega)} + \|r\|_{L^2(\Omega)} \le C \left\|p - p^h\right\|_{L^2(\Omega)} . \tag{3.52}$$

The constant depends on the inverse of β_{is} *from* (3.6)*.*

Proof. Using (3.6), (3.51), and the Cauchy–Schwarz inequality gives

$$\begin{aligned} \beta_{\text{is}} \|r\|_{L^2(\Omega)} &\le \sup_{\boldsymbol{v} \in V \setminus \{\boldsymbol{0}\}} \frac{(\nabla \cdot \boldsymbol{v}, r)}{\|\nabla \boldsymbol{v}\|_{L^2(\Omega)}} = \sup_{\boldsymbol{v} \in V \setminus \{\boldsymbol{0}\}} \frac{(\nabla \boldsymbol{w}, \nabla \boldsymbol{v})}{\|\nabla \boldsymbol{v}\|_{L^2(\Omega)}} \\ &\le \sup_{\boldsymbol{v} \in V \setminus \{\boldsymbol{0}\}} \frac{\|\nabla \boldsymbol{w}\|_{L^2(\Omega)} \|\nabla \boldsymbol{v}\|_{L^2(\Omega)}}{\|\nabla \boldsymbol{v}\|_{L^2(\Omega)}} = \|\nabla \boldsymbol{w}\|_{L^2(\Omega)} . \end{aligned} \tag{3.53}$$

Inserting $(\boldsymbol{v}, q) = (\boldsymbol{w}, r)$ in (3.51), subtracting both equations, and applying the Cauchy–Schwarz inequality and (3.53) yields

$$\begin{aligned} \|\nabla \boldsymbol{w}\|_{L^2(\Omega)}^2 &= -\left(p - p^h, r\right) \le \left\|p - p^h\right\|_{L^2(\Omega)} \|r\|_{L^2(\Omega)} \\ &\le \frac{1}{\beta_{\text{is}}} \left\|p - p^h\right\|_{L^2(\Omega)} \|\nabla \boldsymbol{w}\|_{L^2(\Omega)} . \end{aligned} \tag{3.54}$$

Combining (3.53) and (3.54) leads to

$$\begin{aligned} \|\nabla \boldsymbol{w}\|_{L^2(\Omega)} + \|r\|_{L^2(\Omega)} &\le \left(1 + \frac{1}{\beta_{\text{is}}}\right) \|\nabla \boldsymbol{w}\|_{L^2(\Omega)} \\ &\le \frac{1}{\beta_{\text{is}}} \left(1 + \frac{1}{\beta_{\text{is}}}\right) \left\|p - p^h\right\|_{L^2(\Omega)} . \end{aligned}$$

□

Theorem 3.4.8 (L^2 estimate of the pressure error)**.** *Assume that the solution of* (3.5) *satisfies* $(\boldsymbol{u}, p) \in H^{k+1}(\Omega)^d \times H^{l+1}(\Omega)$ *and that the stabilization parameters satisfy* (3.44) *and* (3.45) *with* $\delta_1 \le 1/C_{\text{inv}}^2$*. Then there holds the error estimate*

$$\left\|p - p^h\right\|_{L^2(\Omega)} \le C \left(\nu h^k \|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + h^{l+1} \|p\|_{H^{l+1}(\Omega)}\right) .$$

Proof. Let $(\boldsymbol{w}, r)$ be the solution of (3.51). Let $\mathcal{I}^h \boldsymbol{w} \in V^h$ be an interpolant of $\boldsymbol{w}$ satisfying (3.10). Inserting $q = p - p^h$ in (3.51) gives

$$\begin{aligned} \left\|p - p^h\right\|_{L^2(\Omega)}^2 &= -\left(\nabla \cdot \boldsymbol{w}, p - p^h\right) \\ &= -\left(\nabla \cdot \left(\boldsymbol{w} - \mathcal{I}^h \boldsymbol{w}\right), p - p^h\right) - \left(\nabla \cdot \left(\mathcal{I}^h \boldsymbol{w}\right), p - p^h\right) . \end{aligned} \tag{3.55}$$

Consider now the second term on the right-hand side of (3.55). The Galerkin orthogonality (3.43) with $\boldsymbol{v}^h = \mathcal{I}^h \boldsymbol{w}$ and $q^h = 0$ leads to

$$0 = \nu \left(\nabla \left(\boldsymbol{u} - \boldsymbol{u}^h\right), \nabla \mathcal{I}^h \boldsymbol{w}\right) - \left(\nabla \cdot \left(\mathcal{I}^h \boldsymbol{w}\right), p - p^h\right) .$$

Hence, one obtains with the Cauchy–Schwarz inequality, (3.10), and (3.52)

$$\left|\left(\nabla \cdot \left(\mathcal{I}^h \boldsymbol{w}\right), p - p^h\right)\right| \leq C\nu \left\|\nabla(\boldsymbol{u} - \boldsymbol{u}^h)\right\|_{L^2(\Omega)} \left\|p - p^h\right\|_{L^2(\Omega)}. \tag{3.56}$$

The estimate of the first term on the right-hand side of (3.55) starts with integration by parts, followed by the Cauchy–Schwarz inequality and application of (3.44), (3.45), (3.10), and (3.52)

$$\begin{aligned}
&-\left(\nabla \cdot \left(\boldsymbol{w} - \mathcal{I}^h \boldsymbol{w}\right), p - p^h\right) \\
&= \sum_{K \in \mathcal{T}^h} \left(\boldsymbol{w} - \mathcal{I}^h \boldsymbol{w}, \nabla\left(p - p^h\right)\right)_K - \sum_{E \in \mathcal{E}^h} \left(\left(\boldsymbol{w} - \mathcal{I}^h \boldsymbol{w}\right) \cdot \boldsymbol{n}_E, \left[\left|p - p^h\right|\right]_E\right)_E \\
&\leq \left(\sum_{K \in \mathcal{T}^h} \delta_K^{-1} \left\|\boldsymbol{w} - \mathcal{I}^h \boldsymbol{w}\right\|_{L^2(K)}^2\right)^{1/2} \left(\sum_{K \in \mathcal{T}^h} \delta_K \left\|\nabla\left(p - p^h\right)\right\|_{L^2(K)}^2\right)^{1/2} \\
&\quad + \left(\sum_{E \in \mathcal{E}^h} \gamma_E^{-1} \left\|\boldsymbol{w} - \mathcal{I}^h \boldsymbol{w}\right\|_{L^2(E)}^2\right)^{1/2} \left(\sum_{E \in \mathcal{E}^h} \gamma_E \left\|\left[\left|p - p^h\right|\right]_E\right\|_{L^2(E)}^2\right)^{1/2} \\
&\leq C\nu^{1/2} \left(\frac{1}{\delta_0^{1/2}} + \frac{1}{\gamma_0^{1/2}}\right) \left\|\nabla \boldsymbol{w}\right\|_{L^2(\Omega)} \left\|\left(\boldsymbol{u} - \boldsymbol{u}^h, p - p^h\right)\right\|_{\text{pspg}} \\
&\leq C\nu^{1/2} \left(\frac{1}{\delta_0^{1/2}} + \frac{1}{\gamma_0^{1/2}}\right) \left\|p - p^h\right\|_{L^2(\Omega)} \left\|\left(\boldsymbol{u} - \boldsymbol{u}^h, p - p^h\right)\right\|_{\text{pspg}}.
\end{aligned} \tag{3.57}$$

Combining the estimates (3.55), (3.56), and (3.57) yields

$$\left\|p - p^h\right\|_{L^2(\Omega)} \leq C\nu^{1/2} \left\|\left(\boldsymbol{u} - \boldsymbol{u}^h, p - p^h\right)\right\|_{\text{pspg}},$$

where the constant C depends on $\delta_0^{-1/2}$ and $\gamma_0^{-1/2}$. Thus, the final estimate follows from Theorem 3.4.6. $\square$

Theorem 3.4.9 (L^2 estimate of the velocity error)**.** *Let the stabilization parameters satisfy* (3.44) *and* (3.45) *with* $\delta_1 \leq 1/C_{\text{inv}}^2$ *and let the Stokes problem* (3.2) *be regular. Assume that the solution of* (3.5) *satisfies* $(\boldsymbol{u}, p) \in H^{k+1}(\Omega)^d \times H^{l+1}(\Omega)$, *then there holds the error estimate*

$$\left\|\boldsymbol{u} - \boldsymbol{u}^h\right\|_{L^2(\Omega)} \leq C \left(h^{k+1} \left\|\boldsymbol{u}\right\|_{H^{k+1}(\Omega)} + \frac{h^{l+2}}{\nu} \left\|p\right\|_{H^{l+1}(\Omega)}\right).$$

Proof. We start as in the proof of Theorem 3.3.5 since, up to (3.28), the proof is independent of the analyzed method. We shall use the fact that, in view of (3.11), (3.23), and (3.45), the interpolant r^I satisfies

$$\left(\sum_{K\in\mathcal{T}^h} \left\|\nabla r^I\right\|_{L^2(K)}^2\right)^{1/2} \le C\,\|\nabla r\|_{L^2(\Omega)} \le C\nu\left\|\boldsymbol{u}-\boldsymbol{u}^h\right\|_{L^2(\Omega)}, \tag{3.58}$$

$$\left(\sum_{E\in\mathcal{E}^h} \gamma_E\left\|[\![r-r^I]\!]_E\right\|_{L^2(E)}^2\right)^{1/2} \le C\nu^{1/2}h\left\|\boldsymbol{u}-\boldsymbol{u}^h\right\|_{L^2(\Omega)}. \tag{3.59}$$

To estimate the last two terms in (3.27), we employ the Galerkin orthogonality (3.43). Since $\boldsymbol{z}^I \in V^h$, we may set $\left(\boldsymbol{v}^h, q^h\right) = (\boldsymbol{z}^I, 0)$ in (3.43), which gives

$$\nu\left(\nabla(\boldsymbol{u}-\boldsymbol{u}^h), \nabla\boldsymbol{z}^I\right) - \left(\nabla\cdot\boldsymbol{z}^I, p-p^h\right) = 0. \tag{3.60}$$

Furthermore, for $\left(\boldsymbol{v}^h, q^h\right) = (\boldsymbol{0}, r^I)$, one deduces from (3.43) that

$$\begin{aligned}&\left(\nabla\cdot(\boldsymbol{u}-\boldsymbol{u}^h), r^I\right) + \sum_{E\in\mathcal{E}^h}\gamma_E\left([\![p-p^h]\!]_E, [\![r^I]\!]_E\right)_E \\ &\qquad + \sum_{K\in\mathcal{T}^h}\left(-\nu\Delta(\boldsymbol{u}-\boldsymbol{u}^h) + \nabla(p-p^h), \delta_K\nabla r^I\right)_K = 0.\end{aligned} \tag{3.61}$$

Thus, using the property $\nabla\cdot\boldsymbol{z} = 0$ and the fact that $r \in H^1(\Omega)$, one has

$$\begin{aligned}&\nu(\nabla\boldsymbol{z}^I, \nabla(\boldsymbol{u}-\boldsymbol{u}^h)) - (\nabla\cdot(\boldsymbol{u}-\boldsymbol{u}^h), r^I) \\ &\quad = \left(\nabla\cdot(\boldsymbol{z}^I-\boldsymbol{z}), p-p^h\right) + \sum_{E\in\mathcal{E}^h}\gamma_E\left([\![p-p^h]\!]_E, [\![r^I-r]\!]_E\right)_E \\ &\qquad + \sum_{K\in\mathcal{T}^h}\left(-\nu\Delta(\boldsymbol{u}-\boldsymbol{u}^h) + \nabla(p-p^h), \delta_K\nabla r^I\right)_K.\end{aligned} \tag{3.62}$$

Then, applying the Cauchy–Schwarz inequality, (3.44), (3.25), (3.58), and (3.59), one derives

$$
\begin{aligned}
&\nu(\nabla \boldsymbol{z}^I, \nabla(\boldsymbol{u}-\boldsymbol{u}^h)) - (\nabla\cdot(\boldsymbol{u}-\boldsymbol{u}^h), r^I) \\
&\quad\le \left\|\nabla(\boldsymbol{z}-\boldsymbol{z}^I)\right\|_{L^2(\Omega)} \left\|p-p^h\right\|_{L^2(\Omega)} \\
&\qquad + \sum_{E\in\mathcal{E}^h} \gamma_E \left\|[|p-p^h|]_E\right\|_{L^2(E)} \left\|[|r-r^I|]_E\right\|_{L^2(E)} \\
&\qquad + \sum_{K\in\mathcal{T}^h} \nu\delta_K \left\|\Delta(\boldsymbol{u}-\boldsymbol{u}^h)\right\|_{L^2(K)} \left\|\nabla r^I\right\|_{L^2(K)} \\
&\qquad + \sum_{K\in\mathcal{T}^h} \delta_K \left\|\nabla(p-p^h)\right\|_{L^2(K)} \left\|\nabla r^I\right\|_{L^2(K)} \\
&\quad\le Ch\left(\left\|p-p^h\right\|^2_{L^2(\Omega)} + \nu\sum_{E\in\mathcal{E}^h}\gamma_E\left\|[|p-p^h|]_E\right\|^2_{L^2(E)}\right. \\
&\qquad\qquad \left. +\nu\sum_{K\in\mathcal{T}^h}\delta_K\left\|\nabla(p-p^h)\right\|^2_{L^2(K)}\right)^{1/2} \left\|\boldsymbol{u}-\boldsymbol{u}^h\right\|_{L^2(\Omega)} \\
&\qquad + C\nu h\left(\sum_{K\in\mathcal{T}^h} h_K^2\left\|\Delta(\boldsymbol{u}-\boldsymbol{u}^h)\right\|^2_{L^2(K)}\right)^{1/2} \left\|\boldsymbol{u}-\boldsymbol{u}^h\right\|_{L^2(\Omega)}. \qquad (3.63)
\end{aligned}
$$

To estimate the last term, we employ the triangle inequality and (3.12) to obtain

$$
\begin{aligned}
&h_K\left\|\Delta(\boldsymbol{u}-\boldsymbol{u}^h)\right\|_{L^2(K)} \\
&\quad\le h_K\left\|\Delta(\boldsymbol{u}-I^h\boldsymbol{u})\right\|_{L^2(K)} + h_K\left\|\Delta(I^h\boldsymbol{u}-\boldsymbol{u}^h)\right\|_{L^2(K)} \\
&\quad\le h_K\left\|\Delta(\boldsymbol{u}-I^h\boldsymbol{u})\right\|_{L^2(K)} + C_{\text{inv}}\left\|\nabla(I^h\boldsymbol{u}-\boldsymbol{u}^h)\right\|_{L^2(K)} \\
&\quad\le h_K\left\|\Delta(\boldsymbol{u}-I^h\boldsymbol{u})\right\|_{L^2(K)} + C_{\text{inv}}\left\|\nabla(I^h\boldsymbol{u}-\boldsymbol{u})\right\|_{L^2(K)} + C_{\text{inv}}\left\|\nabla(\boldsymbol{u}-\boldsymbol{u}^h)\right\|_{L^2(K)}.
\end{aligned}
$$

Then (3.8) implies that

$$
\left(\sum_{K\in\mathcal{T}^h} h_K^2\left\|\Delta(\boldsymbol{u}-\boldsymbol{u}^h)\right\|^2_{L^2(K)}\right)^{1/2} \le Ch^k\left\|\boldsymbol{u}\right\|_{H^{k+1}(\Omega)} + C\left\|\nabla(\boldsymbol{u}-\boldsymbol{u}^h)\right\|_{L^2(\Omega)}. \qquad (3.64)
$$

Combining (3.27), (3.28), (3.63), and (3.64) gives

$$
\begin{aligned}
\nu\left\|\boldsymbol{u}-\boldsymbol{u}^h\right\|_{L^2(\Omega)} &\le C\nu^{1/2}h\left\|(\boldsymbol{u}-\boldsymbol{u}^h, p-p^h)\right\|_{\text{pspg}} \\
&\quad + Ch\left\|p-p^h\right\|_{L^2(\Omega)} + C\nu h^{k+1}\left\|\boldsymbol{u}\right\|_{H^{k+1}(\Omega)}
\end{aligned}
$$

and the statement of the theorem follows from Theorems 3.4.6 and 3.4.8. □

One observes the usual scaling properties of the error estimates with respect to ν: small values of ν lead to large bounds for velocity errors due to large weights of the pressure contributions in the error bounds whereas large values of ν lead to large bounds for pressure errors due to the scaling of the velocity terms in the error bounds.

For discontinuous pressure approximations, the jump term in (3.34) can be replaced by a so-called local jump term, as proposed in [81, 60]. In this approach, there is an outer sum over appropriate macro mesh cells and than an inner sum of jumps over edges that are strictly in the interior of the macro mesh cells. Numerical studies of this method can be found in [81] and a finite element error analysis for P_1/P_0 and Q_1/Q_0 in [60]. The analysis for the Q_1/Q_0 case was extended to special anisotropic meshes in [69].

If the PSPG method is used with the P_1/P_0 finite element, then it is possible to compute a divergence-free velocity field in $H_{\mathrm{div}}(\Omega)$, where

$$\begin{aligned} H_{\mathrm{div}}(\Omega) \quad = \quad & \big\{\boldsymbol{v} \ : \ \boldsymbol{v} \in L^2(\Omega), \nabla \cdot \boldsymbol{v} \in L^2(\Omega), \nabla \cdot \boldsymbol{v} = 0, \ \text{and } \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega \\ & \text{in the sense of traces}\big\}. \end{aligned}$$

with an inexpensive post-processing step, see [12]. The idea consists in adding to $\boldsymbol{u}^h$ a correction $\boldsymbol{u}^h_{\mathrm{RT}_0} \in \mathrm{RT}_0$, the Raviart–Thomas space of lowest order, such that $\nabla \cdot \big(\boldsymbol{u}^h + \boldsymbol{u}^h_{\mathrm{RT}_0}\big) = 0$ in $L^2(\Omega)$. Details of this approach and some numerical results can be found also in [56, Remark 4.102, Example 4.103].

The paper [79] studies a stabilization of somewhat general form, which contains as special cases the PSPG method and the inf-sub stable MINI element from [5]. Error estimates are derived for both, the $H^1(\Omega)$ and the $L^2(\Omega)$ norm of the velocity and the pressure. A PSPG method with weak imposition of the boundary condition using a penalty-free Nitsche method was analyzed in [22]. It was shown in [9] that a PSPG-type method, with an appropriate stabilization parameter, can be used to stabilize discrete inf-sup conditions of the dual Darcy problem and of the curl formulation of Maxwell's problem.

Remark 3.4.10 (Anisotropic meshes). The PSPG method for the Q_1/Q_1 pair of finite element spaces on anisotropic quadrilateral grids aligned with the Cartesian coordinate axes was studied in [16]. The definition of the stabilization parameter includes both edge lengths of the quadrilateral cells.

The PSPG method on anisotropic grids was studied for the P_1/P_1 pair of spaces in [73]. A finite element analysis is presented, where the stabilization parameter is of the form

$$\delta_K = \delta \frac{h_{\mathrm{K,min}}}{\nu},$$

with $h_{\mathrm{K,min}}$ being the smaller characteristic length of K obtained via the polar decomposition of the matrix from the affine map from a standard reference cell to K.

A PSPG method on anisotropic grids in boundary layers, in the context of the Oseen equations, was studied in [2]. For the Stokes equations, the stabilization parameter has the form

$$\delta_K = \delta \frac{h_{\mathrm{K,min}}}{C_{\mathrm{inv}}^2 \nu},$$

where $h_{\mathrm{K,min}}$ is some kind of minimal length of the mesh cell K, e.g., the shortest edge for mesh cells of brick form. △

A modification of the PSPG method for continuous discrete pressure that is stable for stabilization parameters $\delta = \delta_0 h^2/\nu$ with arbitrary $\delta_0 > 0$, in contrast to condition (3.39), will be discussed briefly in Section 3.4.5.

3.4.3 The (Symmetric) Galerkin Least Squares (GLS) Method

The (symmetric) Galerkin Least Squares (GLS) method uses, like the PSPG method (3.33) – (3.35), the residual of the strong form of the equation. In contrast to the PSPG method, the operator of the strong form of the equation is applied also to the test functions. Hence, the application of a GLS method is a little bit more expensive than the use of the PSPG method.

The symmetric GLS method was proposed in [54]. It has the following form: Find $(\boldsymbol{u}^h, p^h) \in V^h \times Q^h$ such that

$$A_{\mathrm{sgls}}\left(\left(\boldsymbol{u}^h, p^h\right), \left(\boldsymbol{v}^h, q^h\right)\right) = L_{\mathrm{sgls}}\left(\left(\boldsymbol{v}^h, q^h\right)\right) \quad \forall\ \left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h, \tag{3.65}$$

with

$$\begin{aligned} A_{\mathrm{sgls}}\left((\boldsymbol{u}, p), (\boldsymbol{v}, q)\right) &= \nu\left(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}\right) - \left(\nabla \cdot \boldsymbol{v}, p\right) + \left(\nabla \cdot \boldsymbol{u}, q\right) \\ &\quad + \sum_{E \in \mathcal{E}^h} \gamma_E \left([|p|]_E, [|q|]_E\right)_E + \sum_{K \in \mathcal{T}^h} \left(-\nu \Delta \boldsymbol{u} + \nabla p, \delta_K \left(\nu \Delta \boldsymbol{v} + \nabla q\right)\right)_K, \end{aligned} \tag{3.66}$$

$$L_{\mathrm{sgls}}\left((\boldsymbol{v}, q)\right) = (\boldsymbol{f}, \boldsymbol{v}) + \sum_{K \in \mathcal{T}^h} \left(\boldsymbol{f}, \delta_K \left(\nu \Delta \boldsymbol{v} + \nabla q\right)\right)_K. \tag{3.67}$$

Remark 3.4.11. The discretization (3.65) can be equivalently written in the form

$$\tilde{A}_{\mathrm{sgls}}\left(\left(\boldsymbol{u}^h, p^h\right), \left(\boldsymbol{v}^h, q^h\right)\right) = \tilde{L}_{\mathrm{sgls}}\left(\left(\boldsymbol{v}^h, q^h\right)\right) \quad \forall\ \left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h, \tag{3.68}$$

where $\tilde{A}_{\mathrm{sgls}}\left((\boldsymbol{u}, p), (\boldsymbol{v}, q)\right) = A_{\mathrm{sgls}}\left((\boldsymbol{u}, p), (\boldsymbol{v}, -q)\right)$ and $\tilde{L}_{\mathrm{sgls}}\left((\boldsymbol{v}, q)\right) = L_{\mathrm{sgls}}\left((\boldsymbol{v}, -q)\right)$. It is easy to see that the bilinear form $\tilde{A}_{\mathrm{sgls}}$ is symmetric, which is the reason for calling the discretization (3.65) *symmetric* GLS method. The form (3.68) is typically used in implementations. However, to unify the presentation of the various methods, we consider (3.65) for the analysis. △

To simplify the subsequent considerations, the analysis will be given only for the case of continuous pressure finite element spaces, i.e., $Q^h \subset H^1(\Omega)$. In this case, the pressure jumps across faces in (3.66) vanish. Discontinuous pressure approximations are discussed briefly in Remark 3.4.18.

As before, it will be assumed that the stabilization parameter δ_K satisfies (3.44).

Defining an extended $L^2(\Omega)$ norm for the pressure

$$\|q\|_{\mathrm{ext}} = \left(\frac{1}{\nu} \|q\|^2_{L^2(\Omega)} + \sum_{K \in \mathcal{T}^h} \delta_K \|\nabla q\|^2_{L^2(K)} \right)^{1/2},$$

the norm for the analysis of the symmetric GLS method is given by

$$\|(\boldsymbol{v}, q)\|_{\mathrm{sgls}} = \left(\nu \|\nabla \boldsymbol{v}\|^2_{L^2(\Omega)} + \|q\|^2_{\mathrm{ext}} \right)^{1/2}. \tag{3.69}$$

In contrast to the bilinear form of the PSPG method, the bilinear form A_{sgls} is not coercive. However, we shall show that it satisfies an inf–sup condition, which is sufficient for proving the unique solvability and error estimates for the symmetric GLS method. First, let us prove the following auxiliary result.

Lemma 3.4.12 (Weaker estimate in the spirit of the discrete inf-sup condition)**.** *There are positive constants C_1 and C_2 independent of h such that for all $q \in Q \cap H^1(\Omega)$, it holds*

$$\sup_{\boldsymbol{v}^h \in V^h \setminus \{\boldsymbol{0}\}} \frac{\left(\nabla \cdot \boldsymbol{v}^h, q\right)}{\|\nabla \boldsymbol{v}^h\|_{L^2(\Omega)}} \geq C_1 \|q\|_{L^2(\Omega)} - C_2 \left(\sum_{K \in \mathcal{T}^h} h_K^2 \|\nabla q\|^2_{L^2(K)} \right)^{1/2}. \tag{3.70}$$

Proof. Choose $q \in Q \cap H^1(\Omega) \setminus \{0\}$ arbitrarily but fixed. The idea of the proof consists in constructing a function $\boldsymbol{w}^h \in V^h$ such that an inequality of form (3.70) is already satisfied with $\boldsymbol{w}^h$.

In view of the inf–sup condition (3.6), there exists $\boldsymbol{w} \in V$ such that

$$\nabla \cdot \boldsymbol{w} = q, \quad \|\nabla \boldsymbol{w}\|_{L^2(\Omega)} \leq \frac{1}{\beta_{\mathrm{is}}} \|q\|_{L^2(\Omega)},$$

see [56, Cor. 3.44]. It follows that

$$\frac{(\nabla \cdot \boldsymbol{w}, q)}{\|\nabla \boldsymbol{w}\|_{L^2(\Omega)}} = \frac{(q, q)}{\|\nabla \boldsymbol{w}\|_{L^2(\Omega)}} \geq \beta_{\mathrm{is}} \|q\|_{L^2(\Omega)}. \tag{3.71}$$

Let $\boldsymbol{w}^h = \mathcal{I}^h \boldsymbol{w} \in V^h$ be an interpolant of $\boldsymbol{w}$ satisfying (3.10). Then, using (3.71), integration by parts, the Cauchy–Schwarz inequality, the Cauchy–Schwarz inequality for sums, and (3.10) yields

$$
\begin{aligned}
&\left(\nabla\cdot\boldsymbol{w}^h,q\right)\\
&\quad= \left(\nabla\cdot\left(\boldsymbol{w}^h-\boldsymbol{w}\right),q\right)+\left(\nabla\cdot\boldsymbol{w},q\right)\\
&\quad\ge \left(\boldsymbol{w}-\boldsymbol{w}^h,\nabla q\right)+\beta_{\mathrm{is}}\left\|q\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{w}\right\|_{L^2(\Omega)}\\
&\quad\ge -\left(\sum_{K\in\mathcal{T}^h}h_K^{-2}\left\|\boldsymbol{w}-\boldsymbol{w}^h\right\|_{L^2(K)}^2\right)^{1/2}\left(\sum_{K\in\mathcal{T}^h}h_K^{2}\left\|\nabla q\right\|_{L^2(K)}^2\right)^{1/2}\\
&\qquad +\beta_{\mathrm{is}}\left\|q\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{w}\right\|_{L^2(\Omega)}\\
&\quad\ge -C\left\|\nabla\boldsymbol{w}\right\|_{L^2(\Omega)}\left(\sum_{K\in\mathcal{T}^h}h_K^{2}\left\|\nabla q\right\|_{L^2(K)}^2\right)^{1/2}+\beta_{\mathrm{is}}\left\|q\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{w}\right\|_{L^2(\Omega)}\\
&\quad= \left[\beta_{\mathrm{is}}\left\|q\right\|_{L^2(\Omega)}-C\left(\sum_{K\in\mathcal{T}^h}h_K^{2}\left\|\nabla q\right\|_{L^2(K)}^2\right)^{1/2}\right]\left\|\nabla\boldsymbol{w}\right\|_{L^2(\Omega)}. \qquad (3.72)
\end{aligned}
$$

If the expression in the square brackets in (3.72) is positive, it follows that $\boldsymbol{w}^h\neq\mathbf{0}$ and then using (3.10) and (3.72) yields

$$
\frac{\left(\nabla\cdot\boldsymbol{w}^h,q\right)}{\left\|\nabla\boldsymbol{w}^h\right\|_{L^2(\Omega)}}\ge C\frac{\left(\nabla\cdot\boldsymbol{w}^h,q\right)}{\left\|\nabla\boldsymbol{w}\right\|_{L^2(\Omega)}}\ge C_1\left\|q\right\|_{L^2(\Omega)}-C_2\left(\sum_{K\in\mathcal{T}^h}h_K^{2}\left\|\nabla q\right\|_{L^2(K)}^2\right)^{1/2}. \qquad (3.73)
$$

If the right-hand side of (3.73) (which is a multiple of the expression in the square brackets in (3.72)) is nonpositive, one chooses an arbitrary $\boldsymbol{w}^h\in V^h\setminus\{\mathbf{0}\}$ for which the left-hand side of (3.73) is nonnegative, such that (3.73) holds also in this case. □

Lemma 3.4.13 (Inf-sup condition for the bilinear form A_{sgls})**.** *Let $Q^h\subset H^1(\Omega)$. Let the conditions* (3.44) *on* $\{\delta_K\}$ *be satisfied and let*

$$
\delta_1<\frac{1}{C_{\mathrm{inv}}^2}. \qquad (3.74)
$$

Then, there is a positive constant C such that for all $\left(\boldsymbol{v}^h,q^h\right)\in V^h\times Q^h$, it holds

$$
\sup_{\left(\boldsymbol{w}^h,r^h\right)\in V^h\times Q^h\setminus\{(\mathbf{0},0)\}}\frac{A_{\mathrm{sgls}}\left(\left(\boldsymbol{v}^h,q^h\right),\left(\boldsymbol{w}^h,r^h\right)\right)}{\left\|\left(\boldsymbol{w}^h,r^h\right)\right\|_{\mathrm{sgls}}}\ge C\left\|\left(\boldsymbol{v}^h,q^h\right)\right\|_{\mathrm{sgls}}. \qquad (3.75)
$$

Proof. Consider an arbitrary pair $\left(\boldsymbol{v}^h,q^h\right)\in V^h\times Q^h$. The idea of the proof consists in constructing a pair $\left(\boldsymbol{w}^h,r^h\right)\in V^h\times Q^h\setminus\{(\mathbf{0},0)\}$ that satisfies inequality (3.75).

First, assume that $q^h \neq 0$. According to the proof of Lemma 3.4.12, there is $\boldsymbol{z}^h \in V^h \setminus \{\boldsymbol{0}\}$ such that (3.70) holds for $\boldsymbol{v}^h = \boldsymbol{z}^h$ and $q = q^h$ without the supremum. Note that (3.70) also holds when $\boldsymbol{z}^h$ is multiplied by any positive number. Accordingly, one chooses $\boldsymbol{z}^h$ such that

$$\left\|\nabla \boldsymbol{z}^h\right\|_{L^2(\Omega)} = \frac{1}{\nu}\left\|q^h\right\|_{L^2(\Omega)}. \tag{3.76}$$

Now, the pair for which the satisfaction of (3.75) will be shown is

$$\left(\boldsymbol{w}^h, r^h\right) = \left(\boldsymbol{v}^h - \kappa \boldsymbol{z}^h, q^h\right), \tag{3.77}$$

where κ will be chosen appropriately in the forthcoming analysis. It is

$$\begin{aligned} & A_{\text{sgls}}\left(\left(\boldsymbol{v}^h, q^h\right), \left(\boldsymbol{w}^h, r^h\right)\right) \\ = \; & A_{\text{sgls}}\left(\left(\boldsymbol{v}^h, q^h\right), \left(\boldsymbol{v}^h, q^h\right)\right) + \kappa A_{\text{sgls}}\left(\left(\boldsymbol{v}^h, q^h\right), \left(-\boldsymbol{z}^h, 0\right)\right). \end{aligned} \tag{3.78}$$

Both terms on the right-hand side of this identity will be studied separately.

With the Cauchy–Schwarz inequality, one obtains

$$\begin{aligned} & A_{\text{sgls}}\left(\left(\boldsymbol{v}^h, q^h\right), \left(-\boldsymbol{z}^h, 0\right)\right) \\ = \; & -\nu\left(\nabla \boldsymbol{v}^h, \nabla \boldsymbol{z}^h\right) + \left(\nabla \cdot \boldsymbol{z}^h, q^h\right) - \sum_{K\in\mathcal{T}^h} \delta_K\left(-\nu\Delta \boldsymbol{v}^h + \nabla q^h, \nu\Delta \boldsymbol{z}^h\right)_K \\ \geq \; & -\nu\left\|\nabla \boldsymbol{v}^h\right\|_{L^2(\Omega)}\left\|\nabla \boldsymbol{z}^h\right\|_{L^2(\Omega)} + \left(\nabla \cdot \boldsymbol{z}^h, q^h\right) \\ & +\nu^2 \sum_{K\in\mathcal{T}^h} \delta_K\left(\Delta \boldsymbol{v}^h, \Delta \boldsymbol{z}^h\right)_K - \nu \sum_{K\in\mathcal{T}^h} \delta_K\left(\nabla q^h, \Delta \boldsymbol{z}^h\right)_K. \end{aligned} \tag{3.79}$$

Each term on the right-hand side of (3.79) will be estimated from below.

Using (3.70) (without supremum) with $(\boldsymbol{v}^h, q) = (\boldsymbol{z}^h, q^h)$, (3.76), and (3.44) yields

$$\begin{aligned} \left(\nabla \cdot \boldsymbol{z}^h, q^h\right) \; \geq \; & \left(C_1\left\|q^h\right\|_{L^2(\Omega)} - C_2\left(\sum_{K\in\mathcal{T}^h} h_K^2\left\|\nabla q^h\right\|_{L^2(K)}^2\right)^{1/2}\right)\left\|\nabla \boldsymbol{z}^h\right\|_{L^2(\Omega)} \\ = \; & \frac{C_1}{\nu}\left\|q^h\right\|_{L^2(\Omega)}^2 - \frac{C_2}{\nu}\left(\sum_{K\in\mathcal{T}^h} h_K^2\left\|\nabla q^h\right\|_{L^2(K)}^2\right)^{1/2}\left\|q^h\right\|_{L^2(\Omega)} \\ \geq \; & \frac{C_1}{\nu}\left\|q^h\right\|_{L^2(\Omega)}^2 - \frac{C_2}{\delta_0^{1/2}}\left(\sum_{K\in\mathcal{T}^h} \delta_K\left\|\nabla q^h\right\|_{L^2(K)}^2\right)^{1/2}\frac{1}{\nu^{1/2}}\left\|q^h\right\|_{L^2(\Omega)}. \end{aligned}$$

Applying the Cauchy–Schwarz inequality, the inverse inequality (3.12), (3.44), the

Cauchy–Schwarz inequality for sums, and (3.76) yields

$$\begin{aligned}
\nu^2 \sum_{K\in\mathcal{T}^h} \delta_K \left(\Delta \boldsymbol{v}^h, \Delta \boldsymbol{z}^h\right)_K &\geq -\nu^2 \sum_{K\in\mathcal{T}^h} \delta_K \left\|\Delta \boldsymbol{v}^h\right\|_{L^2(K)} \left\|\Delta \boldsymbol{z}^h\right\|_{L^2(K)} \\
&\geq -\nu^2 C_{\text{inv}}^2 \sum_{K\in\mathcal{T}^h} \delta_K h_K^{-2} \left\|\nabla \boldsymbol{v}^h\right\|_{L^2(K)} \left\|\nabla \boldsymbol{z}^h\right\|_{L^2(K)} \\
&\geq -\nu C_{\text{inv}}^2 \delta_1 \sum_{K\in\mathcal{T}^h} \left\|\nabla \boldsymbol{v}^h\right\|_{L^2(K)} \left\|\nabla \boldsymbol{z}^h\right\|_{L^2(K)} \\
&\geq -\nu C_{\text{inv}}^2 \delta_1 \left\|\nabla \boldsymbol{v}^h\right\|_{L^2(\Omega)} \left\|\nabla \boldsymbol{z}^h\right\|_{L^2(\Omega)} \\
&= -C_{\text{inv}}^2 \delta_1 \nu^{1/2} \left\|\nabla \boldsymbol{v}^h\right\|_{L^2(\Omega)} \frac{1}{\nu^{1/2}} \left\|q^h\right\|_{L^2(\Omega)}.
\end{aligned}$$

The estimate of the third term uses the same tools

$$\begin{aligned}
&-\nu \sum_{K\in\mathcal{T}^h} \delta_K \left(\nabla q^h, \Delta \boldsymbol{z}^h\right)_K \\
&\geq \quad -\nu \left(\sum_{K\in\mathcal{T}^h} \delta_K \left\|\nabla q^h\right\|_{L^2(K)}^2\right)^{1/2} \left(\sum_{K\in\mathcal{T}^h} \delta_K \left\|\Delta \boldsymbol{z}^h\right\|_{L^2(K)}^2\right)^{1/2} \\
&\geq \quad -C_{\text{inv}} \delta_1^{1/2} \left(\sum_{K\in\mathcal{T}^h} \delta_K \left\|\nabla q^h\right\|_{L^2(K)}^2\right)^{1/2} \frac{1}{\nu^{1/2}} \left\|q^h\right\|_{L^2(\Omega)}.
\end{aligned}$$

Inserting all estimates in (3.79) and applying (3.76) leads to

$$\begin{aligned}
&A_{\text{sgls}}\left(\left(\boldsymbol{v}^h, q^h\right), \left(-\boldsymbol{z}^h, 0\right)\right) \\
&\geq \quad -\left(1 + C_{\text{inv}}^2 \delta_1\right) \nu^{1/2} \left\|\nabla \boldsymbol{v}^h\right\|_{L^2(\Omega)} \frac{1}{\nu^{1/2}} \left\|q^h\right\|_{L^2(\Omega)} + \frac{C_1}{\nu} \left\|q^h\right\|_{L^2(\Omega)}^2 \\
&\qquad - \left(\frac{C_2}{\delta_0^{1/2}} + C_{\text{inv}} \delta_1^{1/2}\right) \left(\sum_{K\in\mathcal{T}^h} \delta_K \left\|\nabla q^h\right\|_{L^2(K)}^2\right)^{1/2} \frac{1}{\nu^{1/2}} \left\|q^h\right\|_{L^2(\Omega)} \\
&= \quad -C_3\, \nu^{1/2} \left\|\nabla \boldsymbol{v}^h\right\|_{L^2(\Omega)} \frac{1}{\nu^{1/2}} \left\|q^h\right\|_{L^2(\Omega)} + \frac{C_1}{\nu} \left\|q^h\right\|_{L^2(\Omega)}^2 \\
&\qquad -C_4 \left(\sum_{K\in\mathcal{T}^h} \delta_K \left\|\nabla q^h\right\|_{L^2(K)}^2\right)^{1/2} \frac{1}{\nu^{1/2}} \left\|q^h\right\|_{L^2(\Omega)},
\end{aligned}$$

with positive constants C_3 and C_4 that do not depend on ν, but C_4 depends on $\delta_0^{-1/2}$. The application of the Young inequality with some $\varepsilon > 0$ gives

$$\begin{aligned}
& A_{\text{sgls}}\left(\left(\boldsymbol{v}^h, q^h\right), \left(-\boldsymbol{z}^h, 0\right)\right) \\
\geq\ & \left(C_1 - \frac{\varepsilon}{2}\left(C_3 + C_4\right)\right)\frac{1}{\nu}\left\|q^h\right\|_{L^2(\Omega)}^2 - \frac{C_3}{2\varepsilon}\nu\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)}^2 \\
& -\frac{C_4}{2\varepsilon}\sum_{K\in\mathcal{T}^h}\delta_K\left\|\nabla q^h\right\|_{L^2(K)}^2 .
\end{aligned}$$

Choosing now $0 < \varepsilon < 2C_1/(C_3 + C_4)$ leads to

$$\begin{aligned}
& A_{\text{sgls}}\left(\left(\boldsymbol{v}^h, q^h\right), \left(-\boldsymbol{z}^h, 0\right)\right) \\
\geq\ & C_5\frac{1}{\nu}\left\|q^h\right\|_{L^2(\Omega)}^2 - C_6\nu\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)}^2 - C_7\sum_{K\in\mathcal{T}^h}\delta_K\left\|\nabla q^h\right\|_{L^2(K)}^2 ,
\end{aligned} \tag{3.80}$$

with positive constants C_5, C_6, and C_7.

Now, the first term on the right-hand side of (3.78) will be estimated. Using the definition (3.66) gives

$$\begin{aligned}
& A_{\text{sgls}}\left(\left(\boldsymbol{v}^h, q^h\right), \left(\boldsymbol{v}^h, q^h\right)\right) \\
=\ & \nu\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)}^2 + \sum_{K\in\mathcal{T}^h}\delta_K\left(-\nu\Delta\boldsymbol{v}^h + \nabla q^h, \nu\Delta\boldsymbol{v}^h + \nabla q^h\right)_K \\
=\ & \nu\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)}^2 + \sum_{K\in\mathcal{T}^h}\delta_K\left\|\nabla q^h\right\|_{L^2(K)}^2 - \nu^2\sum_{K\in\mathcal{T}^h}\delta_K\left\|\Delta\boldsymbol{v}^h\right\|_{L^2(K)}^2 .
\end{aligned}$$

By using (3.12) and (3.44), one obtains

$$\begin{aligned}
& A_{\text{sgls}}\left(\left(\boldsymbol{v}^h, q^h\right), \left(\boldsymbol{v}^h, q^h\right)\right) \\
\geq\ & \nu\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)}^2 + \sum_{K\in\mathcal{T}^h}\delta_K\left\|\nabla q^h\right\|_{L^2(K)}^2 - \nu C_{\text{inv}}^2\delta_1\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)}^2 \\
=\ & \left(1 - C_{\text{inv}}^2\delta_1\right)\nu\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)}^2 + \sum_{K\in\mathcal{T}^h}\delta_K\left\|\nabla q^h\right\|_{L^2(K)}^2 .
\end{aligned}$$

By the assumption (3.74) on δ_1, the term in the parentheses is positive. Hence, with a positive constant C_8, it is

$$A_{\text{sgls}}\left(\left(\boldsymbol{v}^h, q^h\right), \left(\boldsymbol{v}^h, q^h\right)\right) \geq C_8\nu\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)}^2 + \sum_{K\in\mathcal{T}^h}\delta_K\left\|\nabla q^h\right\|_{L^2(K)}^2 . \tag{3.81}$$

Inserting (3.80) and (3.81) in (3.78) yields

$$\begin{aligned}
& A_{\text{sgls}}\left(\left(\boldsymbol{v}^h, q^h\right), \left(\boldsymbol{w}^h, r^h\right)\right) \\
\geq\ & \left(C_8 - \kappa C_6\right)\nu\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)}^2 + \kappa C_5\frac{1}{\nu}\left\|q^h\right\|_{L^2(\Omega)}^2 \\
& + \left(1 - \kappa C_7\right)\sum_{K\in\mathcal{T}^h}\delta_K\left\|\nabla q^h\right\|_{L^2(K)}^2 .
\end{aligned}$$

Choosing now $0 < \kappa < \min\{C_8/C_6, 1/C_7\}$ leads to the existence of a positive constant C_9 such that

$$A_{\mathrm{sgls}}\left(\left(\boldsymbol{v}^h, q^h\right), \left(\boldsymbol{w}^h, r^h\right)\right) \geq C_9 \left\|\left(\boldsymbol{v}^h, q^h\right)\right\|_{\mathrm{sgls}}^2. \tag{3.82}$$

Considering the denominator of (3.75), using the definition (3.77) of $\left(\boldsymbol{w}^h, r^h\right)$, the triangle inequality, and (3.76) yields

$$\begin{aligned} \left\|\left(\boldsymbol{w}^h, r^h\right)\right\|_{\mathrm{sgls}} &= \left(\nu\left\|\nabla\left(\boldsymbol{v}^h - \kappa\boldsymbol{z}^h\right)\right\|_{L^2(\Omega)}^2 + \left\|q^h\right\|_{\mathrm{ext}}^2\right)^{1/2} \\ &\leq \left(2\nu\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)}^2 + 2\kappa^2\nu\left\|\nabla\boldsymbol{z}^h\right\|_{L^2(\Omega)}^2 + \left\|q^h\right\|_{\mathrm{ext}}^2\right)^{1/2} \\ &= \left(2\nu\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)}^2 + 2\kappa^2\frac{1}{\nu}\left\|q^h\right\|_{L^2(\Omega)}^2 + \left\|q^h\right\|_{\mathrm{ext}}^2\right)^{1/2} \\ &\leq \left(2+2\kappa^2\right)^{1/2}\left\|\left(\boldsymbol{v}^h, q^h\right)\right\|_{\mathrm{sgls}} = C_{10}\left\|\left(\boldsymbol{v}^h, q^h\right)\right\|_{\mathrm{sgls}} \end{aligned} \tag{3.83}$$

with a positive constant C_{10} that is independent of ν.

Combining (3.82) and (3.83) gives the inf-sup condition (3.75).

Finally, if $q_h = 0$, the inf-sup condition (3.75) immediately follows from (3.81). $\square$

The proof of the inf-sup condition for the bilinear form A_{sgls} requires an upper bound of the stabilization parameter, hence this method is not absolutely stable. Note that the bound (3.74) for δ_1 depends on the polynomial degree, compare the note after (3.12).

Lemma 3.4.14 (Existence and uniqueness of a solution of (3.65))**.** *Let the assumptions of Lemma 3.4.13 be satisfied, then the symmetric GLS problem* (3.65) *possesses a unique solution.*

Proof. The existence and uniqueness of the solution follows analogously as in the proof of Lemma 3.4.3 since the inf-sup condition (3.75) implies that the homogeneous symmetric GLS problem has only the trivial solution. $\square$

Lemma 3.4.15 (Consistency and Galerkin orthogonality)**.** *Let the solution of* (3.5) *satisfy* $(\boldsymbol{u}, p) \in H^2(\Omega)^d \times H^1(\Omega)$ *and let* $(\boldsymbol{u}^h, p^h) \in V^h \times Q^h$ *be the solution of the symmetric GLS method* (3.65)*. This method is consistent, i.e., it holds*

$$A_{\mathrm{sgls}}\left((\boldsymbol{u}, p), \left(\boldsymbol{v}^h, q^h\right)\right) = L_{\mathrm{sgls}}\left(\left(\boldsymbol{v}^h, q^h\right)\right) \quad \forall\ \left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h \tag{3.84}$$

and it satisfies the Galerkin orthogonality

$$A_{\mathrm{sgls}}\left(\left(\boldsymbol{u} - \boldsymbol{u}^h, p - p^h\right), \left(\boldsymbol{v}^h, q^h\right)\right) = 0 \quad \forall\ \left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h. \tag{3.85}$$

Proof. The lemma follows in the same way as Lemma 3.4.5. $\square$

Theorem 3.4.16 (Error estimate)**.** *Let the assumptions of Lemma 3.4.13 be satisfied. Assume that the solution of* (3.5) *satisfies* $(\boldsymbol{u}, p) \in H^{k+1}(\Omega)^d \times H^{l+1}(\Omega)$, *then there holds the error estimate*

$$\left\| \left(\boldsymbol{u} - \boldsymbol{u}^h, p - p^h \right) \right\|_{\mathrm{sgls}} \leq C \left(\nu^{1/2} h^k \left\| \boldsymbol{u} \right\|_{H^{k+1}(\Omega)} + \frac{h^{l+1}}{\nu^{1/2}} \left\| p \right\|_{H^{l+1}(\Omega)} \right).$$

Proof. Let I^h and J^h be the interpolation operators satisfying (3.8) and (3.9). From the proof of Lemma 3.4.13, it is known that there is a pair $\left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h$ such that

$$\left\| \left(\boldsymbol{u}^h - I^h \boldsymbol{u}, p^h - J^h p \right) \right\|_{\mathrm{sgls}} \leq C \frac{A_{\mathrm{sgls}} \left(\left(\boldsymbol{u}^h - I^h \boldsymbol{u}, p^h - J^h p \right), \left(\boldsymbol{v}^h, q^h \right) \right)}{\left\| \left(\boldsymbol{v}^h, q^h \right) \right\|_{\mathrm{sgls}}}.$$

With the Galerkin orthogonality (3.85) of the symmetric GLS method, one obtains

$$\left\| \left(\boldsymbol{u}^h - I^h \boldsymbol{u}, p^h - J^h p \right) \right\|_{\mathrm{sgls}} \leq C \frac{A_{\mathrm{sgls}} \left(\left(\boldsymbol{u} - I^h \boldsymbol{u}, p - J^h p \right), \left(\boldsymbol{v}^h, q^h \right) \right)}{\left\| \left(\boldsymbol{v}^h, q^h \right) \right\|_{\mathrm{sgls}}}. \tag{3.86}$$

Now, all terms of the numerator of the right-hand side of (3.86) will be estimated such that the contribution from $\left(\boldsymbol{v}^h, q^h\right)$ can be bounded by $\left\| \left(\boldsymbol{v}^h, q^h\right) \right\|_{\mathrm{sgls}}$. With the Cauchy–Schwarz inequality and (3.7), one obtains

$$\begin{aligned}
\nu \left(\nabla \left(\boldsymbol{u} - I^h \boldsymbol{u} \right), \nabla \boldsymbol{v}^h \right) &\leq \nu \left\| \nabla \left(\boldsymbol{u} - I^h \boldsymbol{u} \right) \right\|_{L^2(\Omega)} \left\| \nabla \boldsymbol{v}^h \right\|_{L^2(\Omega)}, \\
\left(\nabla \cdot \boldsymbol{v}^h, p - J^h p \right) &\leq \left\| p - J^h p \right\|_{L^2(\Omega)} \left\| \nabla \boldsymbol{v}^h \right\|_{L^2(\Omega)}, \\
\left(\nabla \cdot \left(\boldsymbol{u} - I^h \boldsymbol{u} \right), q^h \right) &\leq \left\| \nabla \left(\boldsymbol{u} - I^h \boldsymbol{u} \right) \right\|_{L^2(\Omega)} \left\| q^h \right\|_{L^2(\Omega)}.
\end{aligned}$$

The terms coming from the stabilization are estimated individually, using also the inverse inequality (3.12) and the upper bound (3.44) of the parameter δ_K:

$$\begin{aligned}
&\sum_{K\in\mathcal{T}^h} \delta_K \left(-\nu\Delta\left(\boldsymbol{u}-I^h\boldsymbol{u}\right), \nu\Delta\boldsymbol{v}^h\right)_K \\
&\quad\le\ C\nu\left(\sum_{K\in\mathcal{T}^h} h_K^2 \left\|\Delta\left(\boldsymbol{u}-I^h\boldsymbol{u}\right)\right\|_{L^2(K)}^2\right)^{1/2} \left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)}, \\
&\sum_{K\in\mathcal{T}^h} \delta_K \left(-\nu\Delta\left(\boldsymbol{u}-I^h\boldsymbol{u}\right), \nabla q^h\right)_K \\
&\quad\le\ C\nu^{1/2}\left(\sum_{K\in\mathcal{T}^h} h_K^2 \left\|\Delta\left(\boldsymbol{u}-I^h\boldsymbol{u}\right)\right\|_{L^2(K)}^2\right)^{1/2} \left(\sum_{K\in\mathcal{T}^h} \delta_K \left\|\nabla q^h\right\|_{L^2(K)}^2\right)^{1/2}, \\
&\sum_{K\in\mathcal{T}^h} \delta_K \left(\nabla\left(p-J^hp\right), \nu\Delta\boldsymbol{v}^h\right)_K \\
&\quad\le\ C\nu^{1/2}\left(\sum_{K\in\mathcal{T}^h} \delta_K \left\|\nabla\left(p-J^hp\right)\right\|_{L^2(K)}^2\right)^{1/2} \left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)}, \\
&\sum_{K\in\mathcal{T}^h} \delta_K \left(\nabla\left(p-J^hp\right), \nabla q^h\right)_K \\
&\quad\le\ \left(\sum_{K\in\mathcal{T}^h} \delta_K \left\|\nabla\left(p-J^hp\right)\right\|_{L^2(K)}^2\right)^{1/2} \left(\sum_{K\in\mathcal{T}^h} \delta_K \left\|\nabla q^h\right\|_{L^2(K)}^2\right)^{1/2}.
\end{aligned}$$

Collecting terms and using the definition (3.69) of the symmetric GLS norm yields

$$\begin{aligned}
&A_{\mathrm{sgls}}\left(\left(\boldsymbol{u}-I^h\boldsymbol{u}, p-J^hp\right), \left(\boldsymbol{v}^h, q^h\right)\right) \\
&\quad\le\ C\Bigg[\left\|\left(\boldsymbol{u}-I^h\boldsymbol{u}, p-J^hp\right)\right\|_{\mathrm{sgls}} \\
&\qquad+\left(\nu\sum_{K\in\mathcal{T}^h} h_K^2 \left\|\Delta\left(\boldsymbol{u}-I^h\boldsymbol{u}\right)\right\|_{L^2(K)}^2\right)^{1/2}\Bigg] \left\|\left(\boldsymbol{v}^h, q^h\right)\right\|_{\mathrm{sgls}}.
\end{aligned} \tag{3.87}$$

The triangle inequality gives

$$\left\|\left(\boldsymbol{u}-\boldsymbol{u}^h, p-p^h\right)\right\|_{\mathrm{sgls}} \le \left\|\left(\boldsymbol{u}-I^h\boldsymbol{u}, p-J^hp\right)\right\|_{\mathrm{sgls}} + \left\|\left(\boldsymbol{u}^h-I^h\boldsymbol{u}, p^h-J^hp\right)\right\|_{\mathrm{sgls}}$$

and hence, inserting (3.87) in (3.86), one obtains

$$\begin{aligned}
\left\|\left(\boldsymbol{u}-\boldsymbol{u}^h, p-p^h\right)\right\|_{\mathrm{sgls}} \ &\le\ C\left\|\left(\boldsymbol{u}-I^h\boldsymbol{u}, p-J^hp\right)\right\|_{\mathrm{sgls}} \\
&\quad+C\left(\nu\sum_{K\in\mathcal{T}^h} h_K^2 \left\|\Delta\left(\boldsymbol{u}-I^h\boldsymbol{u}\right)\right\|_{L^2(K)}^2\right)^{1/2}.
\end{aligned}$$

The terms on the right-hand side of this estimate can be estimated using (3.8), (3.9), and (3.44), giving the statement of the theorem. □

Theorem 3.4.17 (L^2 estimate of the velocity error)**.** *Let the assumptions of Lemma 3.4.13 be satisfied and let the Stokes problem* (3.2) *be regular. Assume that the solution of* (3.5) *satisfies* $(\boldsymbol{u}, p) \in H^{k+1}(\Omega)^d \times H^{l+1}(\Omega)$*, then there holds the error estimate*

$$\left\|\boldsymbol{u} - \boldsymbol{u}^h\right\|_{L^2(\Omega)} \leq C\left(h^{k+1}\left\|\boldsymbol{u}\right\|_{H^{k+1}(\Omega)} + \frac{h^{l+2}}{\nu}\left\|p\right\|_{H^{l+1}(\Omega)}\right).$$

Proof. The proof is very similar to the proof of Theorem 3.4.9. First, we again repeat the part of the proof of Theorem 3.3.5 up to (3.28). Second, applying the Galerkin orthogonality (3.85) in an analogous way as in the proof of Theorem 3.4.9, one obtains

$$\begin{aligned}
&\nu(\nabla \boldsymbol{z}^I, \nabla(\boldsymbol{u} - \boldsymbol{u}^h)) - (\nabla \cdot (\boldsymbol{u} - \boldsymbol{u}^h), r^I) \\
&\quad = \left(\nabla \cdot (\boldsymbol{z}^I - \boldsymbol{z}), p - p^h\right) \\
&\qquad + \sum_{K \in \mathcal{T}^h} \left(-\nu\Delta(\boldsymbol{u} - \boldsymbol{u}^h) + \nabla(p - p^h), \delta_K(-\nu\Delta \boldsymbol{z}^I + \nabla r^I)\right)_K. \quad (3.88)
\end{aligned}$$

To estimate the additional terms (in comparison to (3.62)), one may use the estimate

$$\left(\sum_{K \in \mathcal{T}^h} h_K^2 \left\|\Delta \boldsymbol{z}^I\right\|_{L^2(K)}^2\right)^{1/2} \leq Ch\left\|\boldsymbol{u} - \boldsymbol{u}^h\right\|_{L^2(\Omega)}, \quad (3.89)$$

which follows from the triangle inequality, (3.8) and (3.23). Then, the right-hand side of (3.88) can be estimated by the right-hand side of (3.63) (the jump term now vanishes), which leads to the estimate

$$\begin{aligned}
&\nu(\nabla \boldsymbol{z}^I, \nabla(\boldsymbol{u} - \boldsymbol{u}^h)) - (\nabla \cdot (\boldsymbol{u} - \boldsymbol{u}^h), r^I) \\
&\quad \leq C\nu^{1/2} h \left\|p - p^h\right\|_{\text{ext}} \left\|\boldsymbol{u} - \boldsymbol{u}^h\right\|_{L^2(\Omega)} \\
&\qquad + C\nu h \left(\sum_{K \in \mathcal{T}^h} h_K^2 \left\|\Delta(\boldsymbol{u} - \boldsymbol{u}^h)\right\|_{L^2(K)}^2\right)^{1/2} \left\|\boldsymbol{u} - \boldsymbol{u}^h\right\|_{L^2(\Omega)}. \quad (3.90)
\end{aligned}$$

Combining (3.27), (3.28), (3.90), and (3.64) gives

$$\nu\left\|\boldsymbol{u} - \boldsymbol{u}^h\right\|_{L^2(\Omega)} \leq C\nu^{1/2} h \left\|\left(\boldsymbol{u} - \boldsymbol{u}^h, p - p^h\right)\right\|_{\text{sgls}} + C\nu h^{k+1}\left\|\boldsymbol{u}\right\|_{H^{k+1}(\Omega)}$$

and the theorem follows from Theorem 3.4.16. □

Remark 3.4.18 (Discontinuous pressure finite element space)*.* The proof of the inf-sup condition (3.75) relies on (3.70). It can be shown that an inequality of this form holds also for discontinuous pressure spaces, see [43]. Then, for low order spaces, one has to include pressure jumps in the method, as for the PSPG method. The optimal choice of the stabilization parameter for the pressure jumps in (3.66) is $\gamma_E \sim h_E/\nu$, see [54] for details. For high order spaces, the inclusion of such jumps is not necessary. High order means that $P_d \subset V^h$ for simplicial meshes and $Q_2 \subset V^h$ for quadrilateral or hexahedral meshes. With such spaces, the known discrete inf-sup stability of V^h/P_0 or V^h/Q_0 is utilized in the proof. △

In [3], a multiscale enrichment of the velocity finite element space is proposed that leads to a family of stabilized methods. The enrichment functions are defined locally, but the functions of the ansatz space do not vanish on the boundary of the mesh cells. After performing some manipulations and applying static condensation, the resulting method contains the symmetric GLS stabilization term and a jump term at the faces. The stabilization parameter of the jump term is known exactly. One member of the family uses the jump of the Cauchy stress tensor across the faces. This method is called algebraic subgrid scale method (ASGS) in [8], where it was analyzed for the Brinkman equations (Stokes equations plus a zeroth order velocity term in the momentum balance).

An a priori and a posteriori error analysis for the symmetric GLS method with minimal regularity conditions on the solution of the weak problem is presented in [83]. It uses a technique developed in [50].

3.4.4 The Non-Symmetric Galerkin Least Squares Method (Douglas–Wang Method)

A method that looks similar to the symmetric GLS method (3.65) – (3.67) was proposed in [39]: Find $\left(\boldsymbol{u}^h, p^h\right) \in V^h \times Q^h$ such that

$$A_{\text{nsgls}}\left(\left(\boldsymbol{u}^h, p^h\right), \left(\boldsymbol{v}^h, q^h\right)\right) = L_{\text{nsgls}}\left(\left(\boldsymbol{v}^h, q^h\right)\right) \quad \forall \left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h, \quad (3.91)$$

with

$$\begin{aligned} A_{\text{nsgls}}\left((\boldsymbol{u}, p), (\boldsymbol{v}, q)\right) &= \nu\left(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}\right) - \left(\nabla \cdot \boldsymbol{v}, p\right) + \left(\nabla \cdot \boldsymbol{u}, q\right) \\ &+ \sum_{E \in \mathcal{E}^h} \gamma_E \left([|p|]_E, [|q|]_E\right)_E + \sum_{K \in \mathcal{T}^h} \left(-\nu \Delta \boldsymbol{u} + \nabla p, \delta_K \left(-\nu \Delta \boldsymbol{v} + \nabla q\right)\right)_K, \quad (3.92) \\ L_{\text{nsgls}}\left((\boldsymbol{v}, q)\right) &= (\boldsymbol{f}, \boldsymbol{v}) + \sum_{K \in \mathcal{T}^h} \left(\boldsymbol{f}, \delta_K \left(-\nu \Delta \boldsymbol{v} + \nabla q\right)\right)_K. \quad (3.93) \end{aligned}$$

The difference between (3.65) – (3.67) and (3.91) – (3.93) is just the sign in front of $\nu\Delta\boldsymbol{v}$ in the residual-based stabilization terms. The method (3.91) – (3.93) is non-symmetric.

Again, the presentation of the analysis will be restricted to continuous pressure finite element spaces, i.e., $Q^h \subset H^1(\Omega)$. To prove error estimates, we shall again use the assumptions (3.44) on the stabilization parameters, to ensure a correct scaling with respect to h_K and ν. However, an important difference to the previous two methods is that the stability holds without any upper bound on the stabilization parameters, cf. Lemmas 3.4.3, 3.4.13, and 3.4.20.

The following norm will be used

$$\|(\boldsymbol{v},q)\|_{\text{nsgls}} = \left(\nu \|\nabla \boldsymbol{v}\|^2_{L^2(\Omega)} + \sum_{K\in\mathcal{T}^h} \delta_K \|-\nu\Delta\boldsymbol{v} + \nabla q\|^2_{L^2(K)}\right)^{1/2}. \tag{3.94}$$

Lemma 3.4.19 ($\|(\cdot,\cdot)\|_{\text{nsgls}}$ defines a norm)**.** *If $Q^h \subset H^1(\Omega)$, then the expression defined in* (3.94) *is a norm in $V^h \times Q^h$ for any set of positive stabilization parameters $\{\delta_K\}$.*

Proof. Clearly, $\|(\cdot,\cdot)\|_{\text{nsgls}}$ defines a seminorm as a sum of norms and seminorms. It remains to show that $\left\|(\boldsymbol{v}^h,q^h)\right\|_{\text{nsgls}} = 0$ implies $(\boldsymbol{v}^h,q^h) = (\boldsymbol{0},0)$.

From $\left\|(\boldsymbol{v}^h,q^h)\right\|_{\text{nsgls}} = 0$, it follows that $\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)} = 0$, hence that $\boldsymbol{v}^h = \boldsymbol{0}$. Now, one has

$$\sum_{K\in\mathcal{T}^h} \delta_K \left\|\nabla q^h\right\|^2_{L^2(K)} = 0.$$

Since all δ_K are positive, one finds that q^h is piecewise constant. The only piecewise constant function that belongs to $H^1(\Omega)\cap L^2_0(\Omega)$ is $q^h = 0$. □

Lemma 3.4.20 (Existence and uniqueness of a solution of (3.91))**.** *For any set of positive stabilization parameters $\{\delta_K\}$, the non-symmetric GLS problem* (3.91) *with $Q^h \subset H^1(\Omega)$ has a unique solution.*

Proof. For $(\boldsymbol{v}^h,q^h) \in V^h \times Q^h$, one has

$$\begin{aligned} &A_{\text{nsgls}}\left(\left(\boldsymbol{v}^h,q^h\right),\left(\boldsymbol{v}^h,q^h\right)\right) \\ &\quad = \nu\left\|\nabla\boldsymbol{v}^h\right\|^2_{L^2(\Omega)} + \sum_{K\in\mathcal{T}^h} \delta_K \left\|-\nu\Delta\boldsymbol{v}^h + \nabla q^h\right\|^2_{L^2(K)} = \left\|(\boldsymbol{v}^h,q^h)\right\|^2_{\text{nsgls}} \end{aligned} \tag{3.95}$$

and hence the bilinear form given in (3.92) is coercive. Now, the existence and uniqueness of the solution follows in the same way as in the proof of Lemma 3.4.3. □

Lemma 3.4.21 (Consistency and Galerkin orthogonality)**.** *Let the solution of* (3.5) *satisfy $(\boldsymbol{u},p) \in H^2(\Omega)^d \times H^1(\Omega)$ and let $(\boldsymbol{u}^h,p^h) \in V^h \times Q^h$ be the solution of the non-symmetric GLS method* (3.91)*. This method is consistent, i.e., it holds*

$$A_{\text{nsgls}}\left((\boldsymbol{u},p),\left(\boldsymbol{v}^h,q^h\right)\right) = L_{\text{nsgls}}\left(\left(\boldsymbol{v}^h,q^h\right)\right) \quad \forall\ \left(\boldsymbol{v}^h,q^h\right) \in V^h \times Q^h \tag{3.96}$$

and it satisfies the Galerkin orthogonality

$$A_{\text{nsgls}}\left(\left(\boldsymbol{u}-\boldsymbol{u}^h,p-p^h\right),\left(\boldsymbol{v}^h,q^h\right)\right) = 0 \quad \forall\ \left(\boldsymbol{v}^h,q^h\right) \in V^h \times Q^h. \tag{3.97}$$

Proof. The proof follows the lines of the proof of Lemma 3.4.5. □

Lemma 3.4.22 (Estimate of the term with the divergence)**.** *Let $Q^h \subset H^1(\Omega)$ and let the stabilization parameters satisfy* (3.44). *Then, for any $\boldsymbol{v} \in V$, any $(\boldsymbol{z}^h, q^h) \in V^h \times Q^h$ and for any $\varepsilon > 0$, it holds*

$$\left|\left(\nabla \cdot \boldsymbol{v}, q^h\right)\right| \le \varepsilon \left\|(\boldsymbol{z}^h, q^h)\right\|_{\text{nsgls}}^2 + \frac{\nu}{4\varepsilon}\left(\frac{1}{\delta_0} + C_{\text{inv}}^2\right) \sum_{K\in\mathcal{T}^h} h_K^{-2} \left\|\boldsymbol{v}\right\|_{L^2(K)}^2. \quad (3.98)$$

Proof. Applying integration by parts and using that $q^h \in H^1(\Omega)$ yields

$$\left(\nabla \cdot \boldsymbol{v}, q^h\right) = -\left(\boldsymbol{v}, \nabla q^h\right).$$

For each mesh cell K, it is for arbitrary $\boldsymbol{z}^h \in V^h$

$$-\left(\boldsymbol{v}, \nabla q^h\right)_K = -\left(\boldsymbol{v}, -\nu\Delta\boldsymbol{z}^h + \nabla q^h\right)_K + \left(\boldsymbol{v}, -\nu\Delta\boldsymbol{z}^h\right)_K.$$

Using the triangle inequality, the Cauchy–Schwarz inequality, the property (3.44), as well as the Young inequality gives for any $\varepsilon_1 > 0$

$$\begin{aligned}
&\left|\left(\nabla \cdot \boldsymbol{v}, q^h\right)\right| \\
\le\ & \sum_{K\in\mathcal{T}^h} \left|\left(\boldsymbol{v}, -\nu\Delta\boldsymbol{z}^h + \nabla q^h\right)_K\right| + \sum_{K\in\mathcal{T}^h} \left|\left(\boldsymbol{v}, -\nu\Delta\boldsymbol{z}^h\right)_K\right| \\
\le\ & \frac{\nu}{4\delta_0\varepsilon} \sum_{K\in\mathcal{T}^h} h_K^{-2} \left\|\boldsymbol{v}\right\|_{L^2(K)}^2 + \varepsilon \sum_{K\in\mathcal{T}^h} \delta_K \left\|-\nu\Delta\boldsymbol{z}^h + \nabla q^h\right\|_{L^2(K)}^2 \\
& +\varepsilon_1 \sum_{K\in\mathcal{T}^h} h_K^2 \left\|\nu\Delta\boldsymbol{z}^h\right\|_{L^2(K)}^2 + \frac{1}{4\varepsilon_1} \sum_{K\in\mathcal{T}^h} h_K^{-2} \left\|\boldsymbol{v}\right\|_{L^2(K)}^2.
\end{aligned}$$

Utilizing the inverse inequality (3.12) yields

$$\varepsilon_1 \sum_{K\in\mathcal{T}^h} h_K^2 \left\|\nu\Delta\boldsymbol{z}^h\right\|_{L^2(K)}^2 \le \varepsilon_1 C_{\text{inv}}^2 \nu^2 \sum_{K\in\mathcal{T}^h} \left\|\nabla\boldsymbol{z}^h\right\|_{L^2(K)}^2.$$

Choosing $\varepsilon_1 = \varepsilon C_{\text{inv}}^{-2} \nu^{-1}$ and collecting terms gives (3.98). □

Theorem 3.4.23 (Error estimate)**.** *Assume that the solution of* (3.5) *satisfies $(\boldsymbol{u}, p) \in H^{k+1}(\Omega)^d \times H^{l+1}(\Omega)$, that $Q^h \subset H^1(\Omega)$, and that the stabilization parameters satisfy* (3.44)*, then there holds the error estimate*

$$\left\|\left(\boldsymbol{u} - \boldsymbol{u}^h, p - p^h\right)\right\|_{\text{nsgls}} \le C\left(\nu^{1/2} h^k \left\|\boldsymbol{u}\right\|_{H^{k+1}(\Omega)} + \frac{h^{l+1}}{\nu^{1/2}} \left\|p\right\|_{H^{l+1}(\Omega)}\right).$$

Proof. Let I^h and J^h be the interpolation operators satisfying (3.8) and (3.9). Using the coercivity (3.95) and the Galerkin orthogonality (3.97) yields

$$\begin{aligned}
&\left\|\left(\boldsymbol{u}^h - I^h\boldsymbol{u}, p^h - J^h p\right)\right\|_{\text{nsgls}}^2 \\
&= A_{\text{nsgls}}\left(\left(\boldsymbol{u}^h - I^h\boldsymbol{u}, p^h - J^h p\right), \left(\boldsymbol{u}^h - I^h\boldsymbol{u}, p^h - J^h p\right)\right) \\
&= A_{\text{nsgls}}\left(\left(\boldsymbol{u} - I^h\boldsymbol{u}, p - J^h p\right), \left(\boldsymbol{u}^h - I^h\boldsymbol{u}, p^h - J^h p\right)\right).
\end{aligned}$$

Applying the Cauchy–Schwarz inequality, Lemma 3.4.22 with $\boldsymbol{z}^h = \boldsymbol{u}^h - I^h\boldsymbol{u}$ and $\varepsilon = 1/4$, and the Young inequality gives

$$\begin{aligned}
&\left\|\left(\boldsymbol{u}^h - I^h\boldsymbol{u}, p^h - J^h p\right)\right\|_{\mathrm{nsgls}}^2 \\
&\leq \nu \left\|\nabla\left(\boldsymbol{u} - I^h\boldsymbol{u}\right)\right\|_{L^2(\Omega)} \left\|\nabla\left(\boldsymbol{u}^h - I^h\boldsymbol{u}\right)\right\|_{L^2(\Omega)} \\
&\quad + \left\|p - J^h p\right\|_{L^2(\Omega)} \left\|\nabla\left(\boldsymbol{u}^h - I^h\boldsymbol{u}\right)\right\|_{L^2(\Omega)} \\
&\quad + \left|\left(\nabla\cdot\left(\boldsymbol{u} - I^h\boldsymbol{u}\right), p^h - J^h p\right)\right| \\
&\quad + \sum_{K\in\mathcal{T}^h} \delta_K \left\|-\nu\Delta\left(\boldsymbol{u} - I^h\boldsymbol{u}\right) + \nabla(p - J^h p)\right\|_{L^2(K)} \\
&\qquad \times \left\|-\nu\Delta\left(\boldsymbol{u}^h - I^h\boldsymbol{u}\right) + \nabla(p^h - J^h p)\right\|_{L^2(K)} \\
&\leq \frac{1}{2}\left\|\left(\boldsymbol{u}^h - I^h\boldsymbol{u}, p^h - J^h p\right)\right\|_{\mathrm{nsgls}}^2 + 2\left\|\left(\boldsymbol{u} - I^h\boldsymbol{u}, p - J^h p\right)\right\|_{\mathrm{nsgls}}^2 \\
&\quad + \frac{2}{\nu}\left\|p - J^h p\right\|_{L^2(\Omega)}^2 + \nu\left(\frac{1}{\delta_0} + C_{\mathrm{inv}}^2\right)\sum_{K\in\mathcal{T}^h} h_K^{-2}\left\|\boldsymbol{u} - I^h\boldsymbol{u}\right\|_{L^2(K)}^2 .
\end{aligned}$$

The proof is finished by applying the triangle inequality

$$\left\|\left(\boldsymbol{u} - \boldsymbol{u}^h, p - p^h\right)\right\|_{\mathrm{nsgls}} \leq \left\|\left(\boldsymbol{u} - I^h\boldsymbol{u}, p - J^h p\right)\right\|_{\mathrm{nsgls}} + \left\|\left(\boldsymbol{u}^h - I^h\boldsymbol{u}, p^h - J^h p\right)\right\|_{\mathrm{nsgls}}$$

and using (3.44), (3.8), and (3.9). □

Theorem 3.4.24 (L^2 estimate of the pressure error)**.** *Assume that the solution of* (3.5) *satisfies* $(\boldsymbol{u}, p) \in H^{k+1}(\Omega)^d \times H^{l+1}(\Omega)$*, that* $Q^h \subset H^1(\Omega)$*, and that the stabilization parameters satisfy* (3.44)*, then there holds the error estimate*

$$\left\|p - p^h\right\|_{L^2(\Omega)} \leq C\left(\nu h^k \left\|\boldsymbol{u}\right\|_{H^{k+1}(\Omega)} + h^{l+1}\left\|p\right\|_{H^{l+1}(\Omega)}\right).$$

Proof. Like in the proof of Theorem 3.4.8, we start with (3.55). The Galerkin orthogonality (3.97) with $(\boldsymbol{v}^h, q^h) = (\mathcal{I}^h\boldsymbol{w}, 0)$ gives

$$\begin{aligned}
0 &= \nu\left(\nabla\left(\boldsymbol{u} - \boldsymbol{u}^h\right), \nabla\mathcal{I}^h\boldsymbol{w}\right) - \left(\nabla\cdot\left(\mathcal{I}^h\boldsymbol{w}\right), p - p^h\right) \\
&\quad - \sum_{K\in\mathcal{T}^h}\left(-\nu\Delta\left(\boldsymbol{u} - \boldsymbol{u}^h\right) + \nabla\left(p - p^h\right), \delta_K \nu\Delta\mathcal{I}^h\boldsymbol{w}\right)_K .
\end{aligned}$$

Hence, using the Cauchy–Schwarz inequality and applying (3.44), (3.12), (3.10), and (3.52), one obtains

$$\left|\left(\nabla\cdot\left(\mathcal{I}^h\boldsymbol{w}\right), p - p^h\right)\right| \leq C\nu^{1/2}\left\|\left(\boldsymbol{u} - \boldsymbol{u}^h, p - p^h\right)\right\|_{\mathrm{nsgls}}\left\|p - p^h\right\|_{L^2(\Omega)} . \quad (3.99)$$

The estimate (3.57) reduces to

$$\begin{aligned}
&-\left(\nabla\cdot\left(\boldsymbol{w} - \mathcal{I}^h\boldsymbol{w}\right), p - p^h\right) \\
&\leq \frac{C\nu^{1/2}}{\delta_0^{1/2}}\left\|p - p^h\right\|_{L^2(\Omega)}\left(\sum_{K\in\mathcal{T}^h}\delta_K\left\|\nabla\left(p - p^h\right)\right\|_{L^2(K)}^2\right)^{1/2} . \quad (3.100)
\end{aligned}$$

To estimate the last term in (3.100), we apply the triangle inequality, (3.44), and (3.64), which gives

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}^h} \delta_K \left\|\nabla\left(p-p^h\right)\right\|^2_{L^2(K)}\\
&\quad\le\quad 2\left\|\left(\boldsymbol{u}-\boldsymbol{u}^h, p-p^h\right)\right\|^2_{\text{nsgls}} + 2\delta_1\nu \sum_{K\in\mathcal{T}^h} h_K^2 \left\|\Delta(\boldsymbol{u}-\boldsymbol{u}^h)\right\|^2_{L^2(K)}\\
&\quad\le\quad C\left\|\left(\boldsymbol{u}-\boldsymbol{u}^h, p-p^h\right)\right\|^2_{\text{nsgls}} + C\nu h^{2k}\left\|\boldsymbol{u}\right\|^2_{H^{k+1}(\Omega)}.
\end{aligned}
$$

Combining this estimate with (3.55), (3.99), and (3.100) yields

$$
\left\|p-p^h\right\|_{L^2(\Omega)} \le C\nu^{1/2}\left\|\left(\boldsymbol{u}-\boldsymbol{u}^h, p-p^h\right)\right\|_{\text{nsgls}} + C\nu h^k\left\|\boldsymbol{u}\right\|_{H^{k+1}(\Omega)},
$$

where the constant C depends on $\delta_0^{-1/2}$. Applying Theorem 3.4.23 finishes the proof. □

Theorem 3.4.25 (L^2 estimate of the velocity error)**.** *Let the Stokes problem* (3.2) *be regular. Assume that the solution of* (3.5) *satisfies* $(\boldsymbol{u},p)\in H^{k+1}(\Omega)^d\times H^{l+1}(\Omega)$, *that* $Q^h\subset H^1(\Omega)$, *and that the stabilization parameters satisfy* (3.44), *then there holds the error estimate*

$$
\left\|\boldsymbol{u}-\boldsymbol{u}^h\right\|_{L^2(\Omega)} \le C\left(h^{k+1}\left\|\boldsymbol{u}\right\|_{H^{k+1}(\Omega)} + \frac{h^{l+2}}{\nu}\left\|p\right\|_{H^{l+1}(\Omega)}\right).
$$

Proof. We proceed analogously as in the proofs of Theorems 3.4.9 and 3.4.17. Again, the starting point is the identity (3.27), where the first two terms on the right-hand side can be estimated by (3.28). From the Galerkin orthogonality (3.97), one obtains

$$
\begin{aligned}
&\nu(\nabla\boldsymbol{z}^I, \nabla(\boldsymbol{u}-\boldsymbol{u}^h)) - (\nabla\cdot(\boldsymbol{u}-\boldsymbol{u}^h), r^I)\\
&\quad=\quad \left(\nabla\cdot(\boldsymbol{z}^I-\boldsymbol{z}), p-p^h\right)\\
&\qquad\quad + \sum_{K\in\mathcal{T}^h}\left(-\nu\Delta(\boldsymbol{u}-\boldsymbol{u}^h)+\nabla(p-p^h), \delta_K(\nu\Delta\boldsymbol{z}^I+\nabla r^I)\right)_K.
\end{aligned}
$$

Thus, using (3.44), (3.25), (3.58), and (3.89), one derives

$$
\begin{aligned}
&\nu(\nabla\boldsymbol{z}^I, \nabla(\boldsymbol{u}-\boldsymbol{u}^h)) - (\nabla\cdot(\boldsymbol{u}-\boldsymbol{u}^h), r^I)\\
&\le \left\|\nabla(\boldsymbol{z}-\boldsymbol{z}^I)\right\|_{L^2(\Omega)}\left\|p-p^h\right\|_{L^2(\Omega)}\\
&+\left\|\left(\boldsymbol{u}-\boldsymbol{u}^h, p-p^h\right)\right\|_{\text{nsgls}}\left(\sum_{K\in\mathcal{T}^h}\delta_K\left\|\nu\Delta\boldsymbol{z}^I+\nabla r^I\right\|^2_{L^2(K)}\right)^{1/2}\\
&\le Ch\left(\left\|p-p^h\right\|_{L^2(\Omega)}+\nu^{1/2}\left\|\left(\boldsymbol{u}-\boldsymbol{u}^h, p-p^h\right)\right\|_{\text{nsgls}}\right)\left\|\boldsymbol{u}-\boldsymbol{u}^h\right\|_{L^2(\Omega)}.
\end{aligned}
\tag{3.101}
$$

Combining (3.27), (3.28), and (3.101) gives

$$\nu \left\|\boldsymbol{u} - \boldsymbol{u}^h\right\|_{L^2(\Omega)} \leq Ch \left(\left\|p - p^h\right\|_{L^2(\Omega)} + \nu^{1/2} \left\|\left(\boldsymbol{u} - \boldsymbol{u}^h, p - p^h\right)\right\|_{\mathrm{nsgls}} \right)$$

and the theorem follows from Theorems 3.4.23 and 3.4.24. □

In the case of discontinuous pressure approximations, the optimal choice of the stabilization parameter for the pressure jumps is $\gamma_E = \mathcal{O}\left(h_E/\nu\right)$, see [39].

In [4], an extension of the non-symmetric GLS method is proposed. This method possesses jump terms that contain the residual of the stress tensor on the internal edges, i.e., the jump of the normal derivative of the finite element velocity and the jump of the finite element pressure. It is unconditionally stable for a norm where, in comparison with $\|(\cdot,\cdot)\|_{\mathrm{nsgls}}$ defined in (3.94), the Laplacian of the velocity is absent but the residual of the stress tensor at the inner faces is present. An a priori analysis and an a posteriori analysis of this method are provided in [4].

For P_1/P_1 finite elements, the non-symmetric GLS method with a weak imposition of the boundary condition via a penalty-free Nitsche method was studied in [18].

3.4.5 An Absolutely Stable Modification of the PSPG Method

The PSPG method from Section 3.4.2 is only conditionally stable, see the upper bound (3.39) for the stabilization parameter used in Lemma 3.4.3 to prove the coercivity. In [19], an absolutely stable modification of the PSPG method was proposed which we now briefly describe.

The PSPG method (3.33) will be now considered with $Q_h \subset H^1(\Omega)$ and $\delta_K = \delta := \delta_0 h^2/\nu$, which can be used on an uniform grid. In (3.34), the operator Δ is applied elementwise. In [19], it was replaced by the discrete Laplacian $\Delta^h : V \to V^h$ defined by

$$\left(\Delta^h \boldsymbol{u}, \boldsymbol{v}^h\right) = -\left(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}^h\right) \quad \forall\, \boldsymbol{u} \in V,\ \boldsymbol{v}^h \in V^h. \tag{3.102}$$

Then the modified PSPG method reads: Find $\left(\boldsymbol{u}^h, p^h\right) \in V^h \times Q^h$ such that

$$\begin{aligned} \nu\left(\nabla \boldsymbol{u}^h, \nabla \boldsymbol{v}^h\right) - \left(\nabla \cdot \boldsymbol{v}^h, p^h\right) + \left(\nabla \cdot \boldsymbol{u}^h, q^h\right) + \delta\left(-\nu \Delta^h \boldsymbol{u}^h + \nabla p^h, \nabla q^h\right) & \\ = \quad \left(\boldsymbol{f}, \boldsymbol{v}^h\right) + \delta\left(\boldsymbol{f}, \nabla q^h\right) \qquad \forall\ \left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h. & \end{aligned} \tag{3.103}$$

Thus, the modified PSPG method requires the additional solution of problem (3.102), which is a linear system with the mass matrix as system matrix. In practical computations, the mass matrix can be replaced by a lumped mass matrix or local projection. Using (3.102), method (3.103) can be rewritten as

$$\left(-\nu \Delta^h \boldsymbol{u}^h + \nabla p^h, \boldsymbol{v}^h + \delta \nabla q^h\right) + \left(\nabla \cdot \boldsymbol{u}^h, q^h\right) = \left(\boldsymbol{f}, \boldsymbol{v}^h + \delta \nabla q^h\right), \tag{3.104}$$

such that it has the form of a Petrov–Galerkin method.

Recall that the conditional stability of the PSPG method stems from the fact that the coercivity is based on estimating the term $\sum_{K\in\mathcal{T}^h} \delta_K \left(-\nu\Delta \boldsymbol{v}^h, \nabla q^h\right)_K$ by $\left\|\left(\boldsymbol{v}^h, q^h\right)\right\|_{\text{pspg}}^2$. This step inevitably leads to a bound on δ_K. However, replacing Δ by Δ^h, it is possible to get rid of this term by a suitable choice of $\boldsymbol{v}^h$. Indeed, defining $\boldsymbol{v}^h$ as the L^2 projection of $-\delta\nabla q^h$ onto V^h, we see from (3.104) that the respective term disappears. This together with further tools enabled to prove in [19] that, for any $\delta_0 > 0$, the bilinear form corresponding to the modified PSPG method satisfies an inf–sup condition of the type (3.75) with respect to the norm $\nu^{1/2}\left\|\nabla \boldsymbol{v}^h\right\|_{L^2(\Omega)} + \nu^{-1/2}\left\|q^h\right\|_{L^2(\Omega)}$ with a constant dependent on δ_0.

The modified PSPG method (3.103) is obviously not consistent in general. However, this inconsistency is very weak so that the optimal order of convergence with respect to the mentioned norm could be proved in [19].

3.5 Stabilizations Using only the Pressure

This section is dedicated to methods that use only the pressure in the stabilization term. Hence, there is no need to compute the residual and the use of second derivatives of the finite element functions is not necessary. However, many methods connect pressure degrees of freedom that do not belong to the same mesh cell. Consequently, the stencil of the matrix C in (3.4) is denser than for residual-based stabilizations.

After having introduced a framework in Section 3.5.1, a number of methods will be presented briefly. A detailed analysis is provided for a Local Projection Stabilization (LPS) method in Section 3.5.4.

3.5.1 A Framework

An abstract approach for the derivation and analysis of pressure-stabilized schemes was presented in [25], see also [21, Chapter 6.3]. For the Stokes equations, the considered scheme has the form: Find $\left(\boldsymbol{u}^h, p^h\right) \in V^h \times Q^h$ such that for all $\left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h$

$$\nu\left(\nabla \boldsymbol{u}^h, \nabla \boldsymbol{v}^h\right) - \left(\nabla\cdot \boldsymbol{v}^h, p^h\right) + \left(\nabla\cdot \boldsymbol{u}^h, q^h\right) + \delta S\left(\left(\boldsymbol{u}^h, p^h\right), \left(\boldsymbol{v}^h, q^h\right)\right) = \left(\boldsymbol{f}, \boldsymbol{v}^h\right), \tag{3.105}$$

with $\delta > 0$ and $S : \left(V^h \times Q^h\right) \times \left(V^h \times Q^h\right) \to \mathbb{R}$ being a bilinear form that should be chosen such that (3.105) is a stable and consistent discrete scheme. There are two essential assumptions on S. The bilinear form should be bounded with a constant independent of h. Likewise, uniformly in h, there should exist a Hilbert space $\mathcal{H}$, some operator $G^h \in \mathcal{L}\left(V^h \times Q^h, \mathcal{H}\right)$, and a constant $C > 0$ such that for all $\left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h$

$$S\left(\left(\boldsymbol{v}^h, q^h\right), \left(\boldsymbol{v}^h, q^h\right)\right) \geq C\left\|G^h\left(\left(\boldsymbol{v}^h, q^h\right)\right)\right\|_{\mathcal{H}}^2.$$

For the abstract problem considered in [25], more operators, assumptions, etc. were introduced. Then, stability and error estimates, e.g., with respect to the errors in the norms of V and Q were derived.

Let $Q^h \subset H^1(\Omega)$. The application of the abstract theory presented in [25] to the Stokes equations considers pressure stabilizations that use only the pressure. A first example consists in taking

$$S\left(\left(\boldsymbol{u}^h, p^h\right), \left(\boldsymbol{v}^h, q^h\right)\right) = \left(\nabla p^h, \nabla q^h\right),$$

$G^h\left(\boldsymbol{v}^h, q^h\right) = \nabla q^h$, $\mathcal{H} = L^2(\Omega)^d$, and $\delta = \mathcal{O}\left(h^2\right)$, which gives the method of Brezzi–Pitkäranta, see Section 3.5.2. A second example consists in choosing

$$S\left(\left(\boldsymbol{u}^h, p^h\right), \left(\boldsymbol{v}^h, q^h\right)\right) = \left(\left(I - P_{\overline{V^h}}\right) \nabla p^h, \nabla q^h\right),$$

with $P_{\overline{V^h}}$ being a the $L^2(\Omega)$ projection operator onto $\overline{V^h}$, where $\overline{V^h}$ is defined with the same polynomials as V^h but without incorporating the boundary conditions in the definition. In this method, one has $\mathcal{H} = L^2(\Omega)^d$ and $G^h\left(\boldsymbol{v}^h, q^h\right) = \left(I - P_{\overline{V^h}}\right) \nabla q^h$. One obtains the method proposed in [34], see Section 3.5.3. Concerning the choice of δ, one finds in [34], where bounds for the pressure error in different norms than in [25] were proved, that one gets stability for $\delta \geq Ch^2$ and optimal convergence for $\delta = \mathcal{O}\left(h^2\right)$. In [25], see also [21, Chapter 8.13.3], it is shown that for $V^h \times Q^h = P_1/P_1$, stability and optimal convergence are obtained with $\delta = \mathcal{O}(1)$.

For a detailed investigation on how several methods introduced in this section fit into the framework of [25], it is referred to [27].

3.5.2 The Brezzi–Pitkäranta Method

The Brezzi–Pitkäranta method from [26] was the first stabilization method for circumventing the discrete inf-sup condition (3.3). This method was proposed for the P_1/P_1 pair of finite element spaces and it has the form: Find $(\boldsymbol{u}^h, p^h) \in V^h \times Q^h = P_1 \times P_1$ such that

$$\begin{aligned}
\nu\left(\nabla \boldsymbol{u}^h, \nabla \boldsymbol{v}^h\right) - \left(\nabla \cdot \boldsymbol{v}^h, p^h\right) &= \left(\boldsymbol{f}, \boldsymbol{v}^h\right) & \forall\, \boldsymbol{v}^h \in V^h, \\
-\left(\nabla \cdot \boldsymbol{u}^h, q^h\right) - \sum_{K \in \mathcal{T}^h} \left(\nabla p^h, \delta_K^p \nabla q^h\right)_K &= 0 & \forall\, q^h \in Q^h.
\end{aligned} \tag{3.106}$$

Considering a uniform family of triangulations, the optimal order convergence of the solution of (3.106) with respect to $\left\|\nabla\left(\boldsymbol{u} - \boldsymbol{u}^h\right)\right\|_{L^2(\Omega)}$ and $\left\|p - p^h\right\|_{L^2(\Omega)}$ was proved for the stabilization parameter $\delta_K^p = \mathcal{O}\left(h^2\right)$ (for $\nu = 1$). As discussed above, method (3.106) fits into the framework presented in Section 3.5.1.

As it is often noted in the literature, the Brezzi–Pitkäranta method imposes artificial boundary conditions for the finite element pressure. Considering for simplicity $\delta_K^p = \delta$, then the strong form of the continuity equation of (3.106) reads as

$$-\nabla \cdot \boldsymbol{u} + \delta \Delta p = 0.$$

Deriving in the usual way the corresponding weak form leads to

$$-(\nabla\cdot\boldsymbol{u},q)-\delta(\nabla p,\nabla q)+\delta\int_{\partial\Omega}(\nabla p\cdot\boldsymbol{n})\,q\;\mathrm{d}\boldsymbol{s}=0\quad\forall\,q\in Q.$$

Since no boundary integral appears in (3.106), one finds that an artificial boundary condition of the form

$$\delta\left(\nabla p^h\cdot\boldsymbol{n}\right)=0\quad\text{on }\partial\Omega$$

for the discrete pressure is introduced with this method.

A stabilized method of Brezzi–Pitkäranta-type with a nonlinear stabilization parameter is presented in [77], the so-called pressure Laplacian stabilization (PLS) method. The stabilization parameter depends on the residuals of the finite element continuity and the momentum equation.

3.5.3 Stabilization with Global Fluctuations of the Pressure Gradient

In [34], it was shown that for constructing a pressure-stable method, it is not necessary to use the full gradient of the discrete pressure, as in (3.106). Denoting by $\overline{V^h}$ the velocity finite element space with the same polynomials as V^h but without prescribed boundary conditions, then it is proposed in [34] to apply the following method: Find $\left(\boldsymbol{u}^h,p^h,\overline{\nabla p^h}\right)\in V^h\times Q^h\times\overline{V^h}$ such that

$$\begin{aligned}\nu\left(\nabla\boldsymbol{u}^h,\nabla\boldsymbol{v}^h\right)-\left(\nabla\cdot\boldsymbol{v}^h,p^h\right)&=\left(\boldsymbol{f},\boldsymbol{v}^h\right)&&\forall\,\boldsymbol{v}^h\in V^h,\\-\left(\nabla\cdot\boldsymbol{u}^h,q^h\right)-\sum_{K\in\mathcal{T}^h}\left(\nabla p^h-\overline{\nabla p^h},\delta_K^p\nabla q^h\right)_K&=0&&\forall\,q^h\in Q^h,\\\left(\nabla p^h-\overline{\nabla p^h},\overline{\boldsymbol{v}^h}\right)&=0&&\forall\,\boldsymbol{v}^h\in\overline{V^h}.\end{aligned}\tag{3.107}$$

The third equation of (3.107) defines $\overline{\nabla p^h}$ to be the $L^2(\Omega)$ projection of ∇p^h onto $\overline{V^h}$. In this way, one can interprete $\overline{\nabla p^h}$ as being large scales of ∇p^h and the difference $\nabla p^h-\overline{\nabla p^h}$ of being fluctuations. Only the fluctuations appear in the stabilization term of the discrete continuity equation. It was already discussed above that this method fits into the framework described in Section 3.5.1.

A finite element analysis of the method can be found in [34]. This analysis considers a family of quasi-uniform triangulations and $\delta_K^p=\delta$. For $\delta=\mathcal{O}\left(h^2\right)$, the stability of the finite element solution and optimal error estimates for $\left\|\nabla\left(\boldsymbol{u}-\boldsymbol{u}^h\right)\right\|_{L^2(\Omega)}$, $\left\|\nabla\left(p-p^h\right)\right\|_{L^2(\Omega)}$, and $\left\|\nabla p-\overline{\nabla p^h}\right\|_{L^2(\Omega)}$ were proved. Extensions of the analysis that allow the choice of local stabilization parameters and to the steady-state Navier–Stokes equations can be found in [35].

Another analysis of method (3.107) can be found in [66][1]. The error estimate

[1] Reading [66], one is wondering that there is no reference to [34] for method (3.107). From the article's history, one finds that [66] was submitted shortly after [34] was published.

from [66] bounds $\left\|p - p^h\right\|_{L^2(\Omega)}$ whereas the estimate from [34] gives a bound for $h\left\|\nabla\left(p - p^h\right)\right\|_{L^2(\Omega)}$.

A method of type (3.107) was analyzed for the Brinkman equations in [8]. As additional terms, a grad-div stabilization, using fluctuations of the divergence, and jump terms across faces, which involve the Cauchy stress tensor, appear. The analysis covers both limit cases of the Brinkman equations, namely the Stokes and the Darcy equations. Global Fluctuations of the Pressure Gradient

3.5.4 Local Projection Stabilization (LPS) Methods

To assure the stability of the PSPG method (3.33), it would be sufficient to consider the term

$$\sum_{K\in\mathcal{T}^h} \delta_K \left(\nabla p, \nabla q\right)_K \tag{3.108}$$

instead of the residual-based terms in (3.34) and (3.35). This would provide several advantages (e.g., symmetry of the stabilization, simpler implementation, absolute stability) but it would not lead to optimal error estimates. A remedy preserving most of the advantages of (3.108) without compromising the convergence rates of the PSPG method is to apply locally suitable projection operators to ∇p and ∇q in (3.108) so that the consistency error can be estimated in the desired way.

It is convenient to define the mentioned local projections on macroelements. Precisely, one introduces a set $\mathcal{M}^h$ consisting of a finite number of open subsets M of Ω such that $\overline{\Omega} = \cup_{M\in\mathcal{M}^h} \overline{M}$. In contrast to $\mathcal{T}^h$, the sets in $\mathcal{M}^h$ are allowed to overlap. For any $K \in \mathcal{T}^h$, $E \in \mathcal{E}^h$, and $M \in \mathcal{M}^h$ it is assumed that either $K \subset \overline{M}$ or $K \subset \overline{\Omega} \setminus M$ and that either $E \subset \overline{M}$ or $E \subset \Omega \setminus \overline{M}$. Furthermore, for any $M \in \mathcal{M}^h$, one introduces a finite-dimensional space $D_M \subset L^2(M)^d$ and a continuous linear projection operator π_M which maps the space $L^2(M)^d$ onto the space D_M. Then one defines the so-called fluctuation operator $\kappa_M = id - \pi_M$, where id is the identity operator on $L^2(M)^d$. Finally, the term (3.108) is replaced by

$$\sum_{M\in\mathcal{M}^h} \delta_M(\kappa_M(\nabla^h p), \kappa_M(\nabla^h q))_M,$$

where $(\nabla^h q)|_K = \nabla(q|_K)$ for any $K \in \mathcal{T}^h$.

Thus, the local projection stabilization (LPS) method reads: Find $\left(\boldsymbol{u}^h, p^h\right) \in V^h \times Q^h$ such that

$$A_{\text{lps}}\left(\left(\boldsymbol{u}^h, p^h\right), \left(\boldsymbol{v}^h, q^h\right)\right) = \left(\boldsymbol{f}, \boldsymbol{v}^h\right) \quad \forall\ \left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h, \tag{3.109}$$

where the bilinear form $A_{\text{lps}}\ :\ \left(V \times \tilde{Q}\right) \times \left(V \times \tilde{Q}\right) \to \mathbb{R}$ is given by

$$\begin{aligned} A_{\text{lps}}\left((\boldsymbol{u}, p), (\boldsymbol{v}, q)\right) &= \nu\left(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}\right) - \left(\nabla\cdot\boldsymbol{v}, p\right) + \left(\nabla\cdot\boldsymbol{u}, q\right) \\ &\quad + \sum_{E\in\mathcal{E}^h} \gamma_E\left([|p|]_E, [|q|]_E\right)_E + \sum_{M\in\mathcal{M}^h} \delta_M(\kappa_M(\nabla^h p), \kappa_M(\nabla^h q))_M \end{aligned}$$

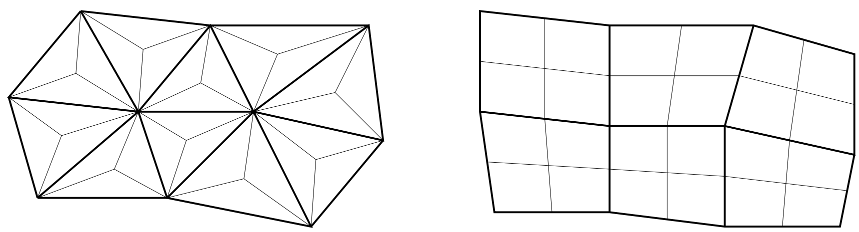

Figure 3.1: Relation between the meshes $\mathcal{M}^h$ (bold lines) and $\mathcal{T}^h$ (bold and fine lines) in the two-level method.

and the space $\tilde{Q}$ was defined in (3.37).

We make analogous assumptions on the stabilization parameters as for the residual-based methods, i.e., it is assumed that the parameters $\{\gamma_E\}$ satisfy (3.45) and that

$$0 < \delta_0 \frac{h_M^2}{\nu} \leq \delta_M \leq \delta_1 \frac{h_M^2}{\nu} \qquad \forall\, M \in \mathcal{M}^h \tag{3.110}$$

with some positive constants δ_0, δ_1 and $h_M := \operatorname{diam}(M)$.

To perform an analysis of the method and prove optimal error estimates, a key assumption is the validity of the inf-sup conditions

$$\sup_{\boldsymbol{v} \in V_M \setminus \{\boldsymbol{0}\}} \frac{(\boldsymbol{v}, \boldsymbol{q})_M}{\|\boldsymbol{v}\|_{L^2(M)}} \geq \beta_{LP} \|\boldsymbol{q}\|_{L^2(M)} \qquad \forall\, \boldsymbol{q} \in D_M,\ M \in \mathcal{M}^h \tag{3.111}$$

with $V_M = \{\boldsymbol{v}^h \in V^h \ :\ \boldsymbol{v}^h = \boldsymbol{0} \text{ in } \Omega \setminus M\}$ and a constant β_{LP} independent of h. This poperty limits possible combinations of spaces V^h and D_M.

Using local projections onto macro mesh cells for pressure stabilization was proposed for the Q_1/Q_0 pair of finite element spaces already in [82]. The original local projection stabilization [13, 14] was designed as a two-level method. Given a triangulation of Ω, the elements of this triangulation are considered as the set $\mathcal{M}^h$. Then this triangulation is refined as depicted in Fig. 3.1 for the two-dimensional case, i.e., each triangle is divided into three triangles by connecting its vertices with the barycenter and each quadrilateral is divided into four quadrilaterals by connecting midpoints of opposite edges. This gives the triangulation $\mathcal{T}^h$. If the space V^h is defined on $\mathcal{T}^h$ like before (i.e., it contains locally (mapped) polynomials of degree $k \geq 1$), then the inf-sup conditions (3.111) hold for $D_M = P_{k-1}(M)^d$.

Another choice of the spaces V^h and D_M (a one-level method) was proposed in [72]. In this case $\mathcal{M}^h = \mathcal{T}^h$ and to satisfy the inf-sup conditions (3.111) with $D_M = P_{k-1}(M)^d$ the space V^h is enriched elementwise by bubble functions.

Finally, let us describe a choice of the spaces V^h and D_M based on a set $\mathcal{M}^h$ consisting of overlapping sets M as proposed in [61]. Assuming that each element of $\mathcal{T}^h$ has at least one vertex in Ω, then for each interior vertex a macroelement consisting of elements of $\mathcal{T}^h$ sharing this vertex is defined. For this set $\mathcal{M}^h$, one can use our standard choice of V^h and local spaces $D_M = P_{k-1}(M)^d$ to satisfy the inf-sup conditions (3.111).

Note that the first two ways of constructing the spaces V^h and D_M lead to a significant increase of the number of degrees of freedom, either due to an enrichment by bubble functions (in the one-level method) or due to a refinement of the given triangulation (in the two-level method). On the other hand, in the variant with overlapping sets M, the number of degrees of freedom remains the same as if one would use, e.g., a residual-based stabilization.

We refer to [61, 72] for details on the definitions of the spaces and for proofs of the inf-sup conditions.

In view of the examples of the spaces D_M, it is reasonable to assume that there exist interpolation operators $j_M : L^2(M)^d \to D_M$ such that, for $m = 0, \dots, k$, one has

$$\|\boldsymbol{q} - j_M \boldsymbol{q}\|_{L^2(M)} \le C\, h_M^m \, \|\boldsymbol{q}\|_{H^m(M)} \qquad \forall\, \boldsymbol{q} \in H^m(M)^d,\ M \in \mathcal{M}^h. \tag{3.112}$$

Finally, let us state a few natural assumptions needed for the subsequent analysis. We assume that there are various positive constants independent of h such that

$$\begin{aligned}
\operatorname{card}\{M' \in \mathcal{M}^h;\ M \cap M' \neq \emptyset\} &\le C_{\mathcal{M}} \qquad \forall\, M \in \mathcal{M}^h, && (3.113)\\
\operatorname{card}\{K \in \mathcal{T}^h;\ K \subset \overline{M}\} &\le C_{\mathcal{T}} \qquad \forall\, M \in \mathcal{M}^h\,, && (3.114)\\
\operatorname{card}\{M \in \mathcal{M}^h;\ K \subset \overline{M}\} &\le C_{\mathcal{T}} \qquad \forall\, K \in \mathcal{T}^h\,, && (3.115)\\
\operatorname{card}\{E \in \mathcal{E}^h;\ E \subset \overline{M}\} &\le C_{\mathcal{E}} \qquad \forall\, M \in \mathcal{M}^h\,, && (3.116)\\
\operatorname{card}\{M \in \mathcal{M}^h;\ E \subset \overline{M}\} &\le C_{\mathcal{E}} \qquad \forall\, E \in \mathcal{E}^h, && (3.117)\\
\|\kappa_M\|_{\mathcal{L}(L^2(M)^d, L^2(M)^d)} &\le C_\kappa \qquad \forall\, M \in \mathcal{M}^h\,, && (3.118)\\
h_M \le C'_{\mathcal{M}}\, h_{M'} &\qquad \forall\, M, M' \in \mathcal{M}^h,\ M \cap M' \neq \emptyset. && (3.119)
\end{aligned}$$

Furthermore, for any $E \in \mathcal{E}^h$ and $M \in \mathcal{M}^h$ with $E \subset \overline{M}$, we assume that

$$h_M \le C'_{\mathcal{E}}\, h_E\,, \tag{3.120}$$

$$\|v\|_{L^2(E)} \le C_e\, (h_M^{-1/2}\, \|v\|_{L^2(M)} + h_M^{1/2}\, \|\nabla v\|_{L^2(M)}) \qquad \forall\, v \in H^1(M). \tag{3.121}$$

Finally, we shall need the inverse inequalities

$$\left\|\nabla \boldsymbol{v}^h\right\|_{L^2(M)} \le \bar{C}_{\mathrm{inv}}\, h_M^{-1}\, \|\boldsymbol{v}^h\|_{L^2(M)} \qquad \forall\, \boldsymbol{v}^h \in V^h,\ M \in \mathcal{M}^h\,. \tag{3.122}$$

Let us now investigate the stability of the LPS method. One obviously has

$$A_{\mathrm{lps}}\left((\boldsymbol{v}, q)\,, (\boldsymbol{v}, q)\right) = |(\boldsymbol{v}, q)|^2_{\mathrm{lps}} \qquad \forall\ (\boldsymbol{v}, q) \in V \times \tilde{Q}, \tag{3.123}$$

where

$$
\begin{aligned}
|(\boldsymbol{v},q)|_{\mathrm{lps}} \;=\; & \Bigg(\nu\,\|\nabla \boldsymbol{v}\|_{L^2(\Omega)}^2 + \sum_{E\in\mathcal{E}^h} \gamma_E\,\|[|q|]_E\|_{L^2(E)}^2 \\
& + \sum_{M\in\mathcal{M}^h} \delta_M\,\left\|\kappa_M(\nabla^h q)\right\|_{L^2(M)}^2\Bigg)^{1/2}.
\end{aligned}
$$

The functional $|\cdot|_{\mathrm{lps}}$ is only a seminorm on $V\times\tilde{Q}$. In what follows we shall prove that the bilinear form A_{lps} is stable on $V^h\times Q^h$ with respect to the norm

$$
\begin{aligned}
\|(\boldsymbol{v},q)\|_{\mathrm{lps}} \;=\; & \Bigg(\nu\,\|\nabla \boldsymbol{v}\|_{L^2(\Omega)}^2 + \sum_{E\in\mathcal{E}^h} \gamma_E\,\|[|q|]_E\|_{L^2(E)}^2 \\
& + \sum_{M\in\mathcal{M}^h} \delta_M\,\left\|\nabla^h q\right\|_{L^2(M)}^2\Bigg)^{1/2}
\end{aligned}
$$

in the sense of an inf-sup condition. The norm $\|\cdot\|_{\mathrm{lps}}$ is an analogue of the PSPG norm (3.38) and the proof that it is a norm on $V\times\tilde{Q}$ is the same as in Lemma 3.4.2. One even has the following result.

Lemma 3.5.1 (Relation to the PSPG norm)**.** *Given stabilization parameters* $\{\delta_M\}$ *and* $\{\gamma_E\}$ *satisfying* (3.110) *and* (3.45), *respectively, one has*

$$
\|(\boldsymbol{v},q)\|_{\mathrm{lps}} = \|(\boldsymbol{v},q)\|_{\mathrm{pspg}} \qquad \forall\ (\boldsymbol{v},q)\in V\times\tilde{Q},
$$

where the norm $\|\cdot\|_{\mathrm{pspg}}$ *is defined using stabilization parameters* $\{\delta_K\}$ *satisfying*

$$
0<\delta_0\frac{h_K^2}{\nu}\le\delta_K\le\delta_1'\frac{h_K^2}{\nu} \qquad \forall\ K\in\mathcal{T}^h \tag{3.124}
$$

with a constant δ_1' *independent of* h *and* ν*.*

Proof. One has

$$
\begin{aligned}
\sum_{M\in\mathcal{M}^h}\delta_M\left\|\nabla^h q\right\|_{L^2(M)}^2 &= \sum_{M\in\mathcal{M}^h}\delta_M \sum_{\substack{K\in\mathcal{T}^h,\\ K\subset\overline{M}}} \|\nabla q\|_{L^2(K)}^2 \\
&= \sum_{K\in\mathcal{T}^h}\ \sum_{\substack{M\in\mathcal{M}^h,\\ K\subset\overline{M}}} \delta_M\,\|\nabla q\|_{L^2(K)}^2 \\
&= \sum_{K\in\mathcal{T}^h}\delta_K\,\|\nabla q\|_{L^2(K)}^2
\end{aligned} \tag{3.125}
$$

with
$$\delta_K := \sum_{\substack{M \in \mathcal{M}^h, \\ K \subset \overline{M}}} \delta_M. \tag{3.126}$$

For any $M \in \mathcal{M}^h$ such that $K \subset \overline{M}$ one gets $\delta_K \geq \delta_M \geq \delta_0 h_M^2/\nu \geq \delta_0 h_K^2/\nu$. On the other hand, using (3.120) and (3.115), it follows that

$$\delta_K \leq \delta_1 \sum_{\substack{M \in \mathcal{M}^h, \\ K \subset \overline{M}}} \frac{h_M^2}{\nu} \leq \delta_1 \left(C'_{\mathcal{E}}\right)^2 C_{\mathcal{T}} \frac{h_K^2}{\nu}.$$

□

Lemma 3.5.2 (Inf-sup condition for the bilinear form A_{lps})**.** *Let the conditions* (3.110) *and* (3.45) *on the stabilization parameters* $\{\delta_M\}$ *and* $\{\gamma_E\}$ *be satisfied and let the inf-sup conditions* (3.111) *hold. Then, there is a positive constant* C *such that for all* $\left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h$*, it holds*

$$\sup_{(\boldsymbol{w}^h, r^h) \in V^h \times Q^h \setminus \{(\mathbf{0},0)\}} \frac{A_{\text{lps}}\left(\left(\boldsymbol{v}^h, q^h\right), \left(\boldsymbol{w}^h, r^h\right)\right)}{\left\|\left(\boldsymbol{w}^h, r^h\right)\right\|_{\text{lps}}} \geq C \left\|\left(\boldsymbol{v}^h, q^h\right)\right\|_{\text{lps}}.$$

Proof. Consider any $\left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h$ and set $\boldsymbol{s} = \nabla^h q^h$. Then $\boldsymbol{s} \in L^2(\Omega)^d$ and, using the identity

$$(\nabla \cdot \boldsymbol{w}, q) + (\boldsymbol{w}, \nabla^h q) = \sum_{E \in \mathcal{E}^h} \left(\boldsymbol{w} \cdot \boldsymbol{n}_E, [|q|]_E\right)_E \quad \forall\, \boldsymbol{w} \in V,\ q \in \tilde{Q}, \tag{3.127}$$

that follows from integration by parts, one obtains

$$\begin{aligned} A_{\text{lps}}((\boldsymbol{v}^h, q^h), (\boldsymbol{z}^h, 0)) \geq (\boldsymbol{z}^h, \boldsymbol{s}) - \nu \left\|\nabla \boldsymbol{v}^h\right\|_{L^2(\Omega)} \left\|\nabla \boldsymbol{z}^h\right\|_{L^2(\Omega)} & \\ - \sum_{E \in \mathcal{E}^h} \left(\boldsymbol{z}^h \cdot \boldsymbol{n}_E, \left[\left|q^h\right|\right]_E\right)_E & \end{aligned} \tag{3.128}$$

for any $\boldsymbol{z}^h \in V^h$. Our aim is to choose the function $\boldsymbol{z}^h$ in such a way that the term $(\boldsymbol{z}^h, \boldsymbol{s})$ provides a control of

$$S := \sum_{M \in \mathcal{M}^h} \delta_M \|\boldsymbol{s}\|_{L^2(M)}^2.$$

For this one can employ the inf-sup conditions (3.111) which imply that, for any $M \in \mathcal{M}^h$, there exists $\boldsymbol{z}_M \in V_M$ such that (cf., e.g., [40])

$$(\boldsymbol{z}_M, \boldsymbol{q})_M = \delta_M (\boldsymbol{s}, \boldsymbol{q})_M \qquad \forall\, \boldsymbol{q} \in D_M, \tag{3.129}$$
$$\|\boldsymbol{z}_M\|_{L^2(M)} \leq \beta_{LP}^{-1} \delta_M \|\boldsymbol{s}\|_{L^2(M)}. \tag{3.130}$$

Since $\pi_M \boldsymbol{s} \in D_M$, one gets

$$\begin{aligned}(\boldsymbol{z}_M, \boldsymbol{s}) &= (\boldsymbol{z}_M, \pi_M \boldsymbol{s})_M + (\boldsymbol{z}_M, \kappa_M \boldsymbol{s})_M \\ &= \delta_M \,(\boldsymbol{s}, \pi_M \boldsymbol{s})_M + (\boldsymbol{z}_M, \kappa_M \boldsymbol{s})_M \\ &= \delta_M \,\|\boldsymbol{s}\|^2_{L^2(M)} - \delta_M \,(\boldsymbol{s}, \kappa_M \boldsymbol{s})_M + (\boldsymbol{z}_M, \kappa_M \boldsymbol{s})_M \,.\end{aligned}$$

Due to (3.130) and the Young inequality, one has

$$\begin{aligned}|\delta_M \,(\boldsymbol{s}, \kappa_M \boldsymbol{s})_M - (\boldsymbol{z}_M, \kappa_M \boldsymbol{s})_M| &\le (\delta_M \,\|\boldsymbol{s}\|_{L^2(M)} + \|\boldsymbol{z}_M\|_{L^2(M)})\,\|\kappa_M \boldsymbol{s}\|_{L^2(M)} \\ &\le \delta_M \,(1 + \beta_{LP}^{-1})\,\|\boldsymbol{s}\|_{L^2(M)}\,\|\kappa_M \boldsymbol{s}\|_{L^2(M)} \\ &\le \frac{\delta_M}{2}\,\|\boldsymbol{s}\|^2_{L^2(M)} + (1 + \beta_{LP}^{-2})\,\delta_M\,\|\kappa_M \boldsymbol{s}\|^2_{L^2(M)}\end{aligned}$$

and hence

$$(\boldsymbol{z}_M, \boldsymbol{s}) \ge \frac{\delta_M}{2}\,\|\boldsymbol{s}\|^2_{L^2(M)} - (1 + \beta_{LP}^{-2})\,\delta_M\,\|\kappa_M \boldsymbol{s}\|^2_{L^2(M)} \,.$$

Thus, setting $\boldsymbol{z}^h = \sum_{M \in \mathcal{M}^h} \boldsymbol{z}_M$, one gets

$$(\boldsymbol{z}^h, \boldsymbol{s}) \ge \frac{1}{2}\,S - (1 + \beta_{LP}^{-2}) \sum_{M \in \mathcal{M}^h} \delta_M\,\|\kappa_M \boldsymbol{s}\|^2_{L^2(M)} \,.$$

In view of (3.113), one has

$$\begin{aligned}\|\nabla \boldsymbol{z}^h\|^2_{L^2(\Omega)} &\le \sum_{M' \in \mathcal{M}^h} \|\nabla \boldsymbol{z}^h\|^2_{L^2(M')} \\ &\le \sum_{M' \in \mathcal{M}^h} \Bigg(\sum_{\substack{M \in \mathcal{M}^h, \\ M \cap M' \neq \emptyset}} \|\nabla \boldsymbol{z}_M\|_{L^2(M')} \Bigg)^2 \\ &\le C_{\mathcal{M}} \sum_{M' \in \mathcal{M}^h} \sum_{M \in \mathcal{M}^h} \|\nabla \boldsymbol{z}_M\|^2_{L^2(M')} \\ &= C_{\mathcal{M}} \sum_{M \in \mathcal{M}^h} \sum_{\substack{M' \in \mathcal{M}^h, \\ M \cap M' \neq \emptyset}} \|\nabla \boldsymbol{z}_M\|^2_{L^2(M')} \\ &\le C_{\mathcal{M}}^2 \sum_{M \in \mathcal{M}^h} \|\nabla \boldsymbol{z}_M\|^2_{L^2(M)} \,.\end{aligned}$$

Using (3.122), (3.130), and (3.110), one derives

$$\nu\,\|\nabla \boldsymbol{z}_M\|^2_{L^2(M)} \le \bar{C}^2_{\mathrm{inv}} \nu h_M^{-2} \|\boldsymbol{z}_M\|^2_{L^2(M)} \le \delta_1 \bar{C}^2_{\mathrm{inv}} \beta_{LP}^{-2} \delta_M \|\boldsymbol{s}\|^2_{L^2(M)}$$

and hence

$$\nu^{1/2}\,\big\|\nabla \boldsymbol{z}^h\big\|_{L^2(\Omega)} \le C_1 S^{1/2}, \tag{3.131}$$

with $C_1 = \delta_1^{1/2} C_{\mathcal{M}} \bar{C}_{\rm inv} \beta_{LP}^{-1}$. Finally, using the Cauchy–Schwarz inequality, (3.121), (3.122), (3.130), (3.110), (3.45), (3.120), (3.116), and (3.117), the last term in (3.128) can be estimated by

$$
\begin{aligned}
\left| \sum_{E\in\mathcal{E}^h} \left(\boldsymbol{z}^h \cdot \boldsymbol{n}_E, \left[|q^h|\right]_E\right)_E \right| &\le \sum_{\substack{E\in\mathcal{E}^h,\ M\in\mathcal{M}^h,\\ E\subset \overline{M}}} \|\boldsymbol{z}_M\|_{L^2(E)} \| \left[|q^h|\right]_E \|_{L^2(E)} \\
&\le C_e\,(1+\bar{C}_{\rm inv})\,\beta_{LP}^{-1} \sum_{\substack{E\in\mathcal{E}^h,\ M\in\mathcal{M}^h,\\ E\subset \overline{M}}} h_M^{-1/2}\,\delta_M\,\|\boldsymbol{s}\|_{L^2(M)} \| \left[|q^h|\right]_E \|_{L^2(E)} \\
&\le C_{\mathcal{E}}\, C_e\,(1+\bar{C}_{\rm inv})\,\beta_{LP}^{-1} \left(C_{\mathcal{E}}' \frac{\delta_1}{\gamma_0}\right)^{1/2} S^{1/2} \left(\sum_{E\in\mathcal{E}^h} \gamma_E \left\| \left[|q^h|\right]_E \right\|_{L^2(E)}^2 \right)^{1/2} .
\end{aligned}
$$

Thus, combining the above inequalities and applying the Young inequality, it follows that

$$
A_{\rm lps}((\boldsymbol{v}^h, q^h), (\boldsymbol{z}^h, 0)) \ge \frac{1}{4} S - C_2 \left|(\boldsymbol{v}^h, q^h)\right|_{\rm lps}^2 ,
$$

where C_2 depends only on $C_{\mathcal{M}}$, $C_{\mathcal{E}}$, $C_{\mathcal{E}}'$, C_e, $\bar{C}_{\rm inv}$, δ_1, γ_0, and β_{LP}. Setting

$$
\boldsymbol{w}^h = 4\,\boldsymbol{z}^h + (1+4\,C_2)\,\boldsymbol{v}^h , \qquad r^h = (1+4\,C_2)\,q^h
$$

and using (3.123), one obtains

$$
A_{\rm lps}((\boldsymbol{v}^h, q^h), (\boldsymbol{w}^h, r^h)) \ge S + \left|(\boldsymbol{v}^h, q^h)\right|_{\rm lps}^2 \ge \left\|(\boldsymbol{v}^h, q^h)\right\|_{\rm lps}^2 .
$$

From (3.131), it follows that

$$
\begin{aligned}
\left\|(\boldsymbol{w}^h, r^h)\right\|_{\rm lps} &\le 4\,\nu^{1/2} \left\|\nabla \boldsymbol{z}^h\right\|_{L^2(\Omega)} + (1+4\,C_2) \left\|(v^h, q^h)\right\|_{\rm lps} \\
&\le (1+4\,C_1+4\,C_2) \left\|(\boldsymbol{v}^h, q^h)\right\|_{\rm lps} ,
\end{aligned}
$$

which proves the theorem. □

We now move on to error estimates. First, let us investigate the consistency of the method.

Lemma 3.5.3 (Consistency error)**.** *Let the solution of* (3.5) *satisfy* $(\boldsymbol{u}, p) \in H_0^1(\Omega)^d \times H^1(\Omega)$ *and let* $(\boldsymbol{u}^h, p^h) \in V^h \times Q^h$ *be the solution of the LPS method* (3.109)*. The LPS method is not consistent and it holds*

$$
A_{\rm lps}\left(\left(\boldsymbol{u}-\boldsymbol{u}^h, p-p^h\right), \left(\boldsymbol{v}^h, q^h\right)\right) = \sum_{M\in\mathcal{M}^h} \delta_M (\kappa_M(\nabla p), \kappa_M(\nabla^h q^h))_M \qquad (3.132)
$$

for all $\left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h$.

Proof. The lemma is a simple consequence of (3.5) and (3.109). □

The term on the right-hand side of (3.132) represents the consistency error and is estimated in the following lemma.

Lemma 3.5.4 (Estimate of the consistency error)**.** *Let* $\{\delta_M\}$ *satisfy* (3.110) *and let* $p \in H^{m+1}(\Omega)$ *with* $0 \leq m \leq k$*. Then, for any* $q^h \in Q^h$*, one has*

$$\sum_{M\in\mathcal{M}^h} \delta_M(\kappa_M(\nabla p), \kappa_M(\nabla^h q^h))_M \leq C\nu^{-1/2} h^{m+1} \|p\|_{H^{m+1}(\Omega)} \left\|(\mathbf{0}, q^h)\right\|_{\mathrm{lps}} .$$

Proof. Applying the Cauchy–Schwarz inequality, (3.110), (3.118), and (3.112), one obtains for any $q^h \in Q^h$

$$\begin{aligned}
&\sum_{M\in\mathcal{M}^h} \delta_M \left(\kappa_M(\nabla p), \kappa_M(\nabla^h q)\right)_M \\
&\quad\leq \sum_{M\in\mathcal{M}^h} \delta_M \left\|\kappa_M(\nabla p - j_M \nabla p)\right\|_{L^2(M)} \left\|\kappa_M(\nabla^h q^h)\right\|_{L^2(M)} \\
&\quad\leq \frac{C_\kappa^2 \delta_1^{1/2}}{\nu^{1/2}} \left(\sum_{M\in\mathcal{M}^h} h_M^2 \left\|\nabla p - j_M \nabla p\right\|_{L^2(M)}^2\right)^{1/2} \left(\sum_{M\in\mathcal{M}^h} \delta_M \left\|\nabla^h q^h\right\|_{L^2(M)}^2\right)^{1/2} \\
&\quad\leq C\nu^{-1/2} \left(\sum_{M\in\mathcal{M}^h} h_M^{2m+2} \left\|\nabla p\right\|_{H^m(M)}^2\right)^{1/2} \left(\sum_{M\in\mathcal{M}^h} \delta_M \left\|\nabla^h q^h\right\|_{L^2(M)}^2\right)^{1/2}
\end{aligned}$$

and the lemma follows using (3.114) and (3.115). □

Theorem 3.5.5 (Error estimate)**.** *Let the solution of* (3.5) *satisfy* $(\boldsymbol{u}, p) \in H^{k+1}(\Omega)^d \times H^{l+1}(\Omega)$ *and let* $\left(\boldsymbol{u}^h, p^h\right) \in V^h \times Q^h$ *be the solution of the LPS problem* (3.109)*. Assume that the stabilization parameters satisfy* (3.110) *and* (3.45) *and that the inf-sup conditions* (3.111) *hold. Then the following error estimate holds*

$$\left\|\left(\boldsymbol{u} - \boldsymbol{u}^h, p - p^h\right)\right\|_{\mathrm{lps}} \leq C\left(\nu^{1/2} h^k \|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + \frac{h^{\min\{k,l\}+1}}{\nu^{1/2}} \|p\|_{H^{l+1}(\Omega)}\right).$$

Proof. Let I^h and J^h be the interpolation operators satisfying (3.8) and (3.9). From the proof of Lemma 3.5.2, it is known that there is a pair $\left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h$ such that

$$\left\|\left(\boldsymbol{u}^h - I^h\boldsymbol{u}, p^h - J^h p\right)\right\|_{\mathrm{lps}} \leq C\frac{A_{\mathrm{lps}}\left(\left(\boldsymbol{u}^h - I^h\boldsymbol{u}, p^h - J^h p\right), \left(\boldsymbol{v}^h, q^h\right)\right)}{\left\|\left(\boldsymbol{v}^h, q^h\right)\right\|_{\mathrm{lps}}}.$$

With Lemmas 3.5.3 and 3.5.4, one obtains

$$\begin{aligned}
\left\|\left(\boldsymbol{u}^h - I^h\boldsymbol{u}, p^h - J^h p\right)\right\|_{\mathrm{lps}} \leq{}& C\frac{A_{\mathrm{lps}}\left(\left(\boldsymbol{u} - I^h\boldsymbol{u}, p - J^h p\right), \left(\boldsymbol{v}^h, q^h\right)\right)}{\left\|\left(\boldsymbol{v}^h, q^h\right)\right\|_{\mathrm{lps}}} \\
&+ C\nu^{-1/2} h^{\min\{k,l\}+1} \|p\|_{H^{\min\{k,l\}+1}(\Omega)} .
\end{aligned}$$

Applying the Cauchy–Schwarz inequality, one gets

$$\begin{aligned} A_{\mathrm{lps}}\left((\boldsymbol{u}-I^h\boldsymbol{u}, p-J^hp), (\boldsymbol{v}^h, q^h)\right) \le \left|(\boldsymbol{u}-I^h\boldsymbol{u}, p-J^hp)\right|_{\mathrm{lps}} \left|(\boldsymbol{v}^h, q^h)\right|_{\mathrm{lps}} \\ -(\nabla\cdot\boldsymbol{v}^h, p-J^hp) + (\nabla\cdot(\boldsymbol{u}-I^h\boldsymbol{u}), q^h) \end{aligned}$$

and

$$-(\nabla\cdot\boldsymbol{v}^h, p-J^hp) \le \nu^{-1/2}\left\|p-J^hp\right\|_{L^2(\Omega)} \left\|(\boldsymbol{v}^h, q^h)\right\|_{\mathrm{lps}}.$$

Using (3.127), the Cauchy–Schwarz inequality, (3.110), (3.45), and (3.8), one derives

$$\begin{aligned} (\nabla\cdot(\boldsymbol{u}-I^h\boldsymbol{u}), q^h) &= -(\boldsymbol{u}-I^h\boldsymbol{u}, \nabla^h q^h) + \sum_{E\in\mathcal{E}^h}\left((\boldsymbol{u}-I^h\boldsymbol{u})\cdot\boldsymbol{n}_E, [|q^h|]_E\right)_E \\ &\le \frac{\nu^{1/2}}{\delta_0^{1/2}}\left(\sum_{K\in\mathcal{T}^h} h_K^{-2}\left\|\boldsymbol{u}-I^h\boldsymbol{u}\right\|_{L^2(K)}^2\right)^{1/2}\left(\sum_{M\in\mathcal{M}^h}\delta_M\left\|\nabla^h q^h\right\|_{L^2(M)}^2\right)^{1/2} \\ &\quad + \frac{\nu^{1/2}}{\gamma_0^{1/2}}\left(\sum_{E\in\mathcal{E}^h} h_E^{-1}\left\|\boldsymbol{u}-I^h\boldsymbol{u}\right\|_{L^2(E)}^2\right)^{1/2}\left(\sum_{E\in\mathcal{E}^h}\gamma_E\left\|[|q^h|]_E\right\|_{L^2(E)}^2\right)^{1/2} \\ &\le C\nu^{1/2}h^k\left\|\boldsymbol{u}\right\|_{H^{k+1}(\Omega)}\left\|(\boldsymbol{v}^h, q^h)\right\|_{\mathrm{lps}}. \end{aligned}$$

Combining the above inequalities and using the triangle inequality, (3.9), (3.118) and Lemma 3.5.1, one obtains

$$\begin{aligned} \left\|(\boldsymbol{u}-\boldsymbol{u}^h, p-p^h)\right\|_{\mathrm{lps}} &\le C\left\|(\boldsymbol{u}-I^h\boldsymbol{u}, p-J^hp)\right\|_{\mathrm{pspg}} \\ &\quad + C\nu^{1/2}h^k\left\|\boldsymbol{u}\right\|_{H^{k+1}(\Omega)} + C\nu^{-1/2}h^{\min\{k,l\}+1}\left\|p\right\|_{H^{l+1}(\Omega)} \end{aligned}$$

and the statement of the theorem follows from (3.48). □

Theorem 3.5.6 (L^2 estimate of the pressure error)**.** *Assume that the solution of* (3.5) *satisfies* $(\boldsymbol{u}, p) \in H^{k+1}(\Omega)^d \times H^{l+1}(\Omega)$*, that the stabilization parameters satisfy* (3.110) *and* (3.45) *and that the inf-sup conditions* (3.111) *hold. Then there holds the error estimate*

$$\left\|p-p^h\right\|_{L^2(\Omega)} \le C\left(\nu h^k\left\|\boldsymbol{u}\right\|_{H^{k+1}(\Omega)} + h^{\min\{k,l\}+1}\left\|p\right\|_{H^{l+1}(\Omega)}\right).$$

Proof. Using $\{\delta_K\}$ defined in (3.126), the proof of Theorem 3.4.8 can be repeated without any changes. Then the statement of the present theorem follows from Lemma 3.5.1 and Theorem 3.5.5. □

Theorem 3.5.7 (L^2 estimate of the velocity error)**.** *Let the stabilization parameters satisfy* (3.110) *and* (3.45)*, let the inf-sup conditions* (3.111) *hold, and let the Stokes problem* (3.2) *be regular. Assume that the solution of* (3.5) *satisfies* $(\boldsymbol{u}, p) \in H^{k+1}(\Omega)^d \times H^{l+1}(\Omega)$*, then there holds the error estimate*

$$\left\|\boldsymbol{u}-\boldsymbol{u}^h\right\|_{L^2(\Omega)} \le C\left(h^{k+1}\left\|\boldsymbol{u}\right\|_{H^{k+1}(\Omega)} + \frac{h^{\min\{k,l\}+2}}{\nu}\left\|p\right\|_{H^{l+1}(\Omega)}\right).$$

Proof. Up to (3.60), the proof of Theorem 3.4.9 remains valid also in this case. Then, instead of (3.61), one obtains from (3.132)

$$
\begin{aligned}
\left(\nabla\cdot(\boldsymbol{u}-\boldsymbol{u}^h), r^I\right) &+ \sum_{E\in\mathcal{E}^h} \gamma_E \left(\left[\!\left|p-p^h\right|\!\right]_E, \left[\!\left|r^I\right|\!\right]_E\right)_E \\
&+ \sum_{M\in\mathcal{M}^h} \delta_M(\kappa_M(\nabla^h(p-p^h)), \kappa_M(\nabla^h r^I))_M \\
&= \sum_{M\in\mathcal{M}^h} \delta_M(\kappa_M(\nabla p), \kappa_M(\nabla^h r^I))_M .
\end{aligned}
$$

Thus, instead of (3.62), one obtains the following expression for the last two terms in (3.27)

$$
\begin{aligned}
&\nu(\nabla \boldsymbol{z}^I, \nabla(\boldsymbol{u}-\boldsymbol{u}^h)) - (\nabla\cdot(\boldsymbol{u}-\boldsymbol{u}^h), r^I) \\
&\quad= \left(\nabla\cdot(\boldsymbol{z}^I-\boldsymbol{z}), p-p^h\right) + \sum_{E\in\mathcal{E}^h} \gamma_E \left(\left[\!\left|p-p^h\right|\!\right]_E, \left[\!\left|r^I-r\right|\!\right]_E\right)_E \\
&\qquad+ \sum_{M\in\mathcal{M}^h} \delta_M(\kappa_M(\nabla^h(p-p^h)), \kappa_M(\nabla^h r^I))_M \\
&\qquad- \sum_{M\in\mathcal{M}^h} \delta_M(\kappa_M(\nabla p), \kappa_M(\nabla^h r^I))_M .
\end{aligned}
$$

Analogously as in (3.63), but using also Lemma 3.5.4, (3.118), (3.125), and (3.124), one derives

$$
\begin{aligned}
&\nu(\nabla \boldsymbol{z}^I, \nabla(\boldsymbol{u}-\boldsymbol{u}^h)) - (\nabla\cdot(\boldsymbol{u}-\boldsymbol{u}^h), r^I) \\
&\quad\le Ch\Big(\left\|p-p^h\right\|_{L^2(\Omega)} + \nu^{1/2}\left\|\left(\boldsymbol{0}, p-p^h\right)\right\|_{\mathrm{lps}} \\
&\qquad\qquad + h^{\min\{k,l\}+1}\left\|p\right\|_{H^{l+1}(\Omega)}\Big)\left\|\boldsymbol{u}-\boldsymbol{u}^h\right\|_{L^2(\Omega)} .
\end{aligned}
$$

Combining this estimate with (3.27) and (3.28), the theorem follows using Theorems 3.5.5 and 3.5.6. □

The LPS method for the Stokes problem was introduced in [13]. A generalization and unified analysis was presented in [72] where the stability with respect to a norm containing the $L^2(\Omega)$ norm of the pressure was established. The techniques presented here are a special case of the analysis published in [64]. As one can see, the LPS method leads to analogous stability and convegence results as residual-based approaches. However, in comparison with residual-based stabilizations, an important advantage of LPS methods is that they do not create additional couplings between various unknowns. A drawback is that the local projections couple pressure degrees of freedom that do not belong to the same mesh cell. Hence, the sparsity pattern of the pressure-pressure matrix C in (3.4) is denser as, e.g., for residual-based discretizations.

Remark 3.5.8 (LPS method with Scott–Zhang-type projector). An LPS method that uses a particular Scott–Zhang-type projector, which is well defined for $L^1(\Omega)$ functions, is proposed in [7]. Like the LPS method with overlapping macroelements, it neither requires nested meshes nor an enrichment of spaces by bubble functions. However, similarly as for the other versions of the LPS method, the projector leads to a wider sparsity pattern of the pressure-pressure matrix. A finite element error analysis of this method and few numerical comparisons with the symmetric GLS method from Section 3.4.3 for P_1/P_1 finite elements, which is in this case equivalent to the PSPG method, are presented in [7]. The method is absolutely stable. There are no assumptions on upper bounds of the stabilization parameter in the analysis and the numerical studies show even a slight improvement of the accuracy for large stabilization parameters. $\triangle$

A stabilizing term of the form

$$\sum_{K\in\mathcal{T}^h} \frac{\alpha}{\nu} h_K^2 \left(\left(I - P^h\right)\left(\nabla p^h\right), \left(I - P^h\right)\left(\nabla q^h\right)\right)_K$$

is proposed in [31] for P_k/P_k finite elements, where P^h is some stable approximation operator from $L^2(\Omega)^d$ into the space of continuous piecewise polynomial functions of degree $k-1$. This operator was chosen in the numerical studies from [31] as an extension of a nodal interpolation operator. The arising method is called term-by-term stabilized method. The differences to already existing methods are discussed in detail in [31]. Depending on the actual choice of P^h, it can be considered as an LPS method that is defined on a single mesh and with standard finite element spaces. If P^h is chosen to be a global $L^2(\Omega)$ projection, then the image of the projection operator is different than for the method from [34]. In [31], a finite element convergence analysis is presented that proves optimal orders for the $L^2(\Omega)$ norm of the velocity gradient and of the pressure.

A two-level LPS method was studied in [76]. Using this method, the pressure gradient from the LPS stabilization term can be locally eliminated, which facilitates the implementation of this LPS method.

In [11], the so-called residual local projection (RELP) method is proposed for low order pairs of finite element spaces. It contains an LPS term for the pressure. An additional pressure-pressure coupling is introduced by jump terms of the stress tensor across faces of the mesh cells. Special cases of the RELP method coincide with methods from [38] and [3]. The finite element error analysis presented in [11] shows optimal convergence for the $L^2(\Omega)$ norms of the velocity gradient and of the pressure. A similar method, where the jumps of the stress tensor are replaced by jumps of the pressure, is proposed and analyzed in [12]. The methods from [11, 12] do not need multiple levels or extra degrees of freedom for computing the local projection and all computations can be performed on the mesh cell level. However, the stencil of some matrix blocks gets enlarged due to the jump terms.

3.5.5 Stabilization with Fluctuations of the Pressure

A pressure-stabilized method that uses fluctuations of the pressure itself, instead of the gradient of the pressure as the methods discussed in Sections 3.5.3 and 3.5.4, was proposed in [38].

Let $Q^h = P_k$ or $Q^h = Q_k$, $k \geq 1$, then the method utilizes the $L^2(\Omega)$ projection $P_{L^2}^{k-1} : Q^h \to P_{k-1}^{\text{disc}}$ onto the discontinuous piecewise polynomial space of degree $k-1$. Since the image space consists of discontinuous finite element functions, the projection operator $P_{L^2}^{k-1}$ can be computed locally, i.e., mesh cell by mesh cell. The discrete continuity equation of the method proposed in [38] reads as follows

$$-\left(\nabla \cdot \boldsymbol{u}^h, q^h\right) - \frac{1}{\nu}\left(p^h - P_{L^2}^{k-1} p^h, q^h - P_{L^2}^{k-1} q^h\right) = 0 \quad \forall\, q^h \in Q^h. \tag{3.133}$$

There is no user-chosen parameter in (3.133).

A finite element analysis of this method for the equal order pairs P_1/P_1 and Q_1/Q_1 of lowest order is performed in [20]. The inverse of the viscosity does not appear in the stabilization term in contrast to (3.133). An extension of the method to the pairs P_1/P_0 and Q_1/Q_0 is also proposed. The analysis shows that in all cases the method is unconditionally stable and optimal error bounds were derived, e.g., linear convergence for $\left\|\nabla\left(\boldsymbol{u} - \boldsymbol{u}^h\right)\right\|_{L^2(\Omega)}$ and $\left\|p - p^h\right\|_{L^2(\Omega)}$.

A similar method, which uses projections in a pressure space defined on a coarser grid, was developed in [65]. The derivation of this method used ideas from the variational multiscale framework.

The stabilization term proposed in [38] can be written in the form

$$c^h\left(p^h, q^h\right) = \frac{\alpha}{\nu} \underline{q}^T \left(\tilde{M} - M\right) \underline{q}, \tag{3.134}$$

where $\alpha \in \mathbb{R}$ ($\alpha = 1$ in [38]), $\underline{p}$ and $\underline{q}$ are the vector representations of p^h and q^h with respect to the standard basis of Q^h, M is the mass matrix with respect to this basis, and $\tilde{M}$ is the mass matrix from the functions arising in the $L^2(\Omega)$ projection.

Subsequently, further methods with stabilization terms of type (3.134) were proposed in [15] and [67]. The method of [15] uses as $\tilde{M}$ an under-integrated mass matrix. Concrete examples for the bilinear form from (3.134) are given for P_1 finite elements in two dimensions, where

$$c^h\left(p^h, q^h\right) = \frac{\alpha}{\nu} \int_\Omega \left(I_1^h(p^h q^h) - p^h q^h\right) \mathrm{d}\boldsymbol{x}$$

and for P_2 finite elements in 2d where

$$c^h\left(p^h, q^h\right) = \frac{\alpha}{\nu} \int_\Omega \left(I_3^h(p^h q^h) - p^h q^h\right) \mathrm{d}\boldsymbol{x}$$

Here, I_k^h, $k \geq 1$, is the Lagrangian interpolation operator onto the space of continuous piecewise polynomial functions of degree k. Optimal estimates for the $L^2(\Omega)$ errors of the velocity gradient and the pressure are derived in [15].

The method from [67] uses two local Gauss integrations to define the matrices, where $\tilde{M}$ is defined by a first order Gaussian integration in each direction. This method is proposed in [67] for P_1/P_1 and Q_1/Q_1 finite elements in two dimensions. It is already observed in [15] that for these cases the method from [67] is equivalent to the method already proposed in [38]. However, the methods from [15] and [38] are not equivalent.

3.5.6 Continuous Interior Penalty Methods

Continuous Interior Penalty (CIP) methods use jumps of the pressure gradient or the normal derivative of the pressure across faces of mesh cells for stabilizing the inf-sup condition. The first method of this class was proposed in [29, 28]. However, the use of jumps across faces of the mesh cells for pressure stabilization dates back to a method proposed in [82]. For the Q_1/Q_0 pair of finite element spaces, this method uses jumps of the pressure itself.

In [29], the stabilization term, which defines the matrix $-C$ in (3.4), has the form

$$\frac{1}{2}\sum_{K\in\mathcal{T}^h}\left(\delta_0 h_K^{s+1}\sum_{E\subset\partial K}\left(\left[\left|\nabla p^h\cdot\boldsymbol{n}_E\right|\right]_E, \left[\left|\nabla q^h\cdot\boldsymbol{n}_E\right|\right]_E\right)_E\right), \tag{3.135}$$

where

$$s = \begin{cases} 2 & \text{if } \nu \geq h, \\ 1 & \text{if } \nu < h. \end{cases}$$

Additionally, a jump term containing the divergence of $\boldsymbol{u}^h$ is included in the method studied in [29]. A finite element analysis for the P_1/P_1 pair of spaces was presented. Assuming that $p \in H^2(\Omega)$, the estimate

$$\begin{aligned} &\left\|\nabla\left(\boldsymbol{u}-\boldsymbol{u}^h\right)\right\|_{L^2(\Omega)} + \frac{1}{\nu}\left\|p-p^h\right\|_{L^2(\Omega)} \\ &\leq \quad Ch\left[\max\left\{\frac{1+\delta_0}{\nu^{1/2}}, 1\right\}\max\left\{h^{s/2}, h^{(2-s)/2}\right\} + \|\boldsymbol{u}\|_{H^2(\Omega)}\right. \\ &\qquad \left. + \max\left\{\frac{1}{\nu^{1/2}}, \frac{1}{\nu}, \frac{\delta_0^{1/2}}{\nu^{1/2}}\right\}\max\left\{h^{s/2}, h^{(2-s)/2}\right\}\|p\|_{H^2(\Omega)}\right] \end{aligned}$$

was proved. It follows that in the case $\nu < h$, the error reduction of $\left\|p - p^h\right\|_{L^2(\Omega)}$ is of order 1.5. Also the case that only $p \in H^1(\Omega)$ holds was studied in [29].

The stabilization term of the method from [28] uses the jumps of the pressure gradient instead of the normal derivative.

Using classical CIP stabilizations, e.g., (3.135), connects pressure degrees of freedom that do not belong to a common mesh cell. Hence, the matrix stencil of C is denser than, e.g., for residual-based stabilizations.

A so-called local CIP method was introduced and analyzed in [30]. The advantage of this method is that it allows static condensation. As result, the matrix stencil of the matrix C is substantially smaller than for the classical CIP methods. The local CIP method uses a so-called macro-mesh $\mathcal{M}^h$, where each mesh cell $M \in \mathcal{M}^h$ consists of a small number of simplicial cells $K \in \mathcal{T}^h$. Then, the stabilization term has the form

$$\sum_{M\in\mathcal{M}^h}\left[\sum_{K\in M}\delta_K h_K\left(\sum_{E\in\partial K, E\subset \mathrm{int}(K)}\left(\left[|\nabla p^h|\right]_E, \left[|\nabla q^h|\right]_E\right)_E\right)\right],$$

where $\mathrm{int}(K)$ is the interior of K and

$$\delta_K = \min\left\{\frac{h_K^2}{\nu}, h_K\right\}.$$

As a particular case of the error analysis presented in [30], one obtains the estimates for P_k/P_k finite elements, $k \geq 1$,

$$\begin{aligned}&\left\|\nabla\left(\boldsymbol{u}-\boldsymbol{u}^h\right)\right\|_{L^2(\Omega)}\\ &\quad\leq\ Ch^k\left(\left(1+\frac{h}{\nu}\right)^{1/2}|\boldsymbol{u}|_{H^{k+1}(\Omega)}+\min\left\{\frac{h^{1/2}}{\nu^{1/2}},\frac{h}{\nu}\right\}|p|_{H^{k+1}(\Omega)}\right)\end{aligned}$$

and

$$\begin{aligned}&\left\|p-p^h\right\|_{L^2(\Omega)}\\ &\quad\leq\ Ch^k\left((\nu+h)^{1/2}|\boldsymbol{u}|_{H^{k+1}(\Omega)}+\left(h+\min\left\{h^{1/2},\frac{h}{\nu^{1/2}}\right\}\right)|p|_{H^{k+1}(\Omega)}\right).\end{aligned}$$

The error reduction for the pressure is of order $k + 0.5$ as long as $\nu < h$. This higher order, even $k + 1$ for $\nu \ll h$, was observed in the numerical studies of [30].

3.6 Connections to Inf-Sup Stable Methods with Bubble Functions

If a pair of finite element spaces for approximating the velocity and pressure does not satisfy the discrete inf-sup condition (3.3), one can construct a stable pair of spaces by adding suitable functions to the velocity space. The velocity space V^h then has the form $V^h = V_1^h \oplus V_2^h$, where V_1^h typically assures the approximation properties of the space V^h and V_2^h guarantees the fulfilment of the inf-sup condition

(3.3) for the given pressure space Q^h. In this section we shall consider only spaces of this type.

The functions contained in the space V_2^h are often called bubble functions. It was realized very soon [78] that there is a close relationship between stabilized methods and Galerkin methods with bubble functions. Namely, if one drops the bubble part of the solution of a Galerkin method with bubble functions, then one sometimes gets functions which represent a solution of a stabilized method. For linear problems, this relationship was also established in an abstract framework via virtual bubbles in [10]. There is a lot of further papers devoted to investigations of the mentioned relationship, see, e.g., [63] for references.

In this section we go a step further and consider also modifications of the conforming discretization for the spaces $V^h = V_1^h \oplus V_2^h$, Q^h to obtain equivalent representations for a wider class of stabilized methods based on the spaces V_1^h, Q^h. Such equivalences are helpful for a better understanding of the properties of stabilized methods and for their theoretical investigations. Moreover, the technique of modified discretizations can be used for designing new stabilized methods. The theory available for the modified discretizations then automatically provides existence and convergence statements for the corresponding stabilized methods.

There are many examples of finite element spaces of the mentioned type. The simplest choice for the spaces V_1^h and Q^h are piecewice constant functions for Q^h and continuous piecewise (bi-, tri-)linear functions for V_1^h. To satisfy the inf-sup condition, it suffices to use a space V_2^h consisting of one vector-valued edge/face-bubble function per each inner edge/face, see [17, 41]. In the triangular/tetrahedral case, spaces Q^h, V_1^h consisting of continuous piecewise linear functions may be stabilized using V_2^h consisting of d vector-valued element bubble functions per each element. This pair of spaces is known as the MINI element, cf. [5]. In two dimensions, the same space V_2^h can be used if V_1^h consists of continuous piecewise quadratic functions and Q^h of discontinuous piecewise linear functions, cf. [36]. A generalization of [5] to the quadrilateral case is described in [74]. Further examples of spaces V_1^h, V_2^h and Q^h can be found, e.g., in [49].

Since $V^h = V_1^h \oplus V_2^h$ (which implies that $V_1^h \cap V_2^h = \{\mathbf{0}\}$), any function $\boldsymbol{v}^h \in V^h$ can be written in the form $\boldsymbol{v}^h = \boldsymbol{v}_1^h + \boldsymbol{v}_2^h$ where the functions $\boldsymbol{v}_1^h \in V_1^h$ and $\boldsymbol{v}_2^h \in V_2^h$ are uniquely determined. When there will be no danger of ambiguity, we shall also use the notations $\boldsymbol{v}_1^h$ and $\boldsymbol{v}_2^h$ for arbitrary functions belonging to V_1^h and V_2^h, respectively. The conforming discretization (3.13) can be equivalently written in the form: Find $\boldsymbol{u}_1^h \in V_1^h$, $\boldsymbol{u}_2^h \in V_2^h$, and $p^h \in Q^h$ such that

$$\nu(\nabla \boldsymbol{u}_1^h, \nabla \boldsymbol{v}_1^h) + \nu(\nabla \boldsymbol{u}_2^h, \nabla \boldsymbol{v}_1^h) - (\nabla \cdot \boldsymbol{v}_1^h, p^h) = (\boldsymbol{f}, \boldsymbol{v}_1^h) \quad \forall\, \boldsymbol{v}_1^h \in V_1^h, \tag{3.136}$$

$$\nu(\nabla \boldsymbol{u}_1^h, \nabla \boldsymbol{v}_2^h) + \nu(\nabla \boldsymbol{u}_2^h, \nabla \boldsymbol{v}_2^h) - (\nabla \cdot \boldsymbol{v}_2^h, p^h) = (\boldsymbol{f}, \boldsymbol{v}_2^h) \quad \forall\, \boldsymbol{v}_2^h \in V_2^h, \tag{3.137}$$

$$-(\nabla \cdot \boldsymbol{u}_1^h, q^h) - (\nabla \cdot \boldsymbol{u}_2^h, q^h) = 0 \quad \forall\, q^h \in Q^h. \tag{3.138}$$

It is assumed that the approximation properties of the space V^h are determined by the space V_1^h and hence the interpolation operator I^h may be assumed to map $V \cap H^{k+1}(\Omega)^d$ into V_1^h. Then it turns out (cf. Lemma 3.6.4 below) that the component $\boldsymbol{u}_1^h$ of $\boldsymbol{u}^h$ has the same asymptotic approximation properties as $\boldsymbol{u}^h$. Therefore, it makes sense to consider $\boldsymbol{u}_1^h$ as an approximation of the velocity $\boldsymbol{u}$ whereas $\boldsymbol{u}_2^h$ serves as a stabilization tool only. Note that one can compute $\boldsymbol{u}_2^h$ from (3.137) as a function of $\boldsymbol{u}_1^h$ and p^h. Substituting this $\boldsymbol{u}_2^h$ into (3.136) and (3.138), one obtains a discrete problem for $\boldsymbol{u}_1^h$ and p^h where the terms $\nu(\nabla\boldsymbol{u}_2^h,\nabla\boldsymbol{v}_1^h)$ and $(\nabla\cdot\boldsymbol{u}_2^h,q^h)$ give rise to stabilization terms.

Let us demonstrate the procedure just described for the MINI element proposed in [5]. In this case the spaces V_1^h and Q^h consist of continuous piecewise linear functions with respect to a simplicial triangulation $\mathcal{T}^h$. Furthermore,

$$V_2^h = \left[\mathrm{span}\{\varphi_K\}_{K\in\mathcal{T}^h}\right]^d , \tag{3.139}$$

where φ_K are scalar element bubble functions defined on K as the product of the barycentric coordinates on K and vanishing outside of K. Thus, $\varphi_K|_K \in P_{d+1}(K)\cap H_0^1(K)$. The proof of the inf-sup stability relies on the construction of a Fortin operator, see [5] or [56, Section 3.6.1] for details. The component $\boldsymbol{u}_2^h$ of $\boldsymbol{u}^h$ can be expressed in the form

$$\boldsymbol{u}_2^h = \sum_{K\in\mathcal{T}^h} \boldsymbol{u}_K\,\varphi_K$$

with uniquely determined numbers $\boldsymbol{u}_K \in \mathbb{R}^d$. To elimininate $\boldsymbol{u}_2^h$ from (3.136)–(3.138), one can employ that

$$(\nabla\boldsymbol{u}_2^h,\nabla\boldsymbol{v}_1^h) = (\nabla\boldsymbol{u}_1^h,\nabla\boldsymbol{v}_2^h) = 0 \qquad \forall\ \boldsymbol{v}_1^h\in V_1^h,\ \boldsymbol{v}_2^h\in V_2^h\,. \tag{3.140}$$

Indeed, since the bubble functions vanish on ∂K and the Laplacian of a linear function vanishes, too, one finds by integration by parts

$$(\nabla\boldsymbol{u}_1^h,\nabla\boldsymbol{v}_2^h)_K = ((\boldsymbol{n}_{\partial K}\cdot\nabla)\boldsymbol{u}_1^h,\boldsymbol{v}_2^h)_{\partial K} - (\Delta\boldsymbol{u}_1^h,\boldsymbol{v}_2^h)_K = 0$$

for any $K\in\mathcal{T}^h$. Similarly, employing that the gradient of a linear function is constant, one gets

$$-(\nabla\cdot\boldsymbol{v}_2^h,p^h)_K = (\boldsymbol{v}_2^h,\nabla p^h)_K = \nabla p^h|_K\cdot\int_K \boldsymbol{v}_2^h\,\mathrm{d}\boldsymbol{x}\,. \tag{3.141}$$

Setting $\boldsymbol{v}_2^h = \boldsymbol{e}^i\,\varphi_K$, $i=1,\dots,d$, where $\boldsymbol{e}^i$ is the unit vector in the direction of the ith coordinate axis, and applying (3.140) and (3.141), one obtains from (3.137)

$$u_{K,i}\,\nu\,\|\nabla\varphi_K\|^2_{L^2(K)} + \partial_{x_i}p^h|_K\int_K\varphi_K\,\mathrm{d}\boldsymbol{x} = (f_i,\varphi_K) = \bar{f}_i^h|_K\int_K\varphi_K\,\mathrm{d}\boldsymbol{x}\,,$$

where $\bar{f}_i^h$ are components of the piecewice constant function $\bar{\boldsymbol{f}}^h$ defined by averaging of $\boldsymbol{f}$ with the weights φ_K, i.e.,

$$\bar{\boldsymbol{f}}^h|_K := \frac{\int_K \boldsymbol{f}\,\varphi_K\,\mathrm{d}\boldsymbol{x}}{\int_K\varphi_K\,\mathrm{d}\boldsymbol{x}}\,,\qquad K\in\mathcal{T}^h\,.$$

Thus, in view of (3.141), the second term in (3.138) becomes

$$
\begin{aligned}
-(\nabla\cdot\boldsymbol{u}_2^h,q^h) &= \sum_{K\in\mathcal{T}^h}\nabla q^h|_K\cdot\boldsymbol{u}_K\int_K\varphi_K\ \mathrm{d}\boldsymbol{x}\\
&= \sum_{K\in\mathcal{T}^h}\nabla q^h|_K\cdot(\bar{\boldsymbol{f}}^h-\nabla p^h)|_K\,\frac{\left(\int_K\varphi_K\ \mathrm{d}\boldsymbol{x}\right)^2}{\nu\,\|\nabla\varphi_K\|^2_{L^2(K)}}\\
&= \sum_{K\in\mathcal{T}^h}(\bar{\boldsymbol{f}}^h-\nabla p^h,\delta_K\nabla q^h)_K\,,
\end{aligned}
$$

where

$$
\delta_K=\frac{\left(\int_K\varphi_K\ \mathrm{d}\boldsymbol{x}\right)^2}{\nu\,\|\nabla\varphi_K\|^2_{L^2(K)}\,|K|}\,.
$$

Therefore, inserting (3.140) in (3.136), one ends up with the following problem for the linear part of the approximate solution: Find $\boldsymbol{u}_1^h\in V_1^h$ and $p^h\in Q^h$ such that

$$
\begin{aligned}
\nu(\nabla\boldsymbol{u}_1^h,\nabla\boldsymbol{v}_1^h)-(\nabla\cdot\boldsymbol{v}_1^h,p^h) &= (\boldsymbol{f},\boldsymbol{v}_1^h) && \forall\ \boldsymbol{v}_1^h\in V_1^h\,,\\
(\nabla\cdot\boldsymbol{u}_1^h,q^h)+\sum_{K\in\mathcal{T}^h}(\nabla p^h,\delta_K\nabla q^h)_K &= \sum_{K\in\mathcal{T}^h}(\bar{\boldsymbol{f}}^h,\delta_K\nabla q^h)_K && \forall\ q^h\in Q^h\,.
\end{aligned}
$$

It is known that $\int_K\varphi_K\ \mathrm{d}\boldsymbol{x}=\mathcal{O}\left(|K|\right)=\mathcal{O}(h_K^d)$ and $\|\nabla\varphi_K\|_{L^2(K)}=\mathcal{O}(h_K^{d/2-1})$, e.g., see [1, Lemma 3.2, Theorem 3.3], and hence δ_K satisfies (3.44). Thus, one finds that the MINI element leads for the linear part of the solution to the PSPG method for $V^h/Q^h=P_1/P_1$ (up to the averaging of the right-hand side), see (3.33).

To recover the PSPG method for other finite elements than the MINI element or to obtain other stabilized methods, it would be convenient to drop some of the terms from (3.136) and (3.137) representing a coupling between the spaces V_1^h and V_2^h. Such modifications of the discrete problem (3.136)–(3.138) were studied in [62] with the aim to reduce the size of the stiffness matrix which may be significantly increased by enriching the velocity space V_1^h by the space V_2^h. Surprisingly, it was shown that not all the terms in (3.136)–(3.138) are necessary for the solvability of the discrete problem and for optimal convergence properties of the approximate solutions. One can even proceed in a more general fashion and to multiply the terms $\nu(\nabla\boldsymbol{u}_2^h,\nabla\boldsymbol{v}_1^h)$, $\nu(\nabla\boldsymbol{u}_1^h,\nabla\boldsymbol{v}_2^h)$, and $\nu(\nabla\boldsymbol{u}_2^h,\nabla\boldsymbol{v}_2^h)$ by some real numbers α_1, α_2, and α_3, respectively. In other words, the bilinear form $\nu(\nabla\boldsymbol{u}^h,\nabla\boldsymbol{v}^h)$ in (3.13) is replaced by the bilinear form

$$
\begin{aligned}
a^h(\boldsymbol{u}^h,\boldsymbol{v}^h)=\nu(\nabla\boldsymbol{u}_1^h,\nabla\boldsymbol{v}_1^h)+\alpha_1\nu(\nabla\boldsymbol{u}_2^h,\nabla\boldsymbol{v}_1^h)&+\alpha_2\nu(\nabla\boldsymbol{u}_1^h,\nabla\boldsymbol{v}_2^h)\\
&+\alpha_3\nu(\nabla\boldsymbol{u}_2^h,\nabla\boldsymbol{v}_2^h)\,. \qquad (3.142)
\end{aligned}
$$

The multiplication by α_3 is considered since numerical experiments suggest that it can reduce the velocity error for small ν. In addition, the right-hand side of (3.13) will be replaced by a functional $\boldsymbol{f}^h \in H^{-1}(\Omega)^d$. In particular, $\boldsymbol{f}^h$ defined by

$$\langle \boldsymbol{f}^h, \boldsymbol{v}^h \rangle = (\boldsymbol{f}, \boldsymbol{v}_1^h) \qquad \forall\, \boldsymbol{v}^h \in V^h \tag{3.143}$$

represents replacing the right-hand side of (3.137) by zero. Note that the relation (3.143) defines a functional $\boldsymbol{f}^h \in [V^h]'$ which can be extended to $\boldsymbol{f}^h \in H^{-1}(\Omega)^d$ according to the Hahn–Banach theorem.

Thus, the following discretization of the Stokes problem will be considered in the following: Find $(\widetilde{\boldsymbol{u}}^h, \widetilde{p}^h) \in V^h \times Q^h$ such that

$$a^h(\widetilde{\boldsymbol{u}}^h, \boldsymbol{v}^h) - (\nabla\cdot \boldsymbol{v}^h, \widetilde{p}^h) + (\nabla\cdot \widetilde{\boldsymbol{u}}^h, q^h) = \langle \boldsymbol{f}^h, \boldsymbol{v}^h \rangle \quad \forall\, \left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h, \tag{3.144}$$

with a^h defined in (3.142).

To analyze the problem (3.144), we shall make additional assumptions on the finite element spaces and the triangulations $\mathcal{T}^h$. We assume that there exists a reference element $\widehat{K}$ such that, for each element $K \in \mathcal{T}^h$, one can introduce a regular one-to-one mapping $F_K : \widehat{K} \to K$ with $F_K(\widehat{K}) = K$. Moreover, it will be assumed that the triangulations $\mathcal{T}^h$ are shape regular in the sense that

$$\|\widehat{\nabla} F_K\|_{L^\infty(\widehat{K})} \le C\, h_K\,, \qquad \|\nabla F_K^{-1}\|_{L^\infty(K)} \le C\, h_K^{-1} \qquad\quad \forall\, K \in \mathcal{T}^h\,.$$

Thus, denoting for any element $K \in \mathcal{T}^h$ and any $v \in L^2(K)$

$$\widehat{v}_K = v \circ F_K\,,$$

one has, for any $K \in \mathcal{T}^h$,

$$C\, h_K^d\, \|\widehat{v}_K\|^2_{L^2(\widehat{K})} \le \|v\|^2_{L^2(K)} \le \widetilde{C}\, h_K^d\, \|\widehat{v}_K\|^2_{L^2(\widehat{K})} \quad \forall\, v \in L^2(K)\,, \tag{3.145}$$

$$C\, h_K^{d-2}\, \|\widehat{\nabla}\widehat{v}_K\|^2_{L^2(\widehat{K})} \le \|\nabla v\|^2_{L^2(K)} \le \widetilde{C}\, h_K^{d-2}\, \|\widehat{\nabla}\widehat{v}_K\|^2_{L^2(\widehat{K})} \quad \forall\, v \in H^1(K)\,. \tag{3.146}$$

It will be assumed that

$$V_1^h = \{\boldsymbol{v} \in H_0^1(\Omega)^d \;:\; \boldsymbol{v} \circ F_K \in \widehat{V}_1 \quad \forall\, K \in \mathcal{T}^h\}\,, \tag{3.147}$$

$$V_2^h \subset \{\boldsymbol{v} \in H_0^1(\Omega)^d \;:\; \boldsymbol{v} \circ F_K \in \widehat{V}_2 \quad \forall\, K \in \mathcal{T}^h\}\,, \tag{3.148}$$

where $\widehat{V}_1, \widehat{V}_2 \subset H^1(\widehat{K})^d$ are finite-dimensional spaces satisfying $\widehat{V}_1 \cap \widehat{V}_2 = \{\boldsymbol{0}\}$. The inclusion in (3.148) is considered to cover the case when the vector bubbles in V_2^h are defined using normal vectors to edges or faces of the triangulation. Then one can prove the following two important results.

Lemma 3.6.1. *The space* $V^h = V_1^h \oplus V_2^h$ *with* V_1^h *and* V_2^h *satisfying* (3.147) *and* (3.148), *respectively, satisfies*

$$\left\|\nabla \boldsymbol{v}_1^h\right\|_{L^2(\Omega)} + \left\|\nabla \boldsymbol{v}_2^h\right\|_{L^2(\Omega)} \le C \left\|\nabla \boldsymbol{v}^h\right\|_{L^2(\Omega)} \qquad \forall\, \boldsymbol{v}^h \in V^h\,. \tag{3.149}$$

Proof. Since, in a finite-dimensional space, any bounded sequence contains a convergent subsequence, it is easy to show by contradiction that

$$0 < \widehat{C}_1 := \inf_{\widehat{\boldsymbol{v}}_1 \in \widehat{V}_1,\, \|\widehat{\boldsymbol{v}}_1\|_{H^1(\widehat{K})}=1} \; \inf_{\widehat{\boldsymbol{v}}_2 \in \widehat{V}_2} \|\widehat{\boldsymbol{v}}_1 + \widehat{\boldsymbol{v}}_2\|_{H^1(\widehat{K})} .$$

This implies that $\widehat{C}_1 \|\widehat{\boldsymbol{v}}_1\|_{H^1(\widehat{K})} \le \|\widehat{\boldsymbol{v}}_1 + \widehat{\boldsymbol{v}}_2\|_{H^1(\widehat{K})}$ for any $\widehat{\boldsymbol{v}}_1 \in \widehat{V}_1$, $\widehat{\boldsymbol{v}}_2 \in \widehat{V}_2$. Thus, it follows from the equivalence of norms in finite-dimensional spaces that $\widehat{C}_1 \|\widehat{\nabla}\widehat{\boldsymbol{v}}_1\|_{L^2(\widehat{K})} \le \widehat{C}_2 \|\widehat{\nabla}(\widehat{\boldsymbol{v}}_1 + \widehat{\boldsymbol{v}}_2)\|_{L^2(\widehat{K})}$ for any $\widehat{\boldsymbol{v}}_1 \in \widehat{V}_1 \cap L^2_0(\widehat{K})^d$, $\widehat{\boldsymbol{v}}_2 \in \widehat{V}_2 \cap L^2_0(\widehat{K})^d$ and hence for any $\widehat{\boldsymbol{v}}_1 \in \widehat{V}_1$, $\widehat{\boldsymbol{v}}_2 \in \widehat{V}_2$. Applying (3.146) and summing over all elements of the triangulation, one gets $\|\nabla \boldsymbol{v}_1^h\|_{L^2(\Omega)} \le C \|\nabla(\boldsymbol{v}_1^h + \boldsymbol{v}_2^h)\|_{L^2(\Omega)}$ for any $\boldsymbol{v}_1^h \in V_1^h$, $\boldsymbol{v}_2^h \in V_2^h$ and the lemma follows. □

Lemma 3.6.2. *Let $\widehat{V}_2 \cap P_0(\widehat{K})^d = \{\mathbf{0}\}$. Then the space V_2^h satisfying* (3.148) *satisfies*

$$\left\|\boldsymbol{v}_2^h\right\|_{L^2(\Omega)} \le C\, h \left\|\nabla \boldsymbol{v}_2^h\right\|_{L^2(\Omega)} \qquad \forall\; \boldsymbol{v}_2^h \in V_2^h . \tag{3.150}$$

Proof. It follows from the equivalence of norms in finite-dimensional spaces that $\|\widehat{\boldsymbol{v}}_2\|_{L^2(\widehat{K})} \le C \|\widehat{\nabla}\widehat{\boldsymbol{v}}_2\|_{L^2(\widehat{K})}$ for any $\widehat{\boldsymbol{v}}_2 \in \widehat{V}_2$. Then (3.150) follows using (3.145) and (3.146). □

Remark 3.6.3. The assumption $\widehat{V}_2 \cap P_0(\widehat{K})^d = \{\mathbf{0}\}$ is satisfied for all common bubble spaces V_2^h. Thus, in particular, (3.149) and (3.150) hold for all the examples of spaces V_1^h and V_2^h presented at the beginning of this section. △

The following lemma shows that, for a finite element discretization of *any* problem, the V_2^h component of the approximate solution can be dropped without influencing the asymptotic convergence properties of the approximate solution.

Lemma 3.6.4 (Estimates for the components of $\boldsymbol{v}^h \in V^h$)**.** *Consider any $\boldsymbol{v} \in V \cap H^{k+1}(\Omega)^d$ and $\boldsymbol{v}^h \in V^h$. Then one has*

$$\begin{aligned}\left\|\nabla(\boldsymbol{v} - \boldsymbol{v}_1^h)\right\|_{L^2(\Omega)} + \left\|\nabla \boldsymbol{v}_2^h\right\|_{L^2(\Omega)} &\le C \left\|\nabla(\boldsymbol{v} - \boldsymbol{v}^h)\right\|_{L^2(\Omega)} \\ &\quad + C\, h^k \left\|\boldsymbol{v}\right\|_{H^{k+1}(\Omega)} ,\end{aligned} \tag{3.151}$$

$$\begin{aligned}\left\|\boldsymbol{v} - \boldsymbol{v}_1^h\right\|_{L^2(\Omega)} + \left\|\boldsymbol{v}_2^h\right\|_{L^2(\Omega)} &\le \left\|\boldsymbol{v} - \boldsymbol{v}^h\right\|_{L^2(\Omega)} \\ &\quad + C \left\{h \left\|\nabla(\boldsymbol{v} - \boldsymbol{v}^h)\right\|_{L^2(\Omega)} + h^{k+1} \left\|\boldsymbol{v}\right\|_{H^{k+1}(\Omega)}\right\}.\end{aligned} \tag{3.152}$$

Proof. Due to (3.150) and (3.149), one has for $m = 0, 1$

$$\left|\boldsymbol{v}_2^h\right|_{H^m(\Omega)} = \left|(\boldsymbol{v}^h - I^h \boldsymbol{v})_2\right|_{H^m(\Omega)} \le C\, h^{1-m} \left\|\nabla(\boldsymbol{v}^h - I^h \boldsymbol{v})\right\|_{L^2(\Omega)}$$

and hence it follows using the triangle inequality that

$$\begin{aligned}\left|\boldsymbol{v} - \boldsymbol{v}_1^h\right|_{H^m(\Omega)} + \left|\boldsymbol{v}_2^h\right|_{H^m(\Omega)} &\le \left|\boldsymbol{v} - \boldsymbol{v}^h\right|_{H^m(\Omega)} \\ &\quad + C\, h^{1-m} \left\{\left\|\nabla(\boldsymbol{v} - \boldsymbol{v}^h)\right\|_{L^2(\Omega)} + \left\|\nabla(\boldsymbol{v} - I^h \boldsymbol{v})\right\|_{L^2(\Omega)}\right\}.\end{aligned}$$

Now (3.151) and (3.152) follow using (3.8). □

Now let us investigate the properties of the discrete problem (3.144).

Theorem 3.6.5 (Existence and uniqueness of a solution of (3.144))**.** *Let the constants α_1, α_2, α_3 used in the definition of a^h satisfy $\alpha_3 > 0$ and $|\alpha_1+\alpha_2| \le 2\sqrt{\alpha_3}$ and let the spaces V^h and Q^h satisfy the discrete inf-sup condition* (3.3)*. Then, for any $\boldsymbol{f}^h \in H^{-1}(\Omega)^d$, the problem* (3.144) *has a unique solution.*

Proof. Denoting $\alpha = (\alpha_1+\alpha_2)/2$, one has for any $\boldsymbol{v}^h \in V^h$

$$\begin{aligned} a^h(\boldsymbol{v}^h,\boldsymbol{v}^h) &= \nu\,(\nabla(\boldsymbol{v}_1^h+\alpha\,\boldsymbol{v}_2^h),\nabla(\boldsymbol{v}_1^h+\alpha\,\boldsymbol{v}_2^h)) + \nu\,(\alpha_3-\alpha^2)\,(\nabla\boldsymbol{v}_2^h,\nabla\boldsymbol{v}_2^h)\\ &= \nu\left\|\nabla(\boldsymbol{v}_1^h+\alpha\,\boldsymbol{v}_2^h)\right\|_{L^2(\Omega)}^2 + \nu\,(\alpha_3-\alpha^2)\left\|\nabla\boldsymbol{v}_2^h\right\|_{L^2(\Omega)}^2 \end{aligned}$$

and hence it follows from (3.149) and the triangle inequality that, for some $C > 0$,

$$C\,\nu\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)}^2 \le a^h(\boldsymbol{v}^h,\boldsymbol{v}^h) \qquad \forall\ \boldsymbol{v}^h \in V^h\,. \tag{3.153}$$

This and the discrete inf-sup condition (3.3) imply that the problem (3.144) has only the trivial solution if $\boldsymbol{f}^h = \boldsymbol{0}$. Consequently, since the problem (3.144) is equivalent to a linear algebraic system with a square matrix, it has a unique solution for any $\boldsymbol{f}^h \in H^{-1}(\Omega)^d$. □

Theorem 3.6.6 (Error estimate)**.** *Let the assumptions of Theorem 3.6.5 be satisfied and let β_{is}^h from* (3.3) *be bounded from below by $\beta_0 > 0$ independent of h. Assume that the solution of* (3.5) *satisfies $(\boldsymbol{u},p) \in H^{k+1}(\Omega)^d \times H^{l+1}(\Omega)$, then one has the error estimate*

$$\begin{aligned} &\nu\,\|\nabla(\boldsymbol{u}-\widetilde{\boldsymbol{u}}^h)\|_{L^2(\Omega)} + \|p-\widetilde{p}^h\|_{L^2(\Omega)} \le C\,\|\boldsymbol{f}-\boldsymbol{f}^h\|_{[V^h]'}\\ &\quad + C\left(\nu h^k\,\|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + h^{l+1}\,\|p\|_{H^{l+1}(\Omega)}\right) + \nu\,|1-\alpha_2|\,h\,\|\boldsymbol{u}\|_{H^2(\Omega)}\,. \end{aligned} \tag{3.154}$$

Proof. Subtracting (3.13) from (3.144), one obtains for $q^h = 0$ and any $\boldsymbol{v}^h \in V^h$

$$\begin{aligned} a^h(\widetilde{\boldsymbol{u}}^h-\boldsymbol{u}^h,\boldsymbol{v}^h) - (\nabla\cdot\boldsymbol{v}^h,\widetilde{p}^h-p^h) &= \langle\boldsymbol{f}^h-\boldsymbol{f},\boldsymbol{v}^h\rangle + \nu\,(1-\alpha_1)\,(\nabla\boldsymbol{u}_2^h,\nabla\boldsymbol{v}_1^h)\\ &\quad + \nu\,(1-\alpha_2)\,(\nabla\boldsymbol{u}_1^h,\nabla\boldsymbol{v}_2^h) + \nu\,(1-\alpha_3)\,(\nabla\boldsymbol{u}_2^h,\nabla\boldsymbol{v}_2^h)\,. \end{aligned} \tag{3.155}$$

One infers applying (3.150) that, for any $\boldsymbol{v}_2^h \in V_2^h$,

$$(\nabla\boldsymbol{u},\nabla\boldsymbol{v}_2^h) \le \|\Delta\boldsymbol{u}\|_{L^2(\Omega)}\left\|\boldsymbol{v}_2^h\right\|_{L^2(\Omega)} \le Ch\,\|\boldsymbol{u}\|_{H^2(\Omega)}\left\|\nabla\boldsymbol{v}_2^h\right\|_{L^2(\Omega)}\,. \tag{3.156}$$

Writing $\boldsymbol{u}_1^h = (\boldsymbol{u}_1^h-\boldsymbol{u})+\boldsymbol{u}$, one gets

$$(\nabla\boldsymbol{u}_1^h,\nabla\boldsymbol{v}_2^h) \le C\,\{\left\|\nabla(\boldsymbol{u}_1^h-\boldsymbol{u})\right\|_{L^2(\Omega)} + h\,\|\boldsymbol{u}\|_{H^2(\Omega)}\}\left\|\nabla\boldsymbol{v}_2^h\right\|_{L^2(\Omega)}\,.$$

Now, denoting

$$A_h = \|\boldsymbol{f}-\boldsymbol{f}^h\|_{[V^h]'} + \nu\left\|\nabla(\boldsymbol{u}-\boldsymbol{u}_1^h)\right\|_{L^2(\Omega)} + \nu\left\|\nabla\boldsymbol{u}_2^h\right\|_{L^2(\Omega)} + \nu\,|1-\alpha_2|\,h\,\|\boldsymbol{u}\|_{H^2(\Omega)}\,,$$

one derives from (3.155) applying (3.149) that

$$a^h(\widetilde{\boldsymbol{u}}^h - \boldsymbol{u}^h, \boldsymbol{v}^h) - (\nabla \cdot \boldsymbol{v}^h, \widetilde{p}^h - p^h) \le C A_h \left\| \nabla \boldsymbol{v}^h \right\|_{L^2(\Omega)} \quad \forall\, \boldsymbol{v}^h \in V^h . \quad (3.157)$$

Using (3.151) and (3.20), one obtains

$$A_h \le \|\boldsymbol{f} - \boldsymbol{f}^h\|_{[V^h]'} + C \left(\nu h^k \|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + h^{l+1} \|p\|_{H^{l+1}(\Omega)} \right) + \nu \, |1 - \alpha_2| \, h \, \|\boldsymbol{u}\|_{H^2(\Omega)} . \quad (3.158)$$

Setting $\boldsymbol{v}^h = \widetilde{\boldsymbol{u}}^h - \boldsymbol{u}^h$ in (3.157) and using the fact that $\boldsymbol{v}^h$ is discretely divergence-free, one gets from (3.153)

$$\nu \, \|\nabla(\widetilde{\boldsymbol{u}}^h - \boldsymbol{u}^h)\|_{L^2(\Omega)} \le C A_h . \quad (3.159)$$

Using the Cauchy–Schwarz inequality and (3.149) gives

$$a^h(\boldsymbol{w}^h, \boldsymbol{v}^h) \le C \, \nu \, \|\nabla \boldsymbol{w}^h\|_{L^2(\Omega)} \|\nabla \boldsymbol{v}^h\|_{L^2(\Omega)} \qquad \forall\, \boldsymbol{w}^h, \boldsymbol{v}^h \in V^h,$$

which together with (3.157) and (3.159) implies that

$$(\nabla \cdot \boldsymbol{v}^h, \widetilde{p}^h - p^h) \le C A_h \, \|\nabla \boldsymbol{v}^h\|_{L^2(\Omega)} \qquad \forall\, \boldsymbol{v}^h \in V^h.$$

Thus, applying (3.3), one gets

$$\|\widetilde{p}^h - p^h\|_{L^2(\Omega)} \le C A_h . \quad (3.160)$$

Now, using the triangle inequality, (3.20), and (3.158)–(3.160), one obtains (3.154). □

Theorem 3.6.7 (L^2 estimate of the velocity error)**.** *Let the assumptions of Theorem 3.6.5 be satisfied and let β_{is}^h from* (3.3) *be bounded from below by $\beta_0 > 0$ independent of h. Let the solution of* (3.5) *satisfy $(\boldsymbol{u}, p) \in H^{k+1}(\Omega)^d \times H^{l+1}(\Omega)$ and let the Stokes problem* (3.2) *be regular. Then there holds the error estimate*

$$\|\boldsymbol{u} - \widetilde{\boldsymbol{u}}^h\|_{L^2(\Omega)} \le \frac{C}{\nu} \|\boldsymbol{f} - \boldsymbol{f}^h\|_{[V_1^h]'} + \frac{Ch}{\nu} \|\boldsymbol{f} - \boldsymbol{f}^h\|_{[V^h]'} + C \left(h^{k+1} \|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + \frac{h^{l+2}}{\nu} \|p\|_{H^{l+1}(\Omega)} \right) + |1 - \alpha_2| \, h^2 \, \|\boldsymbol{u}\|_{H^2(\Omega)} . \quad (3.161)$$

Proof. Let $(\boldsymbol{z}, r) \in V \times Q$ be the solution of the problem (3.22) with $\boldsymbol{u} - \boldsymbol{u}^h$ replaced by $\widetilde{\boldsymbol{u}}^h - \boldsymbol{u}^h$. Then all the relations (3.23)–(3.28) also hold with $\boldsymbol{u} - \boldsymbol{u}^h$ replaced by $\widetilde{\boldsymbol{u}}^h - \boldsymbol{u}^h$. Thus, using the fact that $\widetilde{\boldsymbol{u}}^h - \boldsymbol{u}^h$ is discretely divergence-free, one obtains

$$\begin{aligned} \nu \|\widetilde{\boldsymbol{u}}^h - \boldsymbol{u}^h\|_{L^2(\Omega)}^2 \quad \le \quad & C \nu h \, \|\nabla(\widetilde{\boldsymbol{u}}^h - \boldsymbol{u}^h)\|_{L^2(\Omega)} \|\widetilde{\boldsymbol{u}}^h - \boldsymbol{u}^h\|_{L^2(\Omega)} \\ & + \nu (\nabla \boldsymbol{z}^I, \nabla(\widetilde{\boldsymbol{u}}^h - \boldsymbol{u}^h)), \end{aligned} \quad (3.162)$$

where $\boldsymbol{z}^I$ satisfies

$$\left\|\nabla(\boldsymbol{z}-\boldsymbol{z}^I)\right\|_{L^2(\Omega)} \leq Ch\,\|\boldsymbol{z}\|_{H^2(\Omega)} \leq Ch\,\|\widetilde{\boldsymbol{u}}^h-\boldsymbol{u}^h\|_{L^2(\Omega)}. \tag{3.163}$$

Since $\boldsymbol{z}^I \in V_1^h$, one has

$$\nu(\nabla(\widetilde{\boldsymbol{u}}^h-\boldsymbol{u}^h),\nabla\boldsymbol{z}^I) = a^h(\widetilde{\boldsymbol{u}}^h-\boldsymbol{u}^h,\boldsymbol{z}^I) + (1-\alpha_1)\nu(\nabla(\widetilde{\boldsymbol{u}}_2^h-\boldsymbol{u}_2^h),\nabla\boldsymbol{z}^I)$$

and hence it follows from (3.155) that

$$\begin{aligned}\nu(\nabla(\widetilde{\boldsymbol{u}}^h-\boldsymbol{u}^h),\nabla\boldsymbol{z}^I) =& \langle\boldsymbol{f}^h-\boldsymbol{f},\boldsymbol{z}^I\rangle + (\nabla\cdot\boldsymbol{z}^I,\widetilde{p}^h-p^h) + (1-\alpha_1)\nu(\nabla\widetilde{\boldsymbol{u}}_2^h,\nabla\boldsymbol{z}^I)\\ =& \langle\boldsymbol{f}^h-\boldsymbol{f},\boldsymbol{z}^I\rangle - (\nabla\cdot(\boldsymbol{z}-\boldsymbol{z}^I),\widetilde{p}^h-p^h)\\ &- (1-\alpha_1)\nu(\nabla\widetilde{\boldsymbol{u}}_2^h,\nabla(\boldsymbol{z}-\boldsymbol{z}^I)) + (1-\alpha_1)\nu(\nabla\widetilde{\boldsymbol{u}}_2^h,\nabla\boldsymbol{z}).\end{aligned}$$

Applying (3.163) and (3.156) with $\boldsymbol{u}$ replaced by $\boldsymbol{z}$, one gets

$$\begin{aligned}\nu(\nabla(\widetilde{\boldsymbol{u}}^h-\boldsymbol{u}^h),\nabla\boldsymbol{z}^I) \leq& C\,\|\boldsymbol{f}-\boldsymbol{f}^h\|_{[V_1^h]'}\|\widetilde{\boldsymbol{u}}^h-\boldsymbol{u}^h\|_{L^2(\Omega)}\\ &+ Ch\left(\nu\|\nabla\widetilde{\boldsymbol{u}}_2^h\|_{L^2(\Omega)} + \|\widetilde{p}^h-p^h\|_{L^2(\Omega)}\right)\|\widetilde{\boldsymbol{u}}^h-\boldsymbol{u}^h\|_{L^2(\Omega)}.\end{aligned}$$

Substituting this estimate into (3.162) and using the triangle inequality and (3.149), one obtains

$$\begin{aligned}\nu\|\widetilde{\boldsymbol{u}}^h-\boldsymbol{u}^h\|_{L^2(\Omega)} \leq& C\,\|\boldsymbol{f}-\boldsymbol{f}^h\|_{[V_1^h]'}\\ &+ Ch\left(\nu\|\nabla\boldsymbol{u}_2^h\|_{L^2(\Omega)} + \nu\|\nabla(\widetilde{\boldsymbol{u}}^h-\boldsymbol{u}^h)\|_{L^2(\Omega)} + \|\widetilde{p}^h-p^h\|_{L^2(\Omega)}\right).\end{aligned}$$

Then, (3.161) follows as a consequence of the triangle inequality, (3.151), (3.154), (3.20), and (3.21). □

Remark 3.6.8. If $\boldsymbol{f}^h$ is defined by (3.143), then, for any $\boldsymbol{v}^h \in V^h$, one has $\langle\boldsymbol{f}-\boldsymbol{f}^h,\boldsymbol{v}^h\rangle = (\boldsymbol{f},\boldsymbol{v}_2^h) \leq \|\boldsymbol{f}\|_{L^2(\Omega)}\|\boldsymbol{v}_2^h\|_{L^2(\Omega)}$ and, using (3.150) and (3.149), one deduces that $\boldsymbol{f}^h$ satisfies $\|\boldsymbol{f}-\boldsymbol{f}^h\|_{[V^h]'} \leq Ch\,\|\boldsymbol{f}\|_{L^2(\Omega)}$. Moreover, one has $\|\boldsymbol{f}-\boldsymbol{f}^h\|_{[V_1^h]'} = 0$. Thus, if $k=1$, the problem (3.144) leads to optimal error estimates with respect to h for any constants α_1, α_2, α_3 satisfying $\alpha_3 > 0$ and $|\alpha_1+\alpha_2| \leq 2\sqrt{\alpha_3}$. If $k>1$, optimal error estimates are obtained for $\boldsymbol{f}^h = \boldsymbol{f}$ and $\alpha_2 = 1$. △

Now let us discuss the relation of the modified discretization (3.144) to stabilized methods. For simplicity, we confine ourselves to the two-dimensional case. It is convenient to write the problem (3.144) in the equivalent form

$$\nu(\nabla\widetilde{\boldsymbol{u}}_1^h,\nabla\boldsymbol{v}_1^h) + \alpha_1\nu(\nabla\widetilde{\boldsymbol{u}}_2^h,\nabla\boldsymbol{v}_1^h) - (\nabla\cdot\boldsymbol{v}_1^h,\widetilde{p}^h) = \langle\boldsymbol{f}^h,\boldsymbol{v}_1^h\rangle \quad \forall\,\boldsymbol{v}_1^h \in V_1^h, \tag{3.164}$$

$$\alpha_2\nu(\nabla\widetilde{\boldsymbol{u}}_1^h,\nabla\boldsymbol{v}_2^h) + \alpha_3\nu(\nabla\widetilde{\boldsymbol{u}}_2^h,\nabla\boldsymbol{v}_2^h) - (\nabla\cdot\boldsymbol{v}_2^h,\widetilde{p}^h) = \langle\boldsymbol{f}^h,\boldsymbol{v}_2^h\rangle \quad \forall\,\boldsymbol{v}_2^h \in V_2^h, \tag{3.165}$$

$$-(\nabla\cdot\widetilde{\boldsymbol{u}}_1^h,q^h) - (\nabla\cdot\widetilde{\boldsymbol{u}}_2^h,q^h) = 0 \quad \forall\,q^h \in Q^h. \tag{3.166}$$

First, let us consider the case $\alpha_1 = \alpha_2 = 0$, $\alpha_3 = 1$, and $\boldsymbol{f}^h$ defined by (3.143). Let V_1^h consist of continuous piecewise (bi)linear functions. If Q^h consists of piecewise constant functions, then one can set $V_2^h = \operatorname{span}\{\varphi_E\, \boldsymbol{n}_E\}_{E\in\mathcal{E}^h}$, where $\varphi_E \in H_0^1(\Omega)$ are scalar finite element functions assigned to interior edges E of the triangulation $\mathcal{T}^h$ which have their supports in the two elements adjacent to E and satisfy $\int_E \varphi_E \,\mathrm{d}\boldsymbol{s} \neq 0$, see, e.g., [17, 41] for particular examples of φ_E. The vectors $\boldsymbol{n}_E$ are again fixed normal vectors to the edges E. Defining φ_E in such a way that the interiors of the supports of any two functions φ_E, $\varphi_{E'}$ with $E \neq E'$ are disjoint, one can compute $\boldsymbol{u}_2^h$ from (3.165) and substitute it in (3.166), which gives

$$(\nabla\cdot\widetilde{\boldsymbol{u}}_1^h, q^h) + \sum_{E\in\mathcal{E}^h} \gamma_E\left([|\widetilde{p}^h|]_E\,, [|q^h|]_E\right)_E = 0 \qquad \forall\, q^h \in Q^h\,,$$

where

$$\gamma_E = \frac{|\int_E \varphi_E \,\mathrm{d}\boldsymbol{s}|^2}{\nu\,\|\nabla\varphi_E\|^2_{L^2(\Omega)}\, h_E}\,.$$

The usual scaling argument shows that γ_E satisfies (3.45). Thus, for the considered spaces V^h and Q^h, the modified discretization is equivalent to the PSPG method (3.33) for the spaces V_1^h and Q^h.

If V_1^h is as above and Q^h consists of continuous piecewise (bi)linear functions, one can consider a general space $V_2^h = \operatorname{span}\{\varphi_i^h\, \boldsymbol{t}_i^h\}_{i=1}^{N^h}$ where $\boldsymbol{t}_i^h$ are unit vectors and $\varphi_i^h \in H_0^1(\Omega)$ are finite element functions having their supports in one element or in two elements possessing a common edge. To distinguish which element or elements a function φ_i^h belongs to, points A_i^h different from the vertices of $\mathcal{T}^h$ are introduced. If A_i^h lies in the interior of some element $K \in \mathcal{T}^h$, one requires that $\operatorname{supp}\varphi_i^h \subset K$ and if A_i^h lies on an edge E, one requires that $\operatorname{supp}\varphi_i^h$ lies in the two elements adjacent to E and that $\boldsymbol{t}_i^h$ is parallel to E. For triangular meshes, one assumes that there exist two points $A_i^h \in K$ for each element K. In the quadrilateral case, three points $A_i^h \in K$ are supposed for any K. In other words, one needs two functions φ_i^h per element in the triangular case and three functions φ_i^h per element in the quadrilateral case. In both cases, each function may be common to two elements. Under further assumptions, see [63] for details, which are satisfied for the spaces considered here, it can be shown that this space V_2^h assures the validity of the inf-sup condition (3.3). For example, in the triangular case, the space V_2^h defined in (3.139) and leading to the MINI element can be put into the above general framework. Then, for each element K, one has two bubble functions φ_i^h which coincide and are equal to φ_K. If $\mathcal{T}^h$ consists of quadrilaterals, the stability is assured by four bubble functions on each element, see [74].

In particular, as a special case of the general framework from the previous paragraph, one can use spaces of the type $V_2^h = \operatorname{span}\{\varphi_E\, \boldsymbol{t}_E\}_{E\in\overline{\mathcal{E}}^h}$, where $\boldsymbol{t}_E$ are tangential vectors to the edges E and $\overline{\mathcal{E}}^h$ denotes the set of all edges of the triangulation $\mathcal{T}^h$. The functions φ_E are constructed in such a way that the interiors of their supports are mutually disjoint and they vanish on $\partial\Omega$ also if $E \subset \partial\Omega$, see

[63]. Using again the modified discretization (3.164)–(3.166) with $\alpha_1 = \alpha_2 = 0$, $\alpha_3 = 1$, and $\boldsymbol{f}^h$ defined by (3.143), one infers analogously as above that the piecewise (bi)linear part of the solution to (3.164)–(3.166) satisfies

$$(\nabla \cdot \widetilde{\boldsymbol{u}}_1^h, q^h) + \sum_{E \in \overline{\mathcal{E}}^h} \gamma_E \left(\frac{\partial \widetilde{p}^h}{\partial \boldsymbol{t}_E}, \frac{\partial q^h}{\partial \boldsymbol{t}_E} \right)_E = 0 \qquad \forall\, q^h \in Q^h$$

with

$$\gamma_E = \frac{|\int_\Omega \varphi_E \,\mathrm{d}\boldsymbol{x}|^2}{\nu \, \|\nabla \varphi_E\|^2_{L^2(\Omega)} \, h_E} ,$$

which is a different type of stabilization than those discussed in the preceding sections.

The spaces V_1^h, Q^h consisting of continuous piecewise (bi)linear functions can be also used with a space V_2^h of the type $V_2^h = \mathrm{span}\{\varphi_i^h \, \boldsymbol{t}_i^h\}_{i=1}^{N^h}$. In the triangular case, let V_2^h be the space of the MINI element defined by (3.139) and in the quadrilateral case, it will be assumed that the elements of $\mathcal{T}^h$ are rectangles and, for any element $K \in \mathcal{T}^h$, four functions φ_i^h with disjoint supports in K will be used. The corresponding vectors $\boldsymbol{t}_i^h$ are parallel to the edges of K, see [63] for details. Using the same modified discretization as above and eliminating $\widetilde{\boldsymbol{u}}_2^h$, one obtains

$$(\nabla \cdot \widetilde{\boldsymbol{u}}_1^h, q^h) + \sum_{K \in \mathcal{T}^h} \left(\nabla \widetilde{p}^h, \delta_K \nabla q^h \right)_K = 0 \qquad \forall\, q^h \in Q^h ,$$

where δ_K satisfies (3.44), i.e., one recovers the Brezzi–Pitkäranta method (3.106) for the spaces V_1^h, Q^h. If $\boldsymbol{f}^h = \boldsymbol{f}$ and $\boldsymbol{f}$ is piecewise (bi)linear, one obtains the stabilization

$$(\nabla \cdot \widetilde{\boldsymbol{u}}_1^h, q^h) + \sum_{K \in \mathcal{T}^h} \left(\nabla \widetilde{p}^h - \boldsymbol{f}, \delta_K \nabla q^h \right)_K = 0 \qquad \forall\, q^h \in Q^h ,$$

which corresponds to the PSPG method (3.33) for the spaces V_1^h, Q^h.

Finally, let the spaces V_1^h, Q^h consist of continuous piecewise quadratic functions on triangles. These spaces do not satisfy the inf-sup condition (3.3). Dividing any element $K \in \mathcal{T}^h$ into four equal triangles by connecting midpoints of edges and introducing two vector bubble functions from the MINI element on each subtriangle having a common vertex with K, one obtains a space V_2^h assuring the stability. If one eliminates this space V_2^h from the original conforming discretization (3.136)–(3.138) and assumes that $\boldsymbol{f}$ is piecewise linear, one obtains the symmetric GLS method (3.65) for the spaces V_1^h, Q^h. However, one can also use the modified discretization (3.164)–(3.166) with $\alpha_1 = -1$, $\alpha_2 = \alpha_3 = 1$ and $\boldsymbol{f}^h = \boldsymbol{f}$ which guarantees the same asymptotic convergence rates of the discrete solution as (3.136)–(3.138). Then, eliminating the space V_2^h, one obtains the non-symmetric GLS method (3.91) for the spaces V_1^h, Q^h.

3.7 Numerical Studies

In this section, numerical studies with some of the stabilized methods will be presented: the PSPG method (3.33) – (3.35), the symmetric GLS method (3.65) – (3.67), the non-symmetric GLS method (3.91) – (3.93), and the LPS method that utilizes a modified Scott–Zhang projector, see Remark 3.5.8. In addition, the Brezzi–Pitkäranta method (3.106) with P_1/P_1 finite elements was incorporated in our studies. For the sake of brevity, the results with this method are not presented here since, in our experience, they were not better than, e.g., the results obtained with the PSPG method.

Two examples were studied:

- an example for the Stokes equations (3.2) with prescribed solution, which studies standard errors, their order of convergence, and the dependency on the viscosity coefficient and on the stabilization parameter,
- an example for the stationary Navier–Stokes equations, a flow around a cylinder, which investigates the accuracy of computing quantities at the cylinder that are of physical relevance, the dependency of the results on the discretization of the nonlinear term, and which provides a comparison to results obtained with inf-sup stable pairs of finite elements.

All simulations were performed with the code ParMooN, [44, 84]. Linear systems of equations were solved with the sparse direct solver UMFPACK [37].

Remark 3.7.1 (Comparative numerical studies in the literature). There are few numerical studies that compare several stabilized methods already available in the literature.

- Numerical studies at simple Stokes problems in [77] compare the PLS method, see Section 3.5.2, the symmetric GLS method from Section 3.4.3, and the method with orthogonal subscales from Section 3.5.3.
- A brief numerical comparison of the method based on two local Gauss integrations from [67], which is equivalent to the method from [38], and the symmetric GLS method from Section 3.4.3 for P_1/P_1 finite elements (in this situation the latter method is equivalent to the PSPG method from Section 3.4.2 and the non-symmetric GLS method from Section 3.4.4) can be found in [68]. For a driven cavity problem, it was observed that the pressure approximation close to the boundary is more accurate for the first method.
- A brief comparison of the LPS method mentioned in Remark 3.5.8 and the PSPG method for P_1/P_1 finite elements can be found in [7].

$\triangle$

3.7.1 Stokes Problem with Prescribed Solution

This example studies some of the stabilized discretizations in the framework of the numerical analysis: the Stokes equations (3.2) possess a smooth solution with homogeneous Dirichlet boundary data. In addition, the solution does not depend on the viscosity coefficient ν. Errors in standard norms were monitored. The dependency of the errors and the order of convergence on the viscosity coefficient ν and on the stabilization parameter were investigated.

Consider the domain $\Omega = (0,1)^2$ together with a polynomial solution

$$\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \partial_y \phi \\ -\partial_x \phi \end{pmatrix},$$

where ϕ is the stream function given by

$$\phi(x,y) = 1000\, x^2(1-x)^4 y^3 (1-y)^2.$$

Due to this construction, the solution is divergence-free; furthermore, it has homogeneous Dirichlet boundary values on $\partial\Omega$. The corresponding pressure p therefore should have zero mean value, $p \in L_0^2(\Omega)$. For this example, it is set to be

$$p = \pi^2 \left(xy^3 \cos(2\pi x^2 y) - x^2 y \sin(2\pi xy) \right) + \frac{1}{8}.$$

The right-hand side $\boldsymbol{f}$ in (3.2) is set accordingly. Figure 3.2 shows visualizations of the prescribed solution.

The stabilized methods involved in our studies are already mentioned at the beginning of this section. For all of them, P_k/P_k pairs of finite element spaces were considered with $k \in \{1,2,3\}$. Note that for the P_1/P_1 pair of finite element spaces, the PSPG method, the symmetric GLS method, and the non-symmetric GLS method coincide. The stabilization parameters of all methods have the form $\delta_K = \delta_0 h_K^2/\nu$ and the numerical studies considered for most methods $\delta_0 = 10^i$, $i \in \{-3, -2.5, \ldots, 0\}$. Only for the symmetric GLS method, we found that these stabilization parameters were too large, since an irregular behavior of the monitored errors could be observed, compare Figure 3.4 below. For this method, results obtained with $\delta_0 = 10^i$, $i \in \{-5, -4.5, \ldots, -2\}$, will be presented. Simulations for $\nu = 10^j$, $j \in \{-6, -5, \ldots, 0\}$, were performed.

In our computational studies, unstructured grids of varying fineness have been employed, see Figure 3.2 for an example and Table 3.1 for more details. The grids were generated with GMSH [45]. The convergence order has been computed via the formula $\log(e^H/e^h)/\log(H/h)$, where H and h are the characteristic grid lengths[2] while e^H and e^h are the respective errors on these grids.

In Figures 3.3–3.6, errors as well as convergence orders are shown for the studied methods. The errors are those obtained on the finest grid level 4 and the

[2] In the case of uniform refinement, it is $H = 2h$.

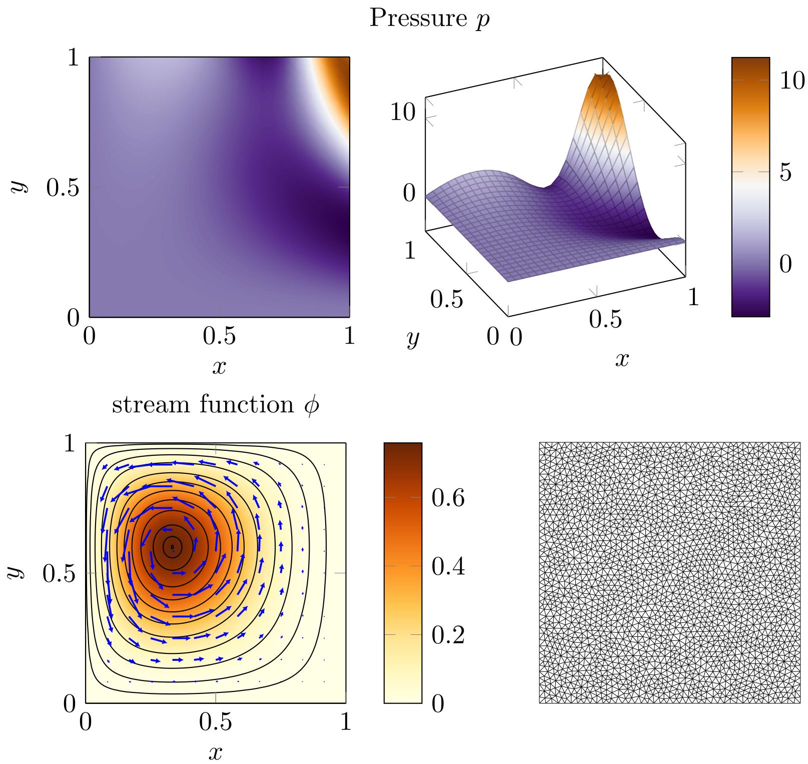

Figure 3.2: Visualizations of the solution of the Stokes example. Top: pressure p. Bottom left: stream function ϕ together with streamlines and arrows of the velocity $\boldsymbol{u}$. Bottom right: For illustration, a generated grid which is coarser than the ones actually used.

Table 3.1: Details on the generated grids used for the Stokes example.

grid level	number of cells	number of degrees of freedom P_1/P_1	P_2/P_2	P_3/P_3
0	38728	58821	233823	525009
1	52464	79545	316479	710805
2	68628	103911	413703	929379
3	86398	130686	520563	1169634
4	106838	161466	643443	1445934

order of convergence was computed with the errors on the two finest levels. The PSPG method, the non-symmetric GLS method, and the LPS method only show a weak dependency on δ_0. This behavior can be expected from the analysis of the non-symmetric GLS method since it is absolutely stable. Furthermore, the errors for the velocity are larger for smaller ν while the errors for the pressure are larger for larger ν. This behavior reflects also the analytical results. These three methods also show a similar behavior in the estimated convergence orders. The orders of error reduction are often higher than expected for small ν, compare also [56, Fig. 4.14]. This effect was observed also for inf-sup stable discretizations, e.g., see [56, Fig. 4.9]. To the best of our knowledge, an explanation for this phenomenon is not known so far. The symmetric GLS method shows a more irregular behavior with respect to the dependency on δ_0, see Figure 3.4 for larger values of δ_0 and higher order finite elements. For small values of δ_0, one can observe the same behavior as it is described above for the other methods.

Figure 3.7 presents a comparison of the methods among each other. For performing this comparison, for each value of the viscosity ν, the most accurate result with respect to the $L^2(\Omega)$ error of the velocity on the finest level was selected for each method, among all values of the stabilization parameter. It can be seen in Figure 3.7 that in many cases, in particular for P_1/P_1 and P_3/P_3 finite elements, the curves are almost on top of each other, i.e., all methods gave very similar results. Only for the P_2/P_2 finite element and small viscosities, the non-symmetric GLS method led to slightly higher velocity errors and the LPS method to notably higher pressure errors than the other methods.

We like to note that we obtained similar results as presented in Figures 3.3–3.7 on structured grids that were generated by refining a coarse grid consisting of two triangles regularly.

In summary, the PSPG, the non-symmetric GLS, and the LPS methods behaved in this example quite similarly. The most remarkable observation was that the instability of the symmetric GLS method for large stabilization parameters became visible already for rather small values of δ_0.

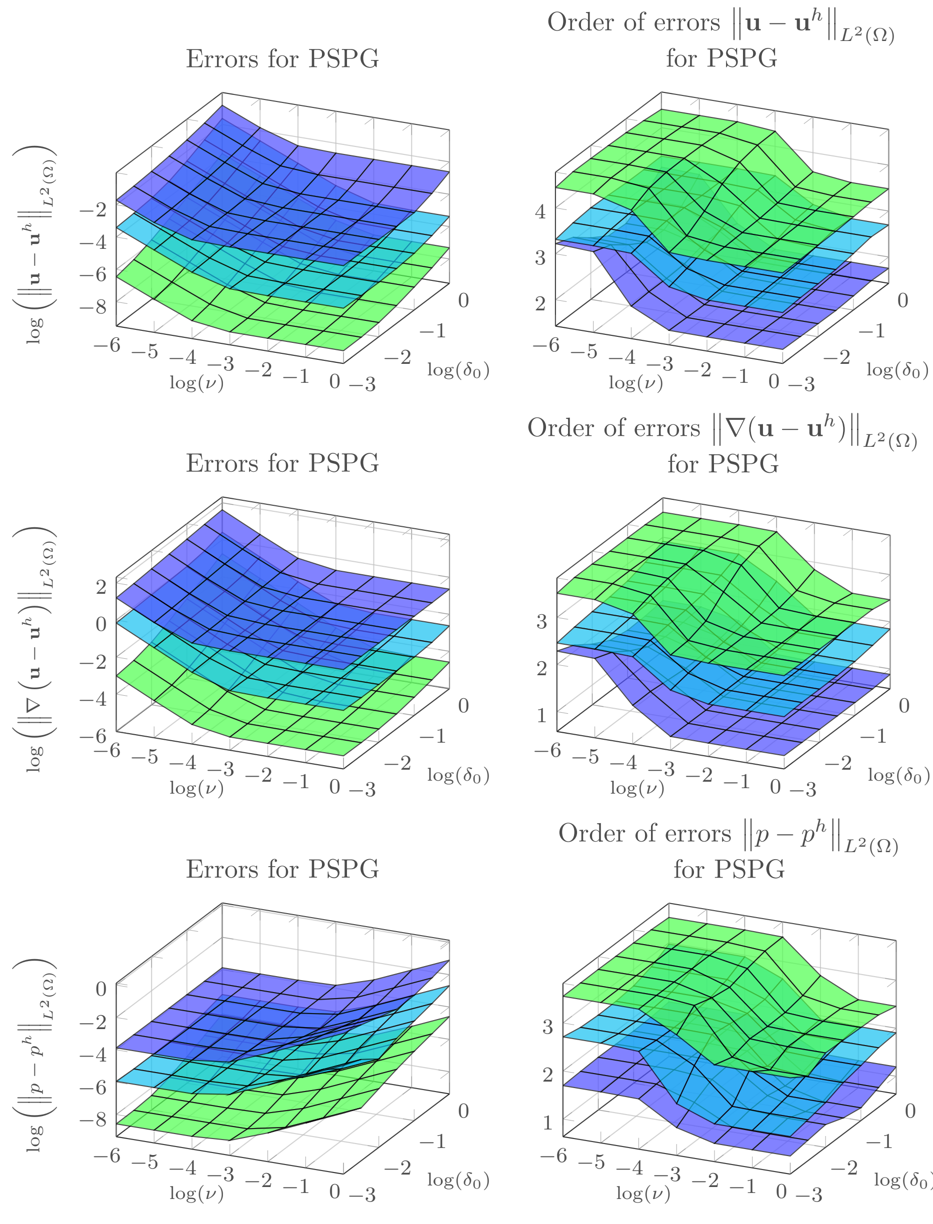

Figure 3.3: The errors (left) and computed orders of convergence (right) with respect to the L^2 (top) and H^1 semi-norm (middle) of the velocity, as well as the L^2 norm of the pressure (bottom) for the PSPG method and P_1/P_1 (blue), P_2/P_2 (cyan), and P_3/P_3 (green).

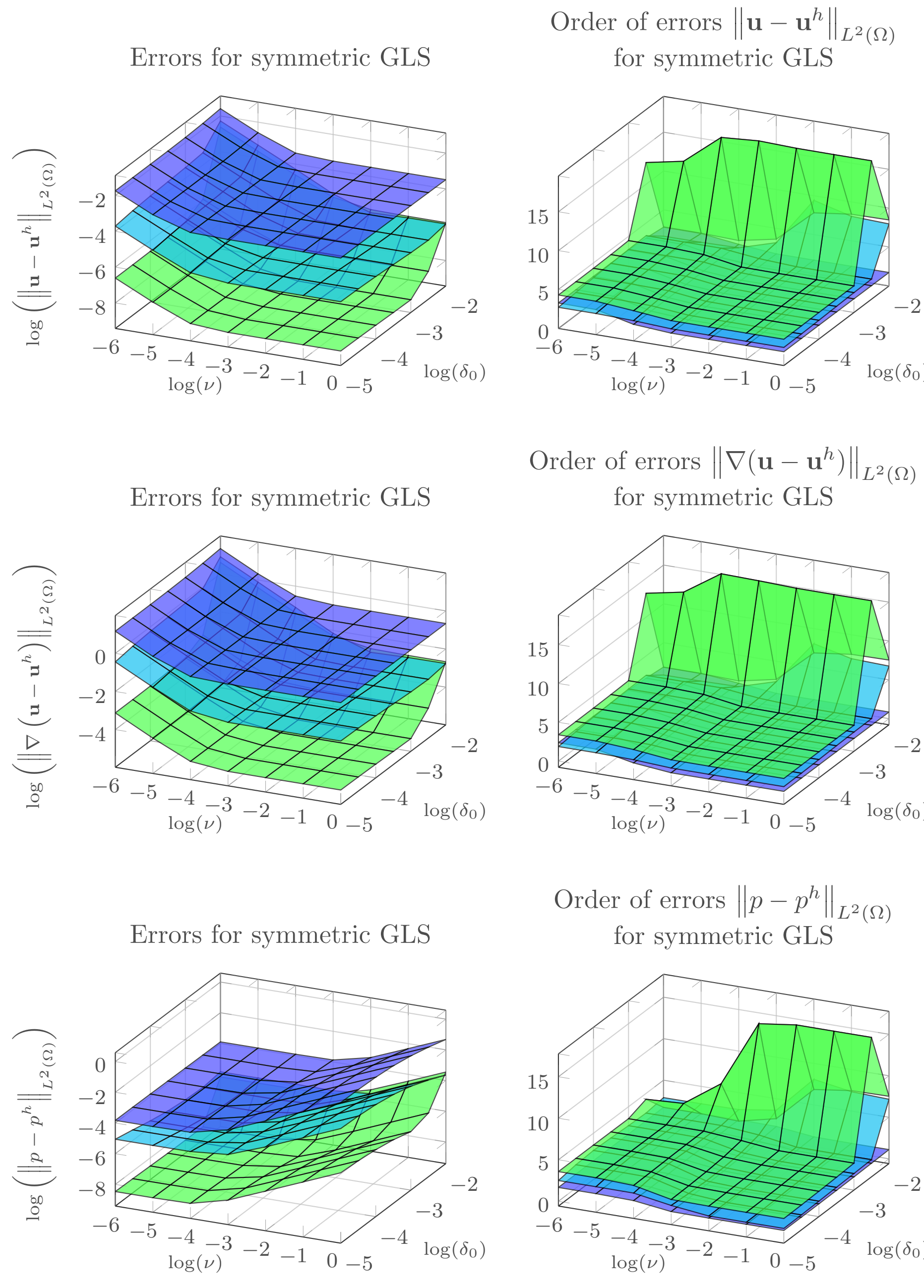

Figure 3.4: The errors (left) and computed orders of convergence (right) with respect to the L^2 (top) and H^1 semi-norm (middle) of the velocity, as well as the L^2 norm of the pressure (bottom) for the symmetric GLS method and P_1/P_1 (blue), P_2/P_2 (cyan), and P_3/P_3 (green).

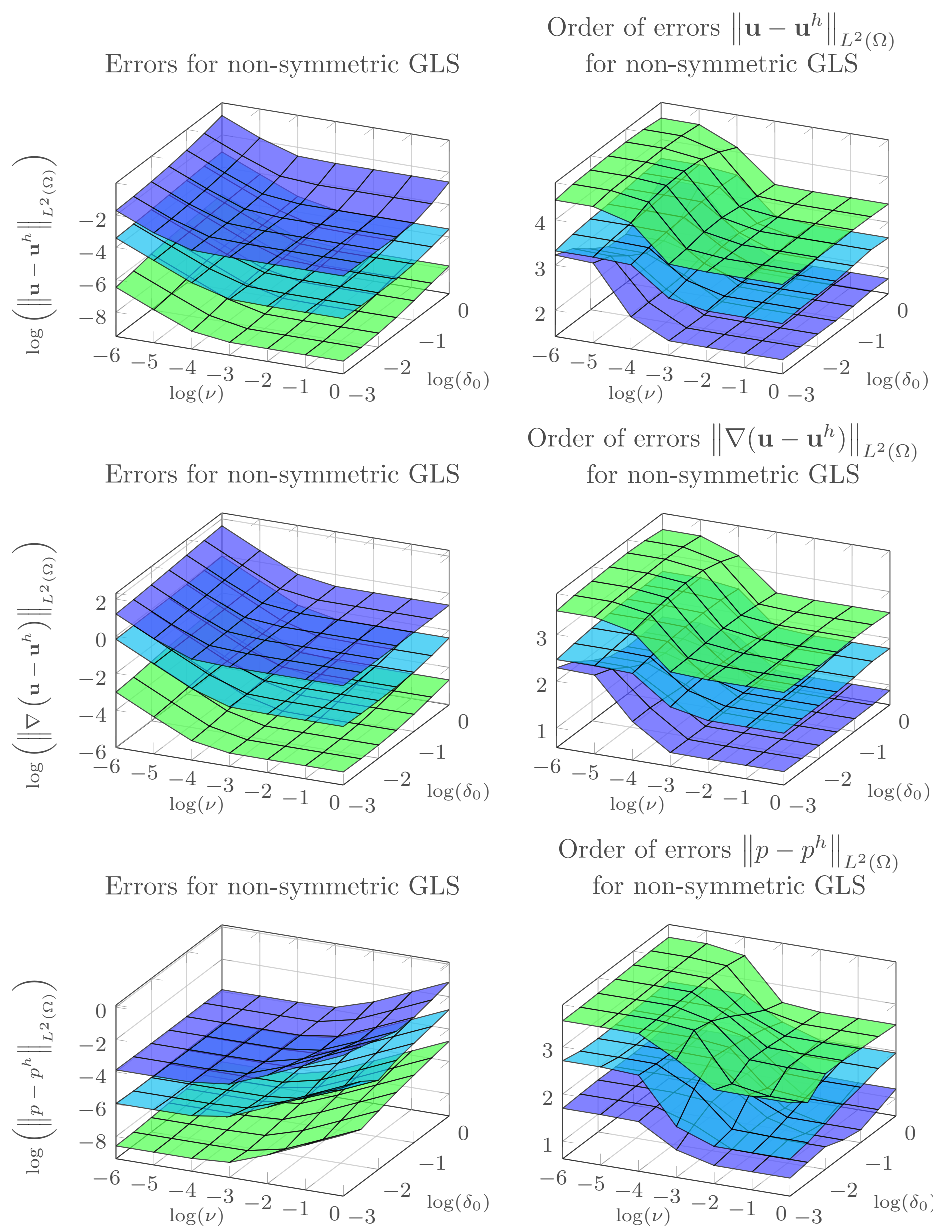

Figure 3.5: The errors (left) and computed orders of convergence (right) with respect to the L^2 (top) and H^1 semi-norm (middle) of the velocity, as well as the L^2 norm of the pressure (bottom) for the non-symmetric GLS method and P_1/P_1 (blue), P_2/P_2 (cyan), and P_3/P_3 (green).

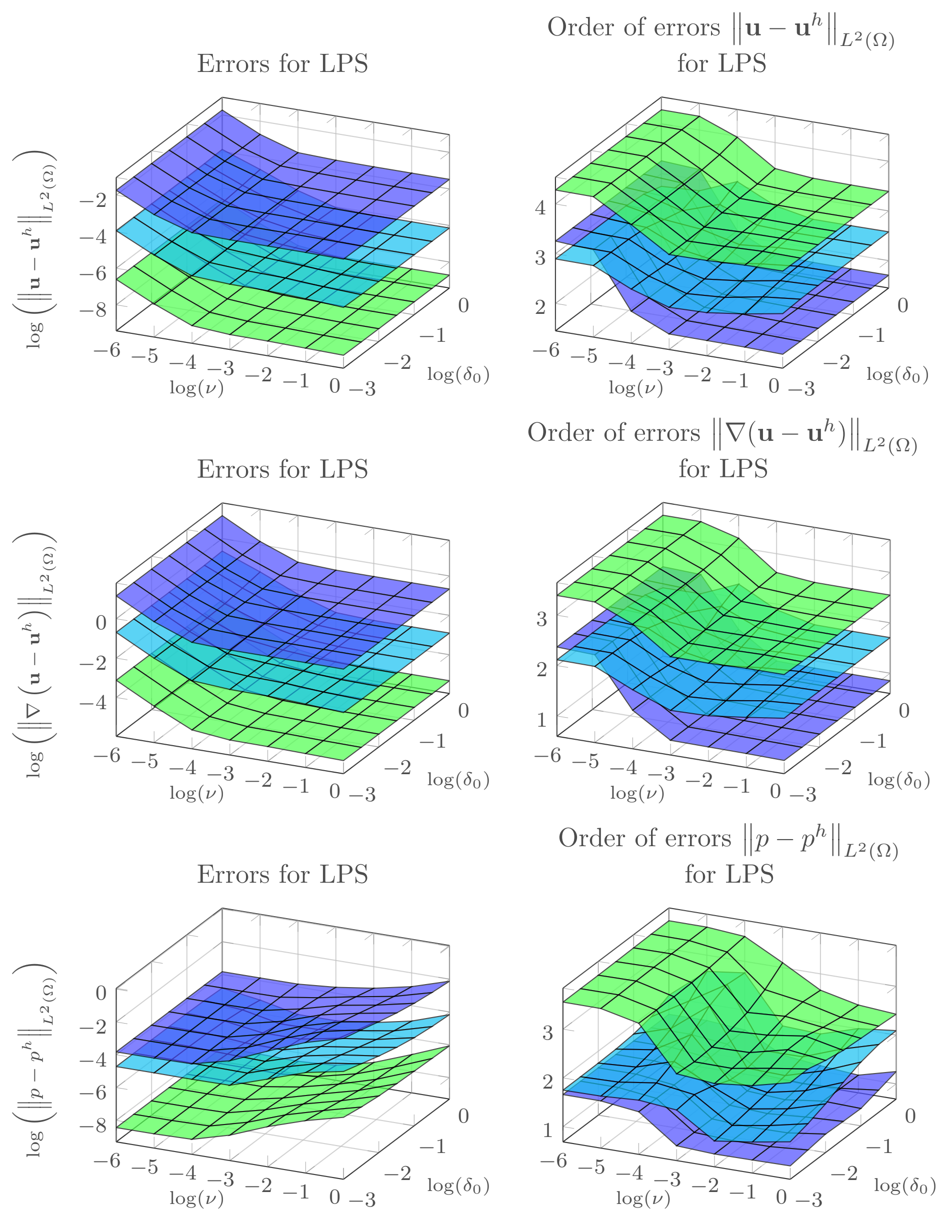

Figure 3.6: The errors (left) and computed orders of convergence (right) with respect to the L^2 (top) and H^1 semi-norm (middle) of the velocity, as well as the L^2 norm of the pressure (bottom) for the LPS method and P_1/P_1 (blue), P_2/P_2 (cyan), and P_3/P_3 (green).

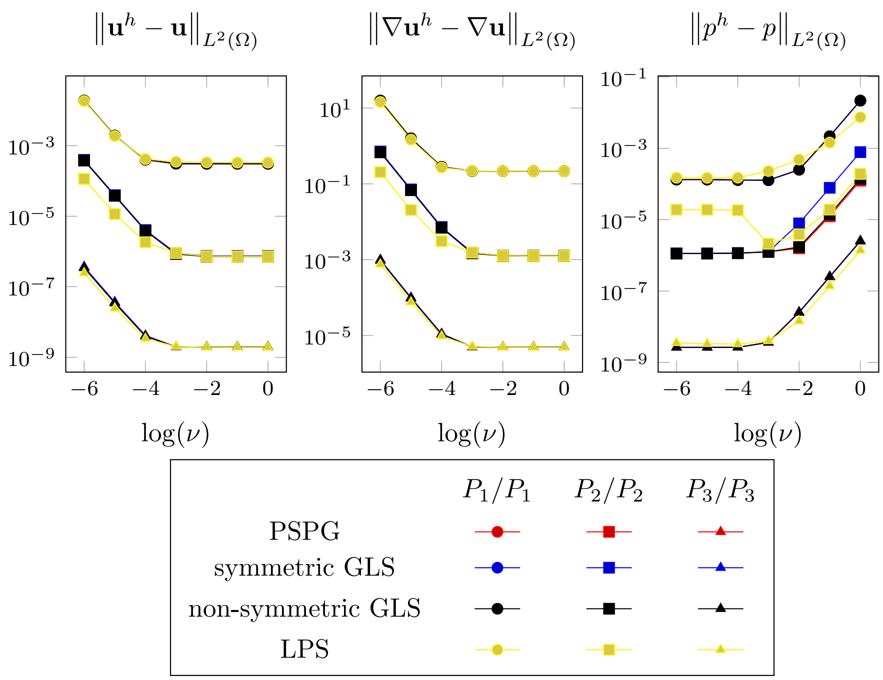

Figure 3.7: The L^2-errors of the velocity (left), its gradient (middle) and of the pressure (right) for the considered methods and different polynomial degrees on the finest level 4. The stabilization parameter δ_0 is always chosen such that the L^2-error of the velocity is smallest among the used values for δ_0. Note that lines are often on top of each other, i. e., the methods led to very similar errors.

3.7.2 A Steady-State Flow around a Cylinder

The second example serves for assessing the stabilized discretizations mentioned at the beginning of this section at a more challenging example. It is given by the stationary Navier–Stokes equations

$$\begin{aligned} -\nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u} &= 0 \quad \text{in } \Omega, \end{aligned} \tag{3.167}$$

and it requires the computation of coefficients which are of importance in applications. Furthermore, comparisons to some inf-sup stable discretizations are also provided.

A standard benchmark problem for (3.167) is the so-called *flow around a cylinder* problem defined in [80]. It is given by $\Omega = (0, 2.2)\times(0, 0.41)\setminus B_{0.1}(0.2, 0.2)$, where $B_r(x, y)$ is a (compact) two-dimensional cylinder (circle) with radius r centered at (x, y), $\nu = 10^{-3}$, and $\boldsymbol{f} = \mathbf{0}$. On the top and bottom boundary as well as at the cylinder homogeneous Dirichlet condition are prescribed. At the outflow boundary $\Gamma_{\text{out}} = \{2.2\} \times [0, 0.41]$, homogeneous Neumann (so-called do-nothing) conditions are imposed while the flow is driven entirely through a parabolic inflow on the left boundary,

$$\boldsymbol{u}(0, y) = \begin{pmatrix} 1.2y(1-y) \\ 0 \end{pmatrix}.$$

Benchmark parameters are the drag and lift coefficients at the cylinder and the pressure difference Δp between the front and the back of the cylinder, see [80] or [56, Ex. D5]. Reference values were computed in [59, 75], see also [56, Ex. D5]:

$$\begin{aligned} c_{\text{drag,ref}} &= 5.57953523384, \quad c_{\text{lift,ref}} = 0.010618948146, \\ \Delta p_{\text{ref}} &= 0.11752016697. \end{aligned}$$

For discretizing the Navier–Stokes equations (3.167), one has to choose the discrete form of the nonlinear term. Several forms were proposed, which are equivalent only if the velocity is weakly divergence-free. However, finite element velocity fields usually do not possess this property. In our numerical studies, the so-called convective form

$$\left(\left(\boldsymbol{u}^h \cdot \nabla\right) \boldsymbol{u}^h, \boldsymbol{v}^h\right),$$

the skew-symmetric form

$$\frac{1}{2}\left[\left(\left(\boldsymbol{u}^h \cdot \nabla\right) \boldsymbol{u}^h, \boldsymbol{v}^h\right) - \left(\left(\boldsymbol{u}^h \cdot \nabla\right) \boldsymbol{v}^h, \boldsymbol{u}^h\right)\right],$$

and the energy momentum and angular momentum conserving (EMAC) form [32]

$$\left(2\mathbb{D}\left(\boldsymbol{u}^h\right) \boldsymbol{u}^h, \boldsymbol{v}^h\right) + \left(\left(\nabla \cdot \boldsymbol{u}^h\right) \boldsymbol{u}^h, \boldsymbol{v}^h\right), \quad \mathbb{D}(\boldsymbol{u}) = \frac{\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T}{2},$$

were tested. Note that the EMAC form has on the one hand several favourable properties with respect to conservation of quantities, but on the other hand, it computes a modified pressure. For calculating the benchmark parameters, a reconstruction of the actual pressure is necessary.

The nonlinear systems were solved with a Picard iteration. It was stopped if the Euclidean norm of the residual vector was smaller than 10^{-10} or after 10 000 iterations.

Results are presented for simulations conducted on unstructured grids, which were generated with GMSH, see Figure 3.8 and Table 3.2. On each grid, the P_k/P_k, $k \in \{1, 2, 3\}$, finite element spaces were applied for the stabilized methods and the P_k/P_{k-1}, $k \in \{2, 3, 4\}$, inf-sup stable Taylor–Hood pairs of finite element spaces. Stabilization parameters of the form $\delta_K = \delta_0 h_K^2/\nu$ with $\delta_0 = 10^i$, $i \in \{-5, -4.5, \ldots, 0\}$, were considered. In all pictures, the results for the stabilization parameter with the smallest error with respect to the drag coefficient is presented.

The accuracy for the computed benchmark parameters is illustrated in Figures 3.9–3.11. For the drag coefficient, Figure 3.9, it can be observed that the

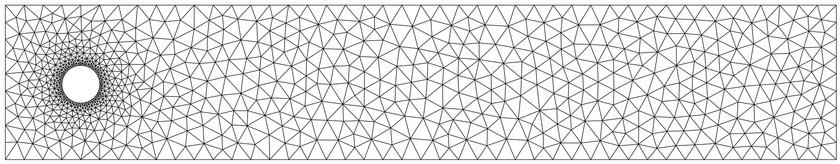

Figure 3.8: Coarse grid, generated with GMSH, used for the flow around a cylinder example.

Table 3.2: Information on the unstructured grids used in the simulations, the coarsest grid (level 0) is shown in Figure 3.8.

grid level	number of cells	number of degrees of freedom P_1/P_1	P_2/P_1	P_2/P_2	P_3/P_2	P_3/P_3	P_4/P_3
0	1629	2697	7753	10281	18595	22752	34324
1	5340	8475	24805	32970	59980	73485	111175
2	11202	17475	51529	68556	125014	153243	232105
3	19076	29493	87307	116214	212180	260163	394281
4	29193	44880	133186	177339	324031	397377	602455
5	41973	64260	191046	254439	465171	570537	865215

results obtained with the convective and skew-symmetric form are usually more accurate than those computed with the EMAC form. Using the inf-sup stable Taylor–Hood pairs of spaces gave often more accurate results than using the pressure-stabilized discretizations. For higher order pairs of spaces, the LPS method was a little bit more accurate than the other methods. For the lift coefficient, Figure 3.10, again the EMAC form led to somewhat less accurate results than the other forms of the discrete convective term. Among the stabilized methods, no substantial differences of the accuracy can be observed. For higher order pairs of spaces, the Taylor–Hood discretization was sometimes somewhat more accurate than the stabilized methods. The results for the pressure difference are shown in Figure 3.11. Again, the results computed with the Taylor–Hood pair of finite elements were usually among the most accurate ones. For the stabilized discretizations, there is no clear picture. Often, the results from the LPS method belong to the better ones.

Information with respect to the number of nonlinear iterations for solving the Navier–Stokes equations is provided in Figure 3.12. Apart of coarse grids, it can be seen that there are only minor differences between the discretization methods. The lowest number of iterations, usually below 20, was needed for the convective form of the convective term and the largest number, generally more than 50, for the EMAC form of the convective term. It should be noted that there are values of

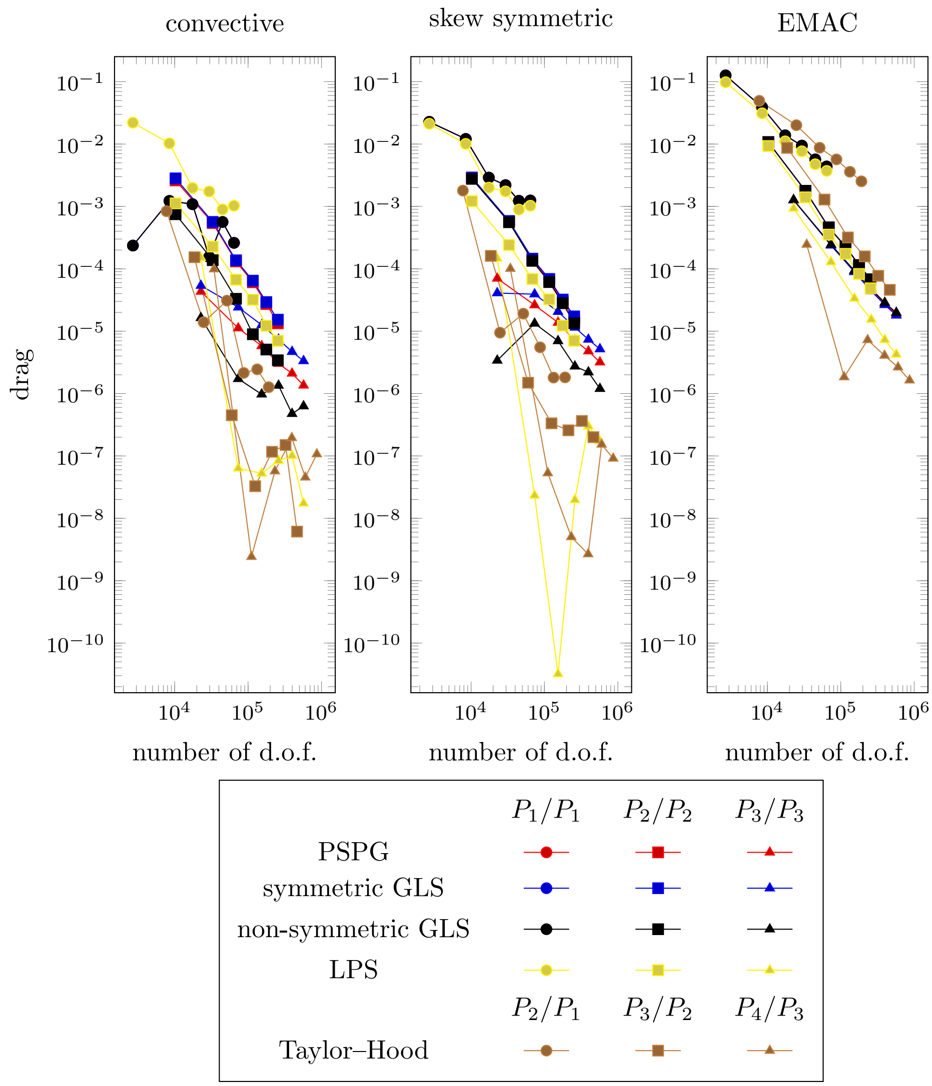

Figure 3.9: Computed absolute differences to reference value for drag using the convective (left), skew symmetric (center) and EMAC (right) nonlinear form on the unstructured grids, see Table 3.2 and Figure 3.8.

δ_0 for some of the pressure-stabilized discretizations where the nonlinear iteration took much more steps than presented in Figure 3.12, even reaching the maximal

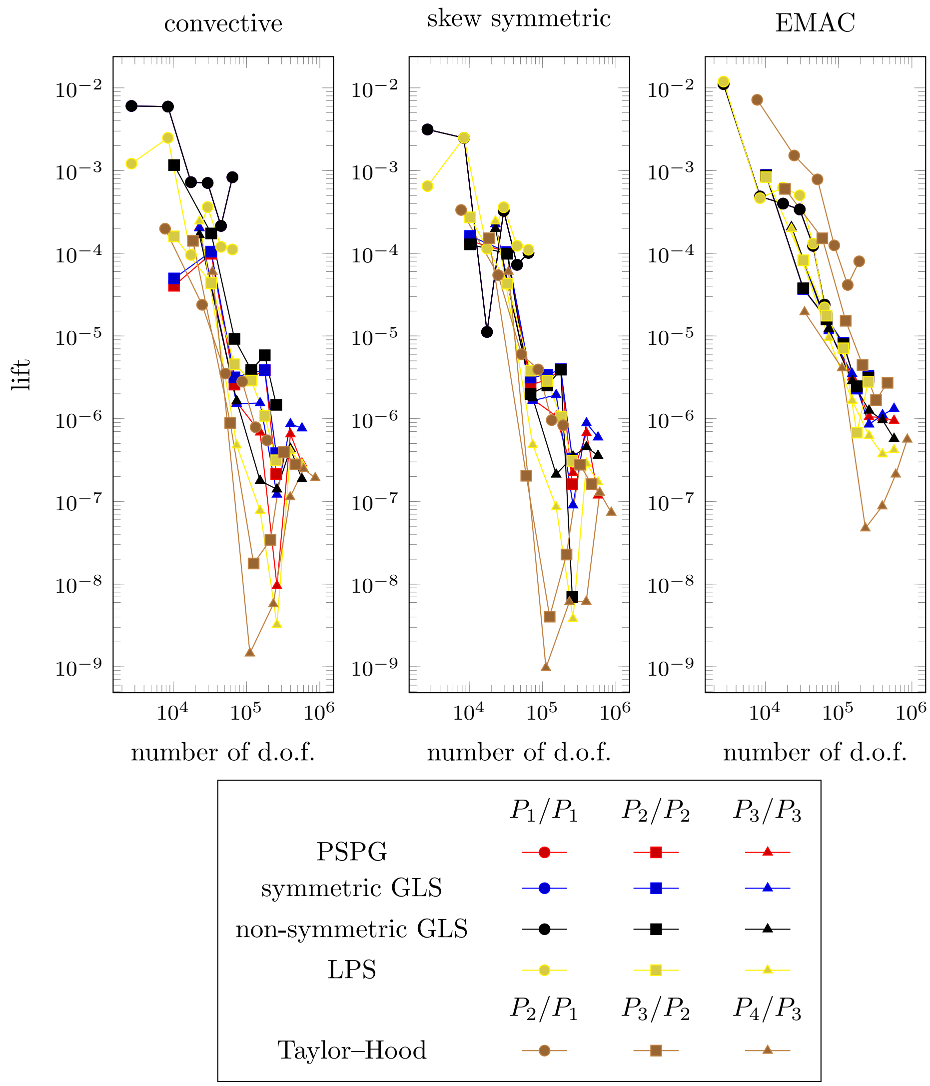

Figure 3.10: Computed absolute differences to reference value for lift using the convective (left), skew symmetric (center) and EMAC (right) nonlinear form on the unstructured grids, see Table 3.2 and Figure 3.8.

prescribed number was observed.

Very similar results as presented in Figures 3.9–3.12 were obtained on the

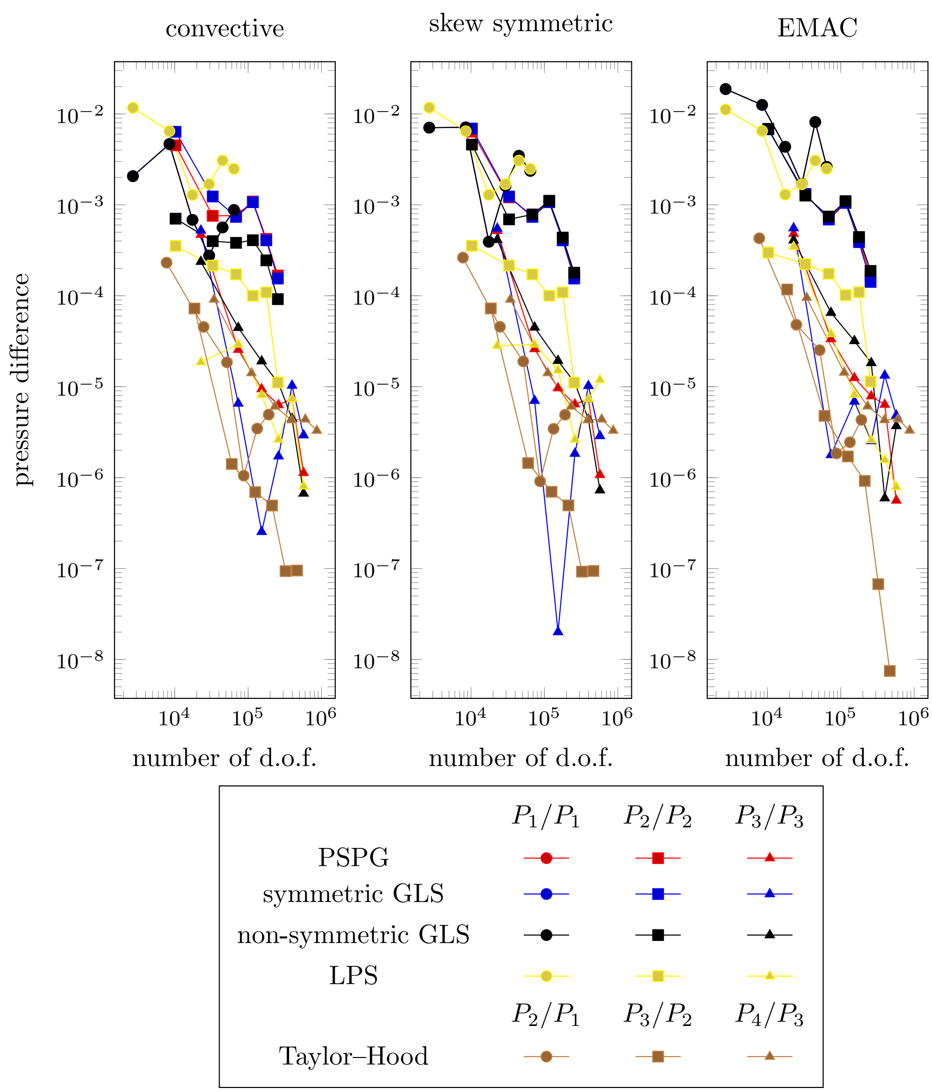

Figure 3.11: Computed absolute differences to reference value for the pressure difference at the cylinder using the convective (left), skew symmetric (center) and EMAC (right) nonlinear form on the unstructured grids, see Table 3.2 and Figure 3.8.

more structured triangular grid from [56, Figure 6.5].

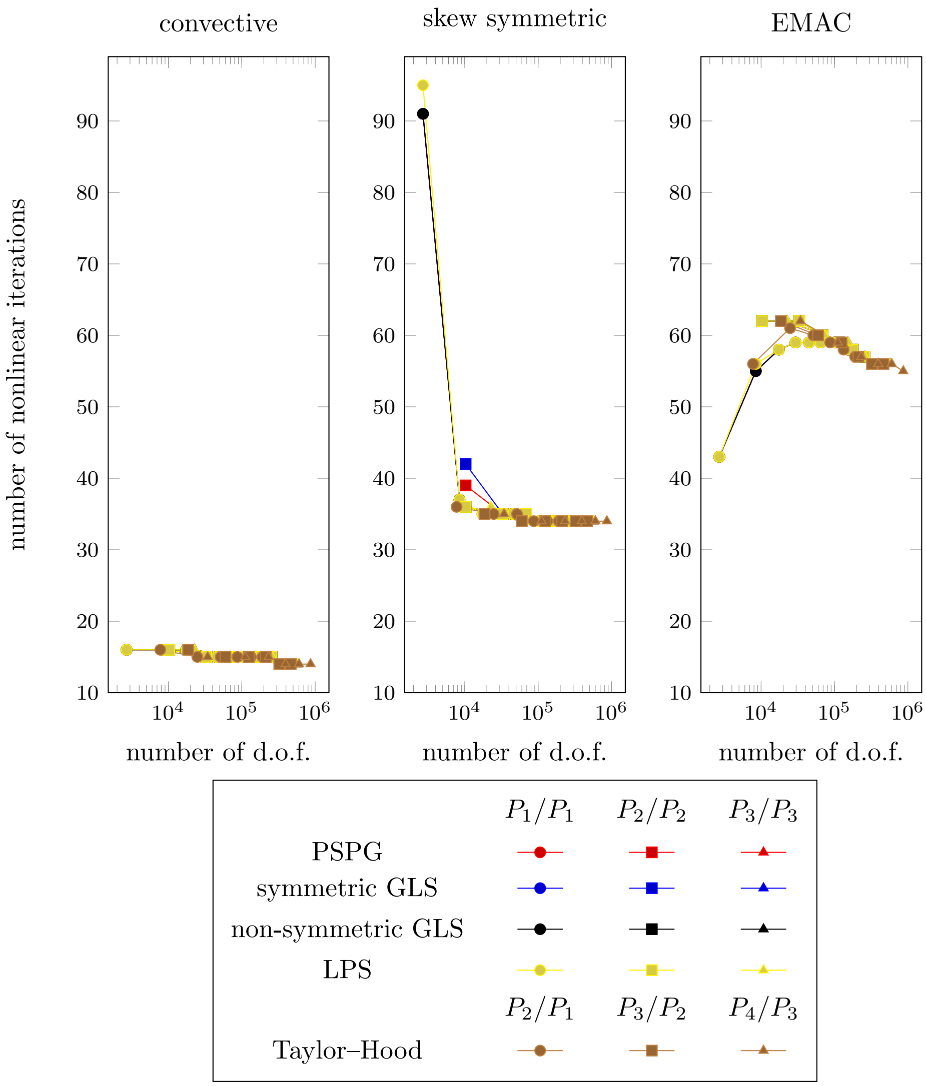

Figure 3.12: The number of nonlinear iterations needed using the convective (left), skew symmetric (center) and EMAC (right) nonlinear form on the unstructured grids, see Table 3.2 and Figure 3.8.

To summarize, no fundamental differences between the pressure-stabilized discretizations could be observed in this example. However, it could be seen that the benchmark parameters computed with the inf-sup stable Taylor–Hood pair of finite element spaces were often more accurate.

3.8 Outlook

The Stokes equations (3.2) are the simplest equations for modeling flows with incompressible fluids. This section provides brief comments concerning the application of pressure-stabilized methods to more complicated equations, like the steady-state or time-dependent Navier–Stokes equations (3.1).

Stabilizations that use only the pressure are independent of the type of equation. In particular, for time-dependent problems, the matrix block C in (3.4) has to be assembled only in the initial time step, if the space Q^h does not change in the whole time interval. Later, only the matrix block A changes, due to the nonlinearity of the Navier–Stokes equations. The assembling procedure is more expensive for residual-based stabilizations, since there, the matrix blocks A, B, and D change whenever a new assembling is performed, because the convective field in the nonlinear term of the residual changes.

The matrix block C has for residual-based stabilizations the standard sparsity pattern that comes from the pressure finite element space Q^h. Pressure-based stabilizations require in general an extended sparsity pattern, since degrees of freedom of Q^h are coupled that do not belong to a common mesh cell.

Implementing residual-based stabilizations for certain temporal discretizations, like the Crank–Nicolson scheme, is somewhat involved, since the residual at former time steps is needed. In this respect the use of BDF schemes is easier.

In connection with optimization for flow problems, one finds in the literature, e.g. [23, Sec. 7.5], that symmetric stabilizations are of advantage, since then optimizing and discretizing commute. Stabilizations that use only the pressure possess this property, whereas the only symmetric residual-based stabilization is the symmetric GLS method. But this method has the drawback of being not absolutely stable.

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