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Numerical analysis of problems in time-dependent domains

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Abstract: This work is concerned with the theoretical analysis of the space-time discontinuous Galerkin method applied to the numerical solution of nonstationary nonlinear convection-diffusion problem in a time-dependent domain. At first, the problem is reformulated by the use of the arbitrary Lagrangian-Eulerian (ALE) method, which replaces the classical partial time derivative by the so-called ALE derivative and an additional convection term. Then the problem is discretized with the use of the ALE space-time discontinuous Galerkin method. On the basis of a technical analysis we obtain an unconditional stability of this method. An important step in the analysis is the generalization of a discrete characteristic function associated with the approximate solution in a time-dependent domain and the derivation of its properties. Further we derive an a priori error estimate of the method in terms of the interpolation error, as well as in terms of h and τ. Finally, some practical applications of the ALE space-time discontinuos Galerkin method in a time-dependent domain are given. We are concerned with the numerical solution of a nonlinear elasticity benchmark problem and moreover with the interaction of compressible viscous flow with elastic structures. The main attention is paid to the modeling of flow induced vocal fold vibrations in a simplified human vocal tract.

Keywords: nonlinear convection-diffusion problem, time-dependent domain, discontinuous Galerkin method, ALE method, stability and error analysis, fluid-structure interaction, linear and nonlinear elasticity

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Introduction

Most of the theoretical mathematical results on the solvability and numerical analysis of nonstationary partial differential equations (PDEs) are obtained under the assumption that a space domain Ω is independent of time t. However, problems in time-dependent domains Ω_t are important in a number of areas of science and technology. We can mention, for example, problems with moving boundaries, when the motion of the boundary $\partial\Omega_t$ is prescribed, or free boundary problems, when the motion of the boundary $\partial\Omega_t$ should be determined together with the solution of the PDEs in consideration. This is particularly the case of fluid-structure interaction (FSI) problems, when the flow is solved in a domain deformed due to the coupling with an elastic structure.

The simulation of an interaction of flow and elastic bodies plays an important role in aerospace industry – aircraft design and safety, in civil engineering – stability of bridges, towers, smokestacks or skyscrapers, in mechanical engineering – rotary bladed machines, etc. On the other hand, FSI is also used in biomedicine. Namely, the flow in blood vessels or flow induced vocal folds vibrations are intensively studied. The FSI problems were studied by a number of different methods in several books, e.g. [17], [18], [65], [66], [73].

There are various approaches to the solution of problems in time-dependent domains as, for example, fictitious domain method [59], or imersed boundary method [12]. Very popular technique is the arbitrary Lagrangian-Eulerian (ALE) method based on a suitable one-to-one ALE mapping of the reference configuration Ω_{ref} (usually Ω_0) onto the current configuration Ω_t. This method is usually applied in connection with conforming finite element space discretization combined with time discretization by the use of a backward difference formula (BDF). From a wide literature we can mention, e.g., works [32], [55], [57], [63]. Paper [49] investigates the stability of the ALE-conforming finite element method for linear parabolic convection-diffusion initial-boundary value problems, whereas papers [4] and [50] are devoted to the error estimation.

For the numerical solution of compressible viscous flow, one of the most attractive techniques appears the discontinuous Galerkin method (DGM). It is based on piecewise polynomial approximations over finite element meshes, in general discontinuous on interfaces between neighbouring elements. This method was applied to the solution of compressible flow first in [10] and then in [11]. It allows a good resolution of boundary and internal layers (including shock waves and contact discontinuities) and has been used for the solution of various types of flow problems ([29], [30], [36], [40], [45], [51], [52], [61]). Theory of the space DGM is a subject of a number of works. We cite only some of them: [2], [3], [16], [19], [26], [31], [32], [47], [54], [56], [58], [64], [72]. It is also possible to refer to the monograph [31] containing a number of references.

In the cited works, the time discretization is carried out with the aid of the BDF of the first or second order. One possibility how to construct a higher order method in time is the application of the DGM using piecewise polynomial approximation in time, which are in general discontinuous at discrete time instants that form a partition of the time interval. This method was used for time discretization combined with conforming finite elements for the space discretization of linear parabolic equations in [1], [23], [35], [37], [38], [70], [71] and [74]. Another approach combines DGM in space with Runge-Kutta time discretization as mentioned in [69].

By the combination of the DGM in space and time we get the space-time discontinuous Galerkin method (STDGM). This method was theoretically analyzed in papers [9], [20], [43], [46], [77] and monograph [31]. In [42] and [62], the BDF-DGM and STDGM are applied to linear and nonlinear dynamic elasticity problems. One of the advantages of the STDGM is the possibility to use different meshes on different time levels.

The mentioned methods have also been extended to the numerical solution of initial-boundary value problems in time-dependent domains using the ALE method. The ALE method combined with the space DGM and BDF in time (ALE-DGM-BDF) was applied with success to the interaction of viscous compressible flow with elastic structures in [21], [44], [53] and [62]. In [22], the ALE-STDGM is applied to the simulation of flow induced airfoil vibrations and the results are compared with the ALE-DGM-BDF approach. It appears that the ALE-STDGM is more robust and accurate. We should note that in [79] the ALE-STDGM technique is applied to inviscid compressible flow.

The ALE-time discontinuous Galerkin semidiscretization of a linear parabolic convection-diffusion problem is analyzed in [14] and [15]. Both papers assume that the transport velocity is divergence free and consider homogeneous Dirichelt boundary condition. In [14], the stability of the ALE-time DGM is proved and [15] is devoted to the error estimation. Papers [6], [7] and [8] are concerned with the stability analysis of the ALE-STDGM applied to a linear convection-diffusion initial-boundary value problem ([7], [8]) as well as to the case with nonlinear convection and diffusion ([6]) with nonhomogeneous Dirichlet boundary condition, using piecewise linear DG time discretization.

The present work is devoted to the theoretical analysis of the ALE-STDGM and its applications to FSI problems. The structure of the thesis is as follows.

In Chapter 1 we introduce the system of Navier-Stokes equations describing viscous compressible flow in a time-dependent domain.

In Chapter 2 we formulate the scalar nonstationary nonlinear convection-diffusion problem equipped with initial condition and nonhomogeneous Dirichlet boundary condition. This problem can be considered as a simplified prototype of the compressible Navier-Stokes system. In this chapter we describe triangulations, ALE mappings and introduce important function spaces and concepts. Then an approximate solution of the nonlinear convection-diffusion problem using the ALE-STDGM is defined.

Chapter 3 is devoted to the stability analysis of the ALE-STDGM with arbitrary polynomial degree in space as well as in time. The method analyzed here corresponds to the technique used in [22] and [42] for the numerical simulation of airfoil vibrations induced by compressible flow. This means that the ALE mapping is constructed successively from one time slab to the next one. The presented stability analysis in this chapter is based on estimates of forms from the definition of the approximate solution. An important tool is the concept of the discrete characteristic function. It was introduced in [23] in the framework of the time discontinuous Galerkin method combined with conforming finite elements applied to a linear parabolic problem. The discrete characteristic function was generalized in connection with the STDGM for nonlinear parabolic problems in [9], [20] and [31]. Here we generalize the concept of the discrete characteristic function and prove its important properties in the case of the ALE-STDGM in time dependent domains. On the basis of a technical analysis we obtain an unconditional stability of this method represented by a bound of the approximate solution in terms of data, without any limitation of the time step in dependence on the size of the triangulations.

In Chapter 4 we derive error estimates for the ALE-STDGM in terms of h (mesh size) and τ (time step). Here we use the standard procedure, which means that we split the error $e = u - U$ (difference between the exact solution u and the approximate solution U) into two parts: ξ and η. The term η approximates the distance of the exact solution u from the space, where the approximate solution is sought, whereas ξ represents the distance between the approximate solution U and the projection of the exact solution u on the space, where the approximate solution belongs. At first error estimates in terms of ξ and η are derived. Again we use the generalized discrete characteristic function and its properties. After that, using results from [31], error estimates in terms of h and τ are proved.

Finally, Chapter 5 is devoted to the application of the ALE-STDGM to the solution of the compressible Navier-Stokes equations in the conservative ALE form in a time-dependent domain coupled with linear or nonlinear elasticity. The developed method is applied to the numerical simulation of air flow in a simplified model of human vocal tract and flow induced vocal folds vibrations. This problem was already solved in [44], where the ALE-DGM-BDF technique for the solution of the compressible Navier-Stokes equations was combined with the linear elasticity approximated in space by conforming finite elements and in time by the Newmark method. In our work, because of the successful solution of the compressible viscous flow by the ALE-STDGM, we discretize the elasticity problems also by the STDGM. The interaction of the flow and elastic structures are interacted via transmission conditions on the interface between the flow domain and the elastic body. In the numerical process this is implemented with the aid of a strong coupling algorithm. Our goal is to consider several nonlinear elasticity models and compare them to the linear elasticity model in simulation of flow induced vocal folds vibrations. We demonstrate that in the studied problem the nonlinear St. Venant-Kirchhoff and neo-Hookean models give more accurate results than the linear elasticity model. The presented numerical experiments show the robustness of the developed method and suggest the importance of using nonlinear elasticity models in such a study.

1. Formulation of the viscous compressible flow problem in time-dependent domains

In this chapter we shall consider unsteady, compressible viscous flow in a bounded, time-dependent domain $\Omega_t \subset \mathbb{R}^2$, $t \in [0,T]$. Compressible viscous flow is described by the system of Navier-Stokes equations, which consists of the continuity equation, the Navier-Stokes equations of motion and the energy equation:

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) &= 0 && (1.1)\\
\frac{\partial \rho v_i}{\partial t} + \operatorname{div}(\rho v_i \boldsymbol{v}) &= \sum_{j=1}^{2} \frac{\partial \tau_{ij}}{\partial x_j}, \quad \text{for } i = 1,2 && (1.2)\\
\frac{\partial E}{\partial t} + \operatorname{div}(E\boldsymbol{v}) &= \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{2} \tau_{ij} v_j - q_i\right), && (1.3)
\end{aligned}
$$

where

$$
\begin{aligned}
\tau_{ij} &= -p\delta_{ij} + \tau_{ij}^V, && (1.4)\\
\tau_{ij}^V &= \lambda \operatorname{div}\boldsymbol{v}\delta_{ij} + 2\mu d_{ij}(\boldsymbol{v}), \quad d_{ij}(\boldsymbol{v}) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), && (1.5)
\end{aligned}
$$

$\tau = \{\tau_{ij}\}$ is the stress tensor and $\tau^V = \{\tau_{ij}^V\}$ denotes the viscous part of the stress tensor. Moreover we use the standard notation: ρ - density, $\boldsymbol{v} = (v_1, v_2)$ - velocity, E - total energy, p - pressure, c_v - specific heat at constant volume, δ_{ij} - Kronecker symbol, $\mu > 0$ and λ - viscosity coefficients. We assume that $\lambda = -2\mu/3$. The heat flux $\boldsymbol{q} = (q_1, q_2)$ satisfies the Fourier law

$$
\boldsymbol{q} = -k\nabla\theta, \tag{1.6}
$$

where $k > 0$ is the heat conductivity assumed to be constant here and θ denotes the absolute temperature.

The above system can be written in the following form:

$$
\frac{\partial \boldsymbol{w}}{\partial t} + \sum_{s=1}^{2} \frac{\partial \boldsymbol{f}_s(\boldsymbol{w})}{\partial x_s} = \sum_{s=1}^{2} \frac{\partial \boldsymbol{R}_s(\boldsymbol{w}, \nabla\boldsymbol{w})}{\partial x_s}, \tag{1.7}
$$

where

$$
\begin{aligned}
\boldsymbol{w} &= (w_1, \dots, w_4)^T = (\rho, \rho v_1, \rho v_2, E)^T \in \mathbb{R}^4,\\
\boldsymbol{w} &= \boldsymbol{w}(x,t),\ x \in \Omega_t,\ t \in (0,T),\\
\boldsymbol{f}_s(\boldsymbol{w}) &= (f_{s1}, \dots, f_{s4})^T = (\rho v_s, \rho v_1 v_s + \delta_{1s}p, \rho v_2 v_s + \delta_{2s}p, (E+p)v_s)^T,\\
\boldsymbol{R}_s(\boldsymbol{w}, \nabla\boldsymbol{w}) &= (R_{s1}, \dots, R_{s4})^T = (0, \tau_{s1}^V, \tau_{s2}^V, \tau_{s1}^V v_1 + \tau_{s2}^V v_2 + k\frac{\partial \theta}{\partial x_s})^T,\ s = 1,2.
\end{aligned}
$$

System (1.7) is completed by the thermodynamical relations

$$p = (\gamma - 1)\left(E - \rho\frac{|\boldsymbol{v}|^2}{2}\right), \quad \theta = \frac{1}{c_v}\left(\frac{E}{\rho} - \frac{|\boldsymbol{v}|^2}{2}\right)$$

and is equipped with the initial condition

$$\boldsymbol{w}(\boldsymbol{x}, 0) = \boldsymbol{w}^0(\boldsymbol{x}), \ \boldsymbol{x} \in \Omega_0.$$

Concerning the boundary conditions, we distinguish following disjoint parts of the boundary $\partial\Omega_t$:

$$\partial\Omega_t = \Gamma_I \cup \Gamma_O \cup \Gamma_{W_t},$$

where Γ_I represents the inlet through which the fluid flows into the domain, Γ_O is the outlet through which the fluid leaves Ω_t and Γ_{W_t} are moving impermeable walls (the parts of which can depend on time t). We assume that Γ_I and Γ_O are fixed. We prescribe the following boundary conditions on individual parts of the boundary:

$$\rho = \rho_D, \quad \boldsymbol{v} = \boldsymbol{v}_D, \quad \sum_{j=1}^{2}\left(\sum_{i=1}^{2}\tau_{ij}^V n_i\right) v_j + k\frac{\partial\theta}{\partial\boldsymbol{n}} = 0 \quad \text{on } \Gamma_I, \tag{1.8}$$

$$\sum_{j=1}^{2}\tau_{ij}^V n_j = 0, \quad \frac{\partial\theta}{\partial\boldsymbol{n}} = 0, \quad i = 1, 2, \quad \text{on } \Gamma_O, \tag{1.9}$$

$$\boldsymbol{v}|_{\Gamma_{W_t}} = \boldsymbol{z}_D = \text{velocity of the moving wall}, \quad \frac{\partial\theta}{\partial\boldsymbol{n}} = 0 \quad \text{on } \Gamma_{W_t}, \tag{1.10}$$

with prescribed data ρ_D, $\boldsymbol{v}_D$, $\boldsymbol{z}_D$. Here $\boldsymbol{n} = (n_1, n_2)$ denotes the outward unit normal to $\partial\Omega_t$ and $\partial/\partial\boldsymbol{n}$ is the derivative in the direction $\boldsymbol{n}$. On Γ_O and Γ_{W_t} only three boundary conditions are specified. The missing condition is completed in the discrete problem by extrapolation.

Now we mention some important properties of vector-valued functions $\boldsymbol{f}_s$ and $\boldsymbol{R}_s$. We set

$$\mathbb{A}_s(w) = \frac{D f_s(w)}{Dw}, \quad s = 1, 2,$$

which are Jacobi matrices of the mappings $\boldsymbol{f}_s$. It is possible to show that

$$\boldsymbol{f}_s(\alpha w) = \alpha \boldsymbol{f}_s(w), \quad \text{for} \quad s > 0,$$

which implies that

$$\boldsymbol{f}_s(w) = \mathbb{A}_s(w) w, \quad s = 1, 2. \tag{1.11}$$

The viscous terms $\boldsymbol{R}_s$ can be written in the form

$$\boldsymbol{R}_s(w, \nabla w) = \sum_{k=1}^{2}\mathbb{K}_{s,k}(w)\frac{\partial w}{\partial x_k}, \quad s = 1, 2, \tag{1.12}$$

where $\mathbb{K}_{s,k}(w) \in \mathbb{R}^{4\times 4}$ for $\boldsymbol{w} \in \mathbb{R}^4$. These results are proved, for example, in [41] or [31].

In Chapter 5 we shall deal with the numerical simulation of the interaction of compressible flow with elastic structures. It is connected with some difficulties, particularly with the fact, that the compressible Navier-Stokes system is strongly nonlinear and that it is considered in a time-dependent domain. These obstacles will be overcome by the discretization using the space-time discontinuous Galerkin method (STDGM) and by the arbitrary Lagrangian-Eulerian (ALE) method.

The next sections are devoted to the theoretical analysis of this technique. Since the compressible Navier-Stokes problem is very complicated, we shall consider an initial-boundary value problem in a time-dependent domain for a scalar nonlinear parabolic convection-diffusion equation, which can be considered as a simplified model of the Navier-Stokes system.

2. Formulation of a continuous model problem and its discretization

In this chapter we introduce the nonlinear convection - diffusion problem in a time dependent domain. Then we reformulate it using the arbitrary Lagrangian-Eulerian (ALE) method. Further we describe suitable function spaces and finally define the approximate solution using the space-time discontinuous Galerkin method (STDGM).

2.1 Continuous problem

We shall be concerned with an initial-boundary value nonlinear convection - diffusion problem in a time-dependent bounded domain $\Omega_t \subset \mathbb{R}^d$ with a Lipschitz-continuous boundary $\partial\Omega_t$, where $d = 2, 3$, $t \in [0, T]$, $T > 0$:

Find a function $u = u(x, t)$ with $x \in \Omega_t$, $t \in (0, T)$ such that

$$\frac{\partial u}{\partial t} + \sum_{s=1}^{d} \frac{\partial f_s(u)}{\partial x_s} - \operatorname{div}(\beta(u)\nabla u) = g \quad \text{in } \Omega_t,\ t \in (0,T), \tag{2.1}$$

$$u = u_D \quad \text{on } \partial\Omega_t,\ t \in (0,T), \tag{2.2}$$

$$u(x,0) = u^0(x), \quad x \in \Omega_0. \tag{2.3}$$

Function $\beta(u)$ yields a nonlinear diffusion term. If we set $\beta(u) = \epsilon$, where $\epsilon \in (0, +\infty)$ is a constant, we arrive at a more standard linear case of equation (2.1). Then the diffusion term becomes $\operatorname{div}(\beta(u)\nabla u) = \epsilon\Delta u$.

Conditions (2.2) and (2.3) represent the Dirichlet boundary condition and the initial condition, respectively, g, u_D and u^0 are given functions and f_s, $s = 1, \dots, d$ are given inviscid fluxes. We assume that $f_s \in C^1(\mathbb{R})$, $f_s(0) = 0$ and

$$|f_s'| \le L_f, \quad s = 1, \dots, d, \tag{2.4}$$

where the constant L_f does not depend on u. Moreover we assume that function β is bounded and Lipschitz-continuous:

$$\beta : \mathbb{R} \to [\beta_0, \beta_1], \quad 0 < \beta_0 < \beta_1 < \infty, \tag{2.5}$$

$$|\beta(u_1) - \beta(u_2)| \le L_\beta |u_1 - u_2| \quad \forall u_1, u_2 \in \mathbb{R}. \tag{2.6}$$

In what follows, we shall use the standard notation $L^2(\omega)$ for the Lebesgue space, $W^{k,p}(\omega)$, $H^k(\omega) = W^{k,2}(\omega)$ for the Sobolev spaces over a bounded domain $\omega \subset \mathbb{R}^d$, $d = 2, 3$, with Lipschitz boundary and the Bochner spaces $L^\infty(0, T; X)$ with a Banach space X. Moreover we define

$$W^{1,\infty}(0,T; W^{1,\infty}(\Omega_t)) = \left\{ f \in L^\infty(0,T; W^{1,\infty}(\Omega_t));\ df/dt \in L^\infty(0,T; W^{1,\infty}(\Omega_t)) \right\},$$

where df/dt denotes the distributional derivative.

If B is a Banach (Hilbert) space, then its norm (scalar product) will be denoted by $\|\cdot\|_B$ ($(\cdot,\cdot)_B$). By $|\cdot|_B$ we denote a seminorm in B. For simplicity we use the notation $\|\cdot\|_{L^2(\omega)} = \|\cdot\|_\omega$, $(\cdot,\cdot)_{L^2(\omega)} = (\cdot,\cdot)_\omega$ and $\|\cdot\|_{L^2(\partial\omega)} = \|\cdot\|_{\partial\omega}$.

2.1.1 ALE formulation

Problem (2.1)–(2.3) can be reformulated with the aid of the so-called arbitrary Lagrangian-Eulerian (ALE) method. Because of simplicity, first we consider a standard ALE formulation prescribed globally in the whole time interval, used in a number of works (cf., e.g., [6], [14], [15], [34], [49], [50], [55], [57]). It is based on a regular one-to-one ALE mapping of the reference domain Ω_0 onto the current configuration Ω_t:

$$\mathcal{A}_t : \overline{\Omega}_0 \to \overline{\Omega}_t, \quad X \in \overline{\Omega}_0 \to x = x(X,t) = \mathcal{A}_t(X) \in \overline{\Omega}_t, \quad t \in [0,T], \tag{2.7}$$

see Figure 2.1.

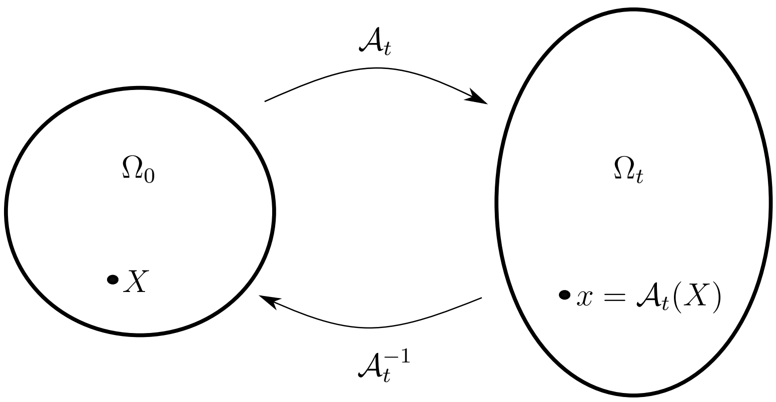

Figure 2.1: ALE mapping $\mathcal{A}_t$ and its inverse $\mathcal{A}_t^{-1}$.

We can also write $\mathcal{A}(X,t) = \mathcal{A}_t(X),\ X \in \overline{\Omega}_0,\ t \in [0,T]$. Usually it is supposed that the ALE mapping is sufficiently regular, e.g., $\mathcal{A}_t \in W^{1,\infty}(0,T; W^{1,\infty}(\Omega_t))$. In further considerations more general property will appear. Now we introduce the domain velocity

$$\tilde{\boldsymbol{z}}(X,t) = \frac{\partial}{\partial t}\mathcal{A}_t(X),\ \boldsymbol{z}(x,t) = \tilde{\boldsymbol{z}}(\mathcal{A}_t^{-1}(x),t),\ t \in [0,T],\ X \in \Omega_0,\ x \in \Omega_t, \tag{2.8}$$

and define the ALE derivative $D_t f = Df/Dt$ of a differentiable function $f = f(x,t)$ for $x \in \Omega_t$ and $t \in [0,T]$ as

$$D_t f(x,t) = \frac{D}{Dt} f(x,t) = \frac{\partial \tilde{f}}{\partial t}(X,t), \tag{2.9}$$

where $\tilde{f}(X,t) = f(\mathcal{A}_t(X),t)$, $X \in \Omega_0$, and $x = \mathcal{A}_t(X) \in \Omega_t$. The use of the chain rule yields the relation

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \boldsymbol{z} \cdot \nabla f, \tag{2.10}$$

which allows us to reformulate problem (2.1)–(2.3) in the ALE form:
Find $u = u(x,t)$ with $x \in \Omega_t,\ t \in (0,T)$ such that

$$\frac{Du}{Dt} + \sum_{s=1}^{d} \frac{\partial f_s(u)}{\partial x_s} - \boldsymbol{z} \cdot \nabla u - \operatorname{div}(\beta(u)\nabla u) = g \quad \text{in } \Omega_t,\ t \in (0,T), \tag{2.11}$$

$$u = u_D \quad \text{on } \partial\Omega_t,\ t \in (0,T), \tag{2.12}$$

$$u(x,0) = u^0(x), \quad x \in \Omega_0. \tag{2.13}$$

2.2 ALE space-time discretization

In the time interval $[0,T]$ we consider a partition $0 = t_0 < t_1 < \cdots < t_M = T$ and set $\tau_m = t_m - t_{m-1},\ I_m = (t_{m-1}, t_m),\ \overline{I}_m = [t_{m-1}, t_m]$ for $m = 1,\dots,M$, $\tau = \max_{m=1,\dots,M} \tau_m$. We assume that $\tau \in (0, \overline{\tau})$, where $\overline{\tau} > 0$. The space-time discontinuous Galerkin method (STDGM) has an advantage that on every time interval $\overline{I}_m = [t_{m-1}, t_m]$ it is possible to consider a different space partition (i. e. triangulation) – see, e. g. [31], [20]. Here we also use this possibility for the application of the STDGM in the framework of the ALE method. It allows us to consider an ALE mapping separately on each time interval $[t_{m-1}, t_m)$ for $m = 1,\dots,M$ and the resulting ALE mapping in $[0,T]$ may be discontinuous at time instants $t_m,\ m = 1,\dots,M-1$. This means that one-sided limits $\mathcal{A}(t_m-) \neq \mathcal{A}(t_m+)$ in general. Similarly the same may hold for the approximate solution. Such situation appears in the numerical solution of fluid-structure interaction problems, when both the ALE mapping and the approximate flow solution are constructed successively on the time intervals $I_m,\ m = 1,\dots,M$, by the space-time discontinuous Galerkin method (see [62]).

As was mentioned above, in what follows, we consider a new generalized ALE technique developed for the simulation of the compressible fluid-structure interaction. This approach is applicable in the framework of the STDGM. To this end, we introduce the following notation.

2.2.1 ALE mappings and triangulations

For every $m = 1,\dots,M$ we consider a standard conforming triangulation $\hat{\mathcal{T}}_{h,t_{m-1}}$ in $\Omega_{t_{m-1}}$, where $h \in (0, \overline{h})$ and $\overline{h} > 0$. This triangulation is formed by a finite number of closed triangles ($d = 2$) or tetrahedra ($d = 3$) with disjoint interiors. We assume that the triangulations $\hat{\mathcal{T}}_{h,t_{m-1}}$ have the standard properties mentioned in [24]. Thus, we assume that the domain $\Omega_{t_{m-1}}$ is polygonal (polyhedral). Further, for each $m = 1,\dots,M$ we introduce a one-to-one ALE mapping

$$\mathcal{A}_{h,t}^{m-1} : \overline{\Omega}_{t_{m-1}} \overset{\text{onto}}{\longrightarrow} \overline{\Omega}_t \text{ for } t \in [t_{m-1}, t_m),\ h \in (0, \overline{h}). \tag{2.14}$$

We assume that $\mathcal{A}_{h,t}^{m-1}$ is in space a piecewise affine mapping on the triangulation $\hat{\mathcal{T}}_{h,t_{m-1}}$, continuous in space variable $X \in \Omega_{t_{m-1}}$ and continuously differentiable in time $t \in [t_{m-1}, t_m)$ and $\mathcal{A}_{h,t_{m-1}}^{m-1} = \mathrm{Id}$ (identical mapping). Hence, we assume that all domains Ω_t are polygonal (polyhedral). For every $t \in [t_{m-1}, t_m)$ we define the conforming triangulation

$$\mathcal{T}_{h,t} = \left\{ K = \mathcal{A}_{h,t}^{m-1}(\hat{K});\ \hat{K} \in \hat{\mathcal{T}}_{h,t_{m-1}} \right\} \text{ in } \Omega_t. \tag{2.15}$$

For an illustration, see Figure 2.2.

At $t = t_m$ we define the one-sided limit $\mathcal{A}_{h,t_m-}^{m-1}$, introduce the triangulation

$$\mathcal{T}_{h,t_m-} = \{ \mathcal{A}_{h,t_m-}^{m-1}(\hat{K});\ \hat{K} \in \hat{\mathcal{T}}_{h,t_{m-1}} \} \text{ in } \overline{\Omega}_{t_m}$$

and suppose that

$$\mathcal{A}_{h,t_m}^{m-1}\left(\overline{\Omega}_{t_{m-1}}\right) = \overline{\Omega}_{t_m}. \tag{2.16}$$

We have $\mathcal{T}_{h,t_{m-1}} = \hat{\mathcal{T}}_{h,t_{m-1}}$, but in general, $\mathcal{T}_{h,t_m-} \neq \hat{\mathcal{T}}_{h,t_m}$.

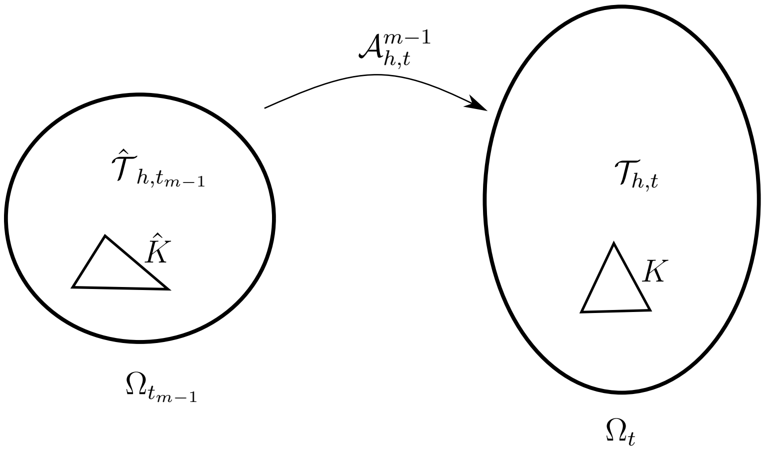

Figure 2.2: ALE mapping $\mathcal{A}_{h,t}^{m-1}$ for $t \in [t_{m-1}, t_m),\ h \in (0, \overline{h})$.

As we see, for every $t \in [0, T]$ we have a family $\{\mathcal{T}_{h,t}\}_{h\in(0,\overline{h})}$ of triangulations of the domain Ω_t. Triangulations $\hat{\mathcal{T}}_{h,t_{m-1}}$ and $\hat{\mathcal{T}}_{h,t_m}$ have different structure and, in general, different number of cells. Triangulations $\mathcal{T}_{h,t}$ and $\mathcal{T}_{h,t_m-}$ have the same structure as $\hat{\mathcal{T}}_{h,t_{m-1}}$ for $t \in [t_{m-1}, t_m]$, but starting from $\hat{\mathcal{T}}_{h,t_m}$ the structure of $\mathcal{T}_{h,t}$ for $t \in [t_m, t_{m+1}]$, may be different from $\hat{\mathcal{T}}_{h,t_{m-1}}$.

2.2.2 Discrete function spaces

In what follows, for every $m = 1, \dots, M$ we consider the space

$$S_h^{p,m-1} = \left\{\varphi \in L^2(\Omega_{t_{m-1}});\ \varphi|_{\hat{K}} \in P^p(\hat{K})\ \forall \hat{K} \in \hat{\mathcal{T}}_{h,t_{m-1}}\right\}, \tag{2.17}$$

where $p \geq 1$ is an integer and $P^p(\hat{K})$ is the space of all polynomials on $\hat{K}$ of degree $\leq p$. Now for every $t \in \overline{I}_m$ we define the space

$$S_h^{t,p,m-1} = \left\{\varphi \in L^2(\Omega_t);\ \varphi \circ \mathcal{A}_{h,t}^{m-1} \in S_h^{p,m-1}\right\}. \tag{2.18}$$

It is possible to see that

$$S_h^{t,p,m-1} = \left\{\varphi \in L^2(\Omega_t);\ \varphi|_K \in P^p(K)\ \forall K \in \mathcal{T}_{h,t}\right\}. \tag{2.19}$$

Of course, $S_h^{t_m,p,m-1} \neq S_h^{p,m}$ in general.

Further, let p, $q \geq 1$ be integers. By $P^q(I_m; S_h^{p,m-1})$ we denote the space of mappings of the time interval I_m into the space $S_h^{p,m-1}$ which are polynomials of degree $\leq q$ in time. We set

$$S_{h,\tau}^{p,q} = \left\{\varphi;\ \varphi(t) \circ \mathcal{A}_{h,t}^{m-1}|_{I_m} \in P^q(I_m; S_h^{p,m-1}),\ m = 1, \dots, M\right\}. \tag{2.20}$$

This means that if $\varphi \in S_{h,\tau}^{p,q}$, then

$$\varphi\left(\mathcal{A}_{h,t}^{m-1}(X), t\right) = \sum_{i=0}^{q} \vartheta_i(X)\, t^i, \tag{2.21}$$
$$\vartheta_i \in S_h^{p,m-1},\ X \in \Omega_{t_{m-1}},\ t \in \overline{I}_m,\ m = 1, \dots, M,\ h \in (0, \overline{h}).$$

An approximate solution of problem (2.11)–(2.13) and test functions will be elements of the space $S_{h,\tau}^{p,q}$.

By D_t we denote the ALE derivative defined by (2.9) for $t \in \bigcup_{m=1}^{M} I_m$.

2.2.3 Some notation and important concepts

Over a triangulation $\mathcal{T}_{h,t}$, for each positive integer k, we define the broken Sobolev space

$$H^k(\Omega_t, \mathcal{T}_{h,t}) = \{v;\, v|_K \in H^k(K) \quad \forall K \in \mathcal{T}_{h,t}\},$$

equipped with the seminorm

$$|v|_{H^k(\Omega_t,\mathcal{T}_{h,t})} = \left(\sum_{K\in\mathcal{T}_{h,t}} |v|^2_{H^k(K)} \right)^{1/2},$$

where $|\cdot|_{H^k(K)}$ denotes the seminorm in the space $H^k(K)$, defined as

$$|v|_{H^k(K)} = \left(\int_K \sum_{|\alpha|=k} |D^\alpha v|^2 \mathrm{d}x \right)^{1/2}.$$

The symbol D^α denotes the d-dimensional derivative defined for example in [31], page 11.

By $\mathcal{F}_{h,t}$ we denote the system of all faces of all elements $K \in \mathcal{T}_{h,t}$. It consists of the set of all inner faces $\mathcal{F}^I_{h,t}$ and the set of all boundary faces $\mathcal{F}^B_{h,t}$:

$$\mathcal{F}_{h,t} = \mathcal{F}^I_{h,t} \cup \mathcal{F}^B_{h,t}.$$

Each $\Gamma \in \mathcal{F}_{h,t}$ will be associated with a unit normal vector $\mathbf{n}_\Gamma$. By $K_\Gamma^{(L)}$ and $K_\Gamma^{(R)} \in \mathcal{T}_{h,t}$ we denote the elements adjacent to the face $\Gamma \in \mathcal{F}^I_{h,t}$. Moreover, for $\Gamma \in \mathcal{F}^B_{h,t}$ the element adjacent to this face will be denoted by $K_\Gamma^{(L)}$. We shall use the convention, that $\mathbf{n}_\Gamma$ is the outer normal to $\partial K_\Gamma^{(L)}$.

Similarly, by $\hat{\mathcal{F}}_{h,t_{m-1}}$ we denote the system of all faces of all elements $\hat{K} \in \hat{\mathcal{T}}_{h,t_{m-1}}$ and it holds, that it consists of the set of all inner and boundary faces:

$$\hat{\mathcal{F}}_{h,t_{m-1}} = \hat{\mathcal{F}}^I_{h,t_{m-1}} \cup \hat{\mathcal{F}}^B_{h,t_{m-1}}.$$

If $v \in H^1(\Omega_t, \mathcal{T}_{h,t})$ and $\Gamma \in \mathcal{F}_{h,t}$, then $v_\Gamma^{(L)}$ and $v_\Gamma^{(R)}$ will denote the traces of v on Γ from the side of elements $K_\Gamma^{(L)}$ and $K_\Gamma^{(R)}$, respectively. We set $h_K = \operatorname{diam} K$ for $K \in \mathcal{T}_{h,t}$, $h(\Gamma) = \operatorname{diam} \Gamma$ for $\Gamma \in \mathcal{F}_{h,t}$ and

$$\begin{aligned}
\langle v \rangle_\Gamma &= \frac{1}{2}\left(v_\Gamma^{(L)} + v_\Gamma^{(R)} \right), \text{ average of traces of } v \text{ on } \Gamma \in \mathcal{F}^I_{h,t}, \\
[v]_\Gamma &= v_\Gamma^{(L)} - v_\Gamma^{(R)}, \text{ jump of traces of } v \text{ on } \Gamma \in \mathcal{F}^I_{h,t}.
\end{aligned}$$

Moreover, by ρ_K we denote the diameter of the largest ball inscribed into $K \in \mathcal{T}_{h,t}$.

2.3 Approximate solution

First we introduce the space semidiscretization of problem (2.11)–(2.13). We assume that u is a sufficiently smooth solution of our problem. If we choose an arbitrary but fixed $t \in (0, T)$, multiply equation (2.11) by a test function

$\varphi \in H^2(\Omega_t, \mathcal{T}_{h,t})$, integrate over any element K and finally sum over all elements $K \in \mathcal{T}_{h,t}$, then for $t \in I_m$ we get

$$\sum_{K\in\mathcal{T}_{h,t}} \int_K \frac{Du}{Dt}\varphi \,\mathrm{d}x + \sum_{K\in\mathcal{T}_{h,t}} \int_K \sum_{s=1}^{d} \frac{\partial f_s(u)}{\partial x_s}\varphi \,\mathrm{d}x \tag{2.22}$$
$$- \sum_{K\in\mathcal{T}_{h,t}} \int_K \sum_{s=1}^{d} z_s \frac{\partial u}{\partial x_s}\varphi \,\mathrm{d}x - \sum_{K\in\mathcal{T}_{h,t}} \int_K \mathrm{div}(\beta(u)\nabla u)\varphi \,\mathrm{d}x = \sum_{K\in\mathcal{T}_{h,t}} \int_K g\varphi \,\mathrm{d}x.$$

Applying Green's theorem to the convection and diffusion terms, introducing the concept of a numerical flux and suitable expressions mutually vanishing, after some manipulation we arrive at the identity

$$(D_t u, \varphi)_{\Omega_t} + A_h(u, \varphi, t) + b_h(u, \varphi, t) + d_h(u, \varphi, t) = l_h(\varphi, t), \tag{2.23}$$

where the forms appearing here are defined for $u, \varphi \in H^2(\Omega_t, \mathcal{T}_{h,t})$, $\Theta \in \mathbb{R}$ and $c_W > 0$ in the following way:

$$a_h(u, \varphi, t) := \sum_{K\in\mathcal{T}_{h,t}} \int_K \beta(u)\nabla u \cdot \nabla\varphi \,\mathrm{d}x \tag{2.24}$$
$$- \sum_{\Gamma\in\mathcal{F}_{h,t}^I} \int_\Gamma \left(\langle\beta(u)\nabla u\rangle \cdot \boldsymbol{n}_\Gamma \,[\varphi] + \Theta\,\langle\beta(u)\nabla\varphi\rangle \cdot \boldsymbol{n}_\Gamma \,[u]\right) \mathrm{d}S$$
$$- \sum_{\Gamma\in\mathcal{F}_{h,t}^B} \int_\Gamma \left(\beta(u)\nabla u \cdot \boldsymbol{n}_\Gamma \,\varphi + \Theta\beta(u)\nabla\varphi \cdot \boldsymbol{n}_\Gamma \,u - \Theta\beta(u)\nabla\varphi \cdot \boldsymbol{n}_\Gamma \,u_D\right) \mathrm{d}S,$$

$$J_h(u, \varphi, t) := c_W \sum_{\Gamma\in\mathcal{F}_{h,t}^I} h(\Gamma)^{-1} \int_\Gamma [u]\,[\varphi] \,\mathrm{d}S + c_W \sum_{\Gamma\in\mathcal{F}_{h,t}^B} h(\Gamma)^{-1} \int_\Gamma u\,\varphi \,\mathrm{d}S, \tag{2.25}$$

$$J_h^B(u, \varphi, t) := c_W \sum_{\Gamma\in\mathcal{F}_{h,t}^B} h(\Gamma)^{-1} \int_\Gamma u\,\varphi \,\mathrm{d}S, \tag{2.26}$$

$$A_h(u, \varphi, t) = a_h(u, \varphi, t) + \beta_0\, J_h(u, \varphi, t), \tag{2.27}$$

$$b_h(u, \varphi, t) := - \sum_{K\in\mathcal{T}_{h,t}} \int_K \sum_{s=1}^{d} f_s(u) \frac{\partial\varphi}{\partial x_s} \,\mathrm{d}x \tag{2.28}$$
$$+ \sum_{\Gamma\in\mathcal{F}_{h,t}^I} \int_\Gamma H(u_\Gamma^{(L)}, u_\Gamma^{(R)}, \boldsymbol{n}_\Gamma)\,[\varphi] \,\mathrm{d}S + \sum_{\Gamma\in\mathcal{F}_{h,t}^B} \int_\Gamma H(u_\Gamma^{(L)}, u_\Gamma^{(L)}, \boldsymbol{n}_\Gamma)\,\varphi \,\mathrm{d}S,$$

$$d_h(u, \varphi, t) := - \sum_{K\in\mathcal{T}_{h,t}} \int_K \sum_{s=1}^{d} z_s \frac{\partial u}{\partial x_s}\varphi \,\mathrm{d}x = - \sum_{K\in\mathcal{T}_{h,t}} \int_K (\boldsymbol{z} \cdot \nabla u)\varphi \,\mathrm{d}x, \tag{2.29}$$

$$l_h(\varphi, t) := \sum_{K\in\mathcal{T}_{h,t}} \int_K g\varphi \,\mathrm{d}x + \beta_0\, c_W \sum_{\Gamma\in\mathcal{F}_{h,t}^B} h(\Gamma)^{-1} \int_\Gamma u_D\,\varphi \,\mathrm{d}S. \tag{2.30}$$

If ω is a measurable set and $\varphi, \psi \in L^2(\omega)$, then we set

$$(\varphi, \psi)_\omega = \int_\omega \varphi\psi \,\mathrm{d}x.$$

Let us note that in integrals over faces we omit the subscript Γ of $\langle\cdot\rangle$ and $[\cdot]$. We consider $\Theta = 1$, $\Theta = 0$ and $\Theta = -1$ and get the symmetric (SIPG), incomplete (IIPG) and nonsymmetric (NIPG) variants of the approximation of the diffusion terms, respectively. The constant c_W will be specified in Section 3.2.1.

In (2.28), H is a numerical flux with the following properties:
(H1) $H(u, v, \boldsymbol{n})$ is defined in $\mathbb{R}^2 \times B_1$, where $B_1 = \{\boldsymbol{n} \in \mathbb{R}^d; |\boldsymbol{n}| = 1\}$, and is *Lipschitz-continuous* with respect to u, v: there exists $L_H > 0$ such that

$$|H(u, v, \mathbf{n}) - H(u^*, v^*, \boldsymbol{n})| \leq L_H(|u - u^*| + |v - v^*|), \quad \text{for } u, v, u^*, v^* \in \mathbb{R}.$$

(H2) H is *consistent*:

$$H(u, u, \boldsymbol{n}) = \sum_{s=1}^{d} f_s(u) n_s, \quad u \in \mathbb{R},\ \boldsymbol{n} = (n_1, \ldots, n_d) \in B_1,$$

(H3) H is *conservative*:

$$H(u, v, \boldsymbol{n}) = -H(v, u, -\boldsymbol{n}), \quad u, v \in \mathbb{R},\ \boldsymbol{n} \in B_1.$$

In what follows, in the stability analysis we shall use properties **(H1)** and **(H2)**. Assumption **(H3)** is important for error estimation.

For a function φ defined in $\bigcup_{m=1}^{M} I_m$ we denote

$$\varphi_m^{\pm} = \varphi(t_m\pm) = \lim_{t \to t_m \pm} \varphi(t), \quad \{\varphi\}_m = \varphi(t_m+) - \varphi(t_m-), \tag{2.31}$$

if the one-sided limits $\varphi_m^{\pm}$ exist.

Now we define an ALE-STDG approximate solution of problem (2.11)–(2.13).

Definition 1. *A function U is an approximate solution of problem (2.11)–(2.13), if $U \in S_{h,\tau}^{p,q}$ and*

$$\int_{I_m} \Big((D_t U, \varphi)_{\Omega_t} + A_h(U, \varphi, t) + b_h(U, \varphi, t) + d_h(U, \varphi, t) \Big)\, \mathrm{d}t \tag{2.32}$$
$$+(\{U\}_{m-1}, \varphi_{m-1}^{+})_{\Omega_{t_{m-1}}} = \int_{I_m} l_h(\varphi, t)\, \mathrm{d}t \quad \forall \varphi \in S_{h,\tau}^{p,q}, \quad m = 1, \ldots, M,$$
$$U_0^- \in S_h^{p,0}, \quad (U_0^- - u^0, v_h)_{\Omega_0} = 0 \quad \forall v_h \in S_h^{p,0}. \tag{2.33}$$

(For $m = 1$ we set $\{U\}_{m-1} = \{U\}_0 := U_0^+ - U_0^-$ with U_0^- given by (2.33)*).*

The ALE-STDG numerical method (2.32)–(2.33) was applied in [22] and [62] in the computer programs for the numerical simulation of a compressible flow in time-dependent domains and fluid-structure interaction.

3. Analysis of the stability

In what follows we shall be concerned with the numerical solution of the ALE problem (2.11)-(2.13) by the space-time discontinuous Galerkin method. In the theoretical analysis a number of various constants will appear. Some important constants in main assertions will be denoted by $C_{T1},\ C_{T2}$, etc. in Theorem 1, Theorem 2, etc. and $C_{L9},\ C_{L10}$, etc. in Lemma 9, Lemma 10, etc. Inside proofs, constants are denoted locally by c, c_1, c_2, c^* etc. The aim of this notation is to show the continuity of individual theorems and lemmas.

3.1 Some auxiliary results

Similarly as in Section 2.1, we define the following Bochner spaces over a time interval $I_m,\ m = 1, \ldots, M$:

$$\begin{aligned}
&W^{1,\infty}(I_m; W^{1,\infty}(\Omega_t)) \\
&\qquad = \left\{ f \in L^\infty(I_m; W^{1,\infty}(\Omega_t));\ df/dt \in L^\infty(I_m; W^{1,\infty}(\Omega_t)) \right\}, \\
&W^{1,\infty}(I_m; W^{1,\infty}(\Omega_{t_{m-1}})) \\
&\qquad = \left\{ f \in L^\infty(I_m; W^{1,\infty}(\Omega_{t_{m-1}}));\ df/dt \in L^\infty(I_m; W^{1,\infty}(\Omega_{t_{m-1}})) \right\}, \\
&W^{1,\infty}(I_m; L^\infty(\Omega_t)) \\
&\qquad = \left\{ f \in L^\infty(I_m; L^\infty(\Omega_t));\ df/dt \in L^\infty(I_m; L^\infty(\Omega_t)) \right\}, \\
&W^{1,\infty}(I_m; L^\infty(\Omega_{t_{m-1}})) \\
&\qquad = \left\{ f \in L^\infty(I_m; L^\infty(\Omega_{t_{m-1}}));\ df/dt \in L^\infty(I_m; L^\infty(\Omega_{t_{m-1}})) \right\},
\end{aligned}$$

where df/dt denotes the distributional derivative.

In the space $H^1(\Omega_t, \mathcal{T}_{h,t})$ we define the norm

$$\|\varphi\|_{DG,t} = \left(\sum_{K \in \mathcal{T}_{h,t}} |\varphi|^2_{H^1(K)} + J_h(\varphi, \varphi, t) \right)^{1/2}. \tag{3.1}$$

Moreover, over $\partial\Omega$ we define the norm

$$\|u_D\|_{DGB,t} = \left(c_W \sum_{\Gamma \in \mathcal{F}^B_{h,t}} h(\Gamma)^{-1} \int_\Gamma |u_D|^2 \,\mathrm{d}S \right)^{1/2} = \left(J^B_h(u_D, u_D, t) \right)^{1/2}. \tag{3.2}$$

As was mentioned in Section 2.2.1, for each $t \in [0, T]$ we consider a system of triangulations $\{\mathcal{T}_{h,t}\}_{h \in (0, \overline{h})}$. We assume that these systems are uniformly shape regular. This means that there exists a positive constant c_R, independent of K, t and h such that

$$\begin{aligned}
&\frac{h_K}{\rho_K} \le c_R \quad \text{for all} \quad K \in \mathcal{T}_{h,t},\ h \in (0, \overline{h}), t \in [t_{m-1}, t_m], \\
&\tau_m \le \tau \in (0, \overline{\tau}),\ m = 1, \ldots, M.
\end{aligned} \tag{3.3}$$

By $(\mathcal{A}^{m-1}_{h,t})^{-1}$ we denote the inverse to the mapping $\mathcal{A}^{m-1}_{h,t}$. The symbols $\frac{\mathrm{d}\mathcal{A}^{m-1}_{h,t}}{\mathrm{d}X}$ and $\frac{\mathrm{d}(\mathcal{A}^{m-1}_{h,t})^{-1}}{\mathrm{d}x}$ denote the Jacobian matrices of $\mathcal{A}^{m-1}_{h,t}$ and $(\mathcal{A}^{m-1}_{h,t})^{-1}$, respectively.

Since mappings $\mathcal{A}_{h,t}^{m-1}$ and $(\mathcal{A}_{h,t}^{m-1})^{-1}$ are piecewise affine, the entries of $\frac{\mathrm{d}\mathcal{A}_{h,t}^{m-1}}{\mathrm{d}X}$ and $\frac{\mathrm{d}(\mathcal{A}_{h,t}^{m-1})^{-1}}{\mathrm{d}x}$ are constant on every element $\hat{K} \in \hat{\mathcal{T}}_{h,t_{m-1}}$ and $K \in \mathcal{T}_{h,t}$, respectively.

Moreover, we define the Jacobians $J(X,t) = \det\frac{\mathrm{d}\mathcal{A}_{h,t}^{m-1}(X)}{\mathrm{d}X}$, $X \in \Omega_{t_{m-1}}$, and $J^{-1}(x,t) = \det\frac{\mathrm{d}(\mathcal{A}_{h,t}^{m-1}(x))^{-1}}{\mathrm{d}x}$, $x \in \Omega_t$. The Jacobians J and J^{-1} are piecewise constant over $\hat{\mathcal{T}}_{h,t_{m-1}}$ and $\mathcal{T}_{h,t}$, respectively. The constant value of J on $\hat{K} \in \hat{\mathcal{T}}_{h,t_{m-1}}$ and of J^{-1} on $K \in \mathcal{T}_{h,t}$ will be denoted by $J_{\hat{K}}$ and J_K^{-1}, respectively. Of course, these terms depend on t and, hence, $J_{\hat{K}} = J_{\hat{K}}(t)$ and $J_K^{-1} = J_K^{-1}(t)$.

In what follows, we assume that

$$\mathcal{A}_{h,t}^{m-1} \in W^{1,\infty}(I_m; W^{1,\infty}(\Omega_{t_{m-1}})), \quad m = 1, \dots, M, \ h \in (0, \overline{h}) \tag{3.4}$$

and

$$(A_{h,t}^{m-1})^{-1} \in W^{1,\infty}(I_m; W^{1,\infty}(\Omega_t)), \quad m = 1, \dots M, \ h \in (0, \overline{h}). \tag{3.5}$$

Obviously, we have $J \in W^{1,\infty}(I_m; L^\infty(\Omega_{t_{m-1}}))$, $J^{-1} \in W^{1,\infty}(I_m; L^\infty(\Omega_t))$. Since $\mathcal{A}_{h,t_{m-1}}^{m-1}$ is the identical mapping and, hence, $J(X, t_{m-1}) = 1$, we assume that there exist constants C_J^-, $C_J^+ > 0$ such that the Jacobians satisfy the conditions

$$\begin{aligned} C_J^- \le J(X,t) \le C_J^+, \quad & X \in \overline{\Omega}_{t_{m-1}}, \ t \in \overline{I}_m, \ m = 1, \dots, M, \ h \in (0, \overline{h}), \quad (3.6)\\ (C_J^+)^{-1} \le J^{-1}(x,t) \le (C_J^-)^{-1}, \quad & x \in \overline{\Omega}_t, \ t \in \overline{I}_m, \ m = 1, \dots, M, \ h \in (0, \overline{h}). \end{aligned}$$

Finally, there exist constants $C_A^-, C_A^+, c_J > 0$ such that

$$\left\| \frac{\mathrm{d}\mathcal{A}_{h,t}^{m-1}(X)}{\mathrm{d}X} \right\| \le C_A^+, \ X \in \overline{\Omega}_{t_{m-1}}, \ t \in \overline{I}_m, \ m = 1, \dots, M, \ h \in (0, \overline{h}), \tag{3.7}$$

$$\left\| \frac{\mathrm{d}(\mathcal{A}_{h,t}^{m-1})^{-1}(x)}{\mathrm{d}x} \right\| \le C_A^-, \ x \in \overline{\Omega}_t, \ t \in \overline{I}_m, \ m = 1, \dots, M, \ h \in (0, \overline{h}), \tag{3.8}$$

$$|J_K'| = \left| \frac{\partial J_K}{\partial t} \right| \le c_J, \ K \in \mathcal{T}_{h,t} \ t \in \overline{I}_m, \ m = 1, \dots, M, \ h \in (0, \overline{h}), \tag{3.9}$$

where $\| \cdot \|$ is the matrix norm induced by the Euclidean norm $| \cdot |$ in $\mathbb{R}^d$.

The above assumptions imply the following properties of the domain velocity: There exists a constant $c_z > 0$ such that

$$|\boldsymbol{z}(x,t)|, \ |\mathrm{div}\boldsymbol{z}(x,t)| \le c_z \quad \text{for } x \in \Omega_t, \ t \in (0,T). \tag{3.10}$$

In what follows, for the sake of simplicity, we use the notation $\mathcal{A}_t$ for the ALE mapping defined in $\bigcup_{m=1}^M I_m$ so that

$$\mathcal{A}_t(X) = \mathcal{A}_{h,t}^{m-1}(X) \quad \text{for } X \in \overline{\Omega}_{t_{m-1}}, \ t \in \overline{I}_m, \ m = 1, \dots, M, \ h \in (0, \overline{h}). \tag{3.11}$$

The symbol $\mathcal{A}_t^{-1}$ will denote the inverse to $\mathcal{A}_t$. This means that $\mathcal{A}_t^{-1} : \overline{\Omega}_t \xrightarrow{\text{onto}} \overline{\Omega}_{t_{m-1}}$ for $t \in \overline{I}_m, \ m = 1, \dots, M$.

Under assumption (3.3), the multiplicative trace inequality and the inverse inequality hold: There exist constants $c_M, c_I > 0$ independent of v, h, t and K such that

$$\begin{aligned} &\|v\|_{L^2(\partial K)}^2 \le c_M \left(\|v\|_{L^2(K)} \, |v|_{H^1(K)} + h_K^{-1} \|v\|_{L^2(K)}^2 \right), \quad (3.12)\\ &v \in H^1(K), \, K \in \mathcal{T}_{h,t}, \ h \in (0, \overline{h}), \ t \in [0,T], \end{aligned}$$

and

$$|v|_{H^1(K)} \leq c_I\, h_K^{-1} \|v\|_{L^2(K)}, \quad v \in P^p(K),\, K \in \mathcal{T}_{h,t},\ h \in (0,\bar{h}),\ t \in [0,T]. \quad (3.13)$$

If we use $\varphi := U$ as a test function in (2.32), we get the basic identity

$$\int_{I_m} \Big((D_t U, U)_{\Omega_t} + A_h(U,U,t) + b_h(U,U,t) + d_h(U,U,t) \Big)\, \mathrm{d}t \qquad (3.14)$$
$$+ (\{U\}_{m-1}, U_{m-1}^{+})_{\Omega_{t_{m-1}}} = \int_{I_m} l_h(U,t)\, \mathrm{d}t.$$

3.2 Important estimates

In what follows we need to estimate each term in (3.14). These estimates can be found in Lemmas 1 - 6. Their proofs are mainly based on the multiplicative trace inequality (3.12), inverse inequality (3.13), Young's inequality and assumptions (2.5) of function β.

3.2.1 Coercivity of the diffusion and penalty term

For a sufficiently large constant c_W we obtain the coercivity of the diffusion and penalty terms.

Lemma 1. *Let*

$$c_W \geq \frac{\beta_1^2}{\beta_0^2} c_M (c_I + 1) \quad \textit{for} \quad \Theta = -1 \ (\textit{NIPG}), \qquad (3.15)$$

$$c_W \geq \frac{\beta_1^2}{\beta_0^2} c_M (c_I + 1) \quad \textit{for} \quad \Theta = 0 \ (\textit{IIPG}), \qquad (3.16)$$

$$c_W \geq \frac{16\beta_1^2}{\beta_0^2} c_M (c_I + 1) \quad \textit{for} \quad \Theta = 1 \ (\textit{SIPG}). \qquad (3.17)$$

Then

$$\int_{I_m} \left(a_h(U,U,t) + \beta_0\, J_h(U,U,t) \right)\, \mathrm{d}t \qquad (3.18)$$
$$\geq \frac{\beta_0}{2} \int_{I_m} \|U\|_{DG,t}^2 \, \mathrm{d}t - \frac{\beta_0}{2} \int_{I_m} \|u_D\|_{DGB,t}^2 \, \mathrm{d}t.$$

Proof. 1) Let $\Theta = -1$. Then from the definition of the forms we get

$$a_h(U,U,t) + \beta_0\, J_h(U,U,t)$$
$$= \sum_{K \in \mathcal{T}_{h,t}} \int_K \beta(U) \nabla U \cdot \nabla U \, \mathrm{d}x - \sum_{\Gamma \in \mathcal{F}_{h,t}^B} \int_\Gamma \beta(U) \nabla U \cdot \mathbf{n}_\Gamma\, u_D \, \mathrm{d}S + \beta_0\, J_h(U,U,t)$$

Using assumption (2.5) and the definition of the $\|\cdot\|_{DG,t}$-norm, we have

$$a_h(U,U,t) + \beta_0\, J_h(U,U,t) \geq \beta_0 \|U\|_{DG,t}^2 - \beta_1 \sum_{\Gamma \in \mathcal{F}_{h,t}^B} \int_\Gamma |\nabla U|\, |u_D| \, \mathrm{d}S. \qquad (3.19)$$

Now we have to estimate the last term on the right-hand side of (3.19). Using Young's inequality and the relation $h(\Gamma) \leq h_{K_\Gamma^{(L)}}$, for each $\varepsilon > 0$ we get

$$\beta_1 \sum_{\Gamma \in \mathcal{F}_{h,t}^B} \int_\Gamma |\nabla U| \, |u_D| \, \mathrm{d}S$$
$$\leq \frac{\beta_1 \varepsilon}{2} \sum_{\Gamma \in \mathcal{F}_{h,t}^B} \int_\Gamma h(\Gamma)^{-1} \, |u_D|^2 \, \mathrm{d}S + \frac{\beta_1}{2\varepsilon} \sum_{\Gamma \in \mathcal{F}_{h,t}^B} \int_\Gamma h_{K_\Gamma^{(L)}} \, |\nabla U|^2 \, \mathrm{d}S.$$

If we use the definition of the form J_h^B, we obtain

$$\beta_1 \sum_{\Gamma \in \mathcal{F}_{h,t}^B} \int_\Gamma |\nabla U| \, |u_D| \, \mathrm{d}S$$
$$\leq \frac{\beta_1 \varepsilon}{2 c_W} J_h^B(u_D, u_D) + \frac{\beta_1}{2\varepsilon} \sum_{\Gamma \in \mathcal{F}_{h,t}^B} \int_{\partial K_\Gamma^{(L)}} h_{K_\Gamma^{(L)}} \, |\nabla U|^2 \, \mathrm{d}S.$$

Now we express the first term on the right-hand side with the aid of the definition of the $\| \cdot \|_{DGB,t}$-norm and to the second term we apply the multiplicative trace inequality (3.12) and the inverse inequality (3.13). We get

$$\beta_1 \sum_{\Gamma \in \mathcal{F}_{h,t}^B} \int_\Gamma |\nabla U| \, |u_D| \, \mathrm{d}S \tag{3.20}$$
$$\leq \frac{\beta_1 \varepsilon}{2 c_W} \|u_D\|_{DGB,t}^2 + \frac{\beta_1}{2\varepsilon} c_M (c_I + 1) \sum_{K \in \mathcal{T}_{h,t}} \|\nabla U\|_{L^2(K)}^2.$$

If we use the inequality $\sum_{K \in \mathcal{T}_{h,t}} \|\nabla U\|_{L^2(K)}^2 \leq \|U\|_{DG,t}^2$, which obviously follows from the definition of the $\| \cdot \|_{DG,t}$-norm, we get

$$\beta_1 \sum_{\Gamma \in \mathcal{F}_{h,t}^B} \int_\Gamma |\nabla U| \, |u_D| \, \mathrm{d}S \tag{3.21}$$
$$\leq \frac{\beta_1 \varepsilon}{2 c_W} \|u_D\|_{DGB,t}^2 + \frac{\beta_1}{2\varepsilon} c_M (c_I + 1) \|U\|_{DG,t}^2.$$

Substituting back to (3.19), implies that

$$a_h(U, U, t) + \beta_0 \, J_h(U, U, t)$$
$$\geq \left(\beta_0 - \frac{\beta_1}{2\varepsilon} c_M (c_I + 1) \right) \|U\|_{DG,t}^2 - \frac{\beta_1 \varepsilon}{2 c_W} \|u_D\|_{DGB,t}^2.$$

If we set $\varepsilon = \frac{\beta_1}{\beta_0} c_M (c_I + 1)$, we get the inequality

$$a_h(U, U, t) + \beta_0 \, J_h(U, U, t)$$
$$\geq \frac{\beta_0}{2} \|U\|_{DG,t}^2 - \frac{\beta_1^2 c_M (c_I + 1)}{2 \beta_0 c_W} \|u_D\|_{DGB,t}^2.$$

Finally, this inequality, assumption (3.15) and integration over the interval I_m imply (3.18), what we wanted to prove.

2) Let $\Theta = 0$. From (2.24), assumption (2.5) and the definition of the $\|.\|_{DG,t}$-norm, we get

$$
\begin{aligned}
&a_h(U,U,t) + \beta_0\, J_h(U,U,t) \qquad (3.22)\\
&\geq \beta_0 \|U\|^2_{DG,t} - \beta_1 \sum_{\Gamma\in\mathcal{F}^I_{h,t}} \int_\Gamma |\langle \nabla U\rangle \cdot \mathbf{n}_\Gamma\, [U]|\,\mathrm{d}S - \beta_1 \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma |\nabla U \cdot \mathbf{n}_\Gamma\, U|\,\mathrm{d}S\\
&\geq \beta_0 \|U\|^2_{DG,t}\\
&\quad -\beta_1 \left(\sum_{\Gamma\in\mathcal{F}^I_{h,t}} \int_\Gamma \frac{|\nabla U^{(L)}_\Gamma| + |\nabla U^{(R)}_\Gamma|}{2}\, |[U]|\,\mathrm{d}S + \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma |\nabla U|\,|U|\,\mathrm{d}S \right).
\end{aligned}
$$

Now applying Young's inequality with $\delta > 0$ separately to the first and the second term above in round brackets and using the inequality $(a+b)^2 \leq 2(a^2+b^2)$ valid for $a, b \in \mathbb{R}$, we obtain

$$
\begin{aligned}
&\sum_{\Gamma\in\mathcal{F}^I_{h,t}} \int_\Gamma \frac{|\nabla U^{(L)}_\Gamma| + |\nabla U^{(R)}_\Gamma|}{2}\, |[U]|\,\mathrm{d}S + \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma |\nabla U|\,|U|\,\mathrm{d}S \qquad (3.23)\\
&\leq \frac{1}{2} \sum_{\Gamma\in\mathcal{F}^I_{h,t}} \int_\Gamma \frac{h(\Gamma)}{\delta c_W} \frac{\left(|\nabla U^{(L)}_\Gamma| + |\nabla U^{(R)}_\Gamma|\right)^2}{4}\,\mathrm{d}S + \frac{1}{2} \sum_{\Gamma\in\mathcal{F}^I_{h,t}} \int_\Gamma \frac{\delta c_W}{h(\Gamma)} |[U]|^2\,\mathrm{d}S\\
&\quad + \frac{1}{2} \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma \frac{h(\Gamma)}{\delta c_W} \left|\nabla U^{(L)}_\Gamma\right|^2\,\mathrm{d}S + \frac{1}{2} \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma \frac{\delta c_W}{h(\Gamma)} |U|^2 \mathrm{d}S\\
&\leq \frac{1}{2\delta c_W} \sum_{\Gamma\in\mathcal{F}^I_{h,t}} \int_\Gamma h(\Gamma) \frac{|\nabla U^{(L)}_\Gamma|^2 + |\nabla U^{(R)}_\Gamma|^2}{2}\,\mathrm{d}S\\
&\quad + \frac{1}{2\delta c_W} \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma h_{K^{(L)}_\Gamma} \left|\nabla U^{(L)}_\Gamma\right|^2\,\mathrm{d}S + \frac{\delta}{2} J_h(U,U,t).
\end{aligned}
$$

Using the inequality $h(\Gamma) \leq h_K$ for $\Gamma \subset \partial K$ we get

$$
\begin{aligned}
&\frac{1}{2\delta c_W} \sum_{\Gamma\in\mathcal{F}^I_{h,t}} \int_\Gamma h(\Gamma) \frac{|\nabla U^{(L)}_\Gamma|^2 + |\nabla U^{(R)}_\Gamma|^2}{2}\,\mathrm{d}S \qquad (3.24)\\
&\quad + \frac{1}{2\delta c_W} \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma h_{K^{(L)}_\Gamma} \left|\nabla U^{(L)}_\Gamma\right|^2\,\mathrm{d}S + \frac{\delta}{2} J_h(U,U,t)\\
&\leq \frac{1}{4\delta c_W} \sum_{\Gamma\in\mathcal{F}^I_{h,t}} \int_\Gamma \left(h_{K^{(L)}_\Gamma} |\nabla U^{(L)}_\Gamma|^2 + h_{K^{(R)}_\Gamma} |\nabla U^{(R)}_\Gamma|^2 \right)\mathrm{d}S\\
&\quad + \frac{1}{2\delta c_W} \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma h_{K^{(L)}_\Gamma} \left|\nabla U^{(L)}_\Gamma\right|^2\,\mathrm{d}S + \frac{\delta}{2} J_h(U,U,t)\\
&\leq \frac{1}{2\delta c_W} \sum_{K\in\mathcal{T}_{h,t}} \int_{\partial K} h_K |\nabla U|^2\,\mathrm{d}S + \frac{\delta}{2} J_h(U,U,t).
\end{aligned}
$$

The multiplicative trace inequality and the inverse inequality imply that

$$
\begin{aligned}
&\int_{\partial K} h_K |\nabla U|^2\,\mathrm{d}S = h_K \|\nabla U\|^2_{L^2(\partial K)} \qquad (3.25)\\
&\leq c_M(1+c_I) \|\nabla U\|^2_{L^2(K)} = c_M(1+c_I) |U|^2_{H^1(K)}.
\end{aligned}
$$

Now, summarizing (3.23)–(3.25) yields

$$\sum_{\Gamma\in\mathcal{F}^I_{h,t}} \int_\Gamma \frac{|\nabla U^{(L)}_\Gamma| + |\nabla U^{(R)}_\Gamma|}{2} \, |[U]| \, \mathrm{d}S + \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma |\nabla U| \, |U| \, \mathrm{d}S \tag{3.26}$$
$$\leq \frac{1}{2\delta c_W} \sum_{K\in\mathcal{T}_{h,t}} \int_{\partial K} h_K |\nabla U|^2 \, \mathrm{d}x + \frac{\delta}{2} J_h(U,U,t)$$
$$\leq c_M(1+c_I)\frac{1}{2\delta c_W} \sum_{K\in\mathcal{T}_{h,t}} |U|^2_{H^1(K)} + \frac{\delta}{2} J_h(U,U,t).$$

This and (3.22) imply that

$$a_h(U,U,t) + \beta_0 \, J_h(U,U,t) \tag{3.27}$$
$$\geq \beta_0 \|U\|^2_{DG,t} - \frac{\beta_1 c_M(1+c_I)}{2\delta c_W} \sum_{K\in\mathcal{T}_{h,t}} |U|^2_{H^1(K)} - \frac{\beta_1\delta}{2} J_{h,t}(U,U,t).$$

If we set $\delta = \frac{\beta_0}{\beta_1}$, we find that

$$a_h(U,U,t) + \beta_0 \, J_h(U,U,t) \tag{3.28}$$
$$\geq \beta_0 \|U\|^2_{DG,t} - \frac{\beta_1^2 c_M(1+c_I)}{2\beta_0 c_W} \sum_{K\in\mathcal{T}_{h,t}} |U|^2_{H^1(K)} - \frac{\beta_0}{2} J_h(U,U,t).$$

Using assumption (3.16) for the constant c_W and the definition (3.1) of the $\|\cdot\|_{DG,t}$-norm, we find that

$$a_h(U,U,t) + \beta_0 \, J_h(U,U,t) \geq \frac{\beta_0}{2} \|U\|^2_{DG,t}. \tag{3.29}$$

Integrating both sides over the interval I_m, we finally get (3.18).

3) Let $\Theta = 1$. From assumption (2.5) and the definition of the $\|\cdot\|_{DG,t}$-norm, we get

$$a_h(U,U,t) + \beta_0 \, J_h(U,U,t) \tag{3.30}$$
$$\geq \beta_0 \|U\|^2_{DG,t}$$
$$-2\beta_1 \sum_{\Gamma\in\mathcal{F}^I_{h,t}} \int_\Gamma |\langle\nabla U\rangle \cdot \mathbf{n}_\Gamma \, [U]| \, \mathrm{d}S - 2\beta_1 \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma |\nabla U \cdot \mathbf{n}_\Gamma \, U| \, \mathrm{d}S$$
$$-\beta_1 \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma |\nabla U \cdot \mathbf{n}_\Gamma u_D| \, \mathrm{d}S$$
$$\geq \beta_0 \|U\|^2_{DG,t}$$
$$-2\beta_1 \left(\sum_{\Gamma\in\mathcal{F}^I_{h,t}} \int_\Gamma \frac{|\nabla U^{(L)}_\Gamma| + |\nabla U^{(R)}_\Gamma|}{2} \, |[U]| \, \mathrm{d}S + \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma |\nabla U| \, |U| \, \mathrm{d}S \right)$$
$$-\beta_1 \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma |\nabla U| \, |u_D| \, \mathrm{d}S.$$

Expression in round brackets has already been estimated in (3.26). It follows from (3.30) and (3.26) that

$$
\begin{aligned}
&a_h(U,U,t) + \beta_0\, J_h(U,U,t) \qquad (3.31)\\
&\geq \beta_0 \|U\|^2_{DG,t} - \frac{\beta_1 c_M (1+c_I)}{\delta c_W} \sum_{K\in\mathcal{T}_{h,t}} |U|^2_{H^1(K)} - \beta_1 \delta J_h(U,U,t)\\
&\quad -\beta_1 \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma |\nabla U|\,|u_D| \,\mathrm{d}S.
\end{aligned}
$$

The last term on the right-hand side has been estimated in (3.21) with arbitrary $\varepsilon > 0$. Substituting it into (3.31), we obtain

$$
\begin{aligned}
&a_h(U,U,t) + \beta_0\, J_h(U,U,t) \qquad (3.32)\\
&\geq \beta_0 \|U\|^2_{DG,t} - \frac{\beta_1 c_M (c_I+1)}{\delta c_W} \sum_{K\in\mathcal{T}_{h,t}} |U|^2_{H^1(K)} - \beta_1 \delta J_h(U,U,t)\\
&\quad -\frac{\beta_1 \varepsilon}{2 c_W} \|u_D\|^2_{DGB,t} - \frac{\beta_1}{2\varepsilon} c_M (c_I+1) \|U\|^2_{DG,t}.
\end{aligned}
$$

If we set $\delta := \frac{\beta_0}{4\beta_1}$ and $\varepsilon := 4\frac{\beta_1}{\beta_0} c_M(c_I+1)$, we find that

$$
\begin{aligned}
&a_h(U,U,t) + \beta_0\, J_h(U,U,t) \qquad (3.33)\\
&\geq \beta_0 \|U\|^2_{DG,t} - \frac{4\beta_1^2 c_M (c_I+1)}{\beta_0 c_W} \sum_{K\in\mathcal{T}_{h,t}} |U|^2_{H^1(K)} - \frac{\beta_0}{4} J_h(U,U,t)\\
&\quad -\frac{2\beta_1^2}{\beta_0 c_W} c_M (c_I+1) \|u_D\|^2_{DGB,t} - \frac{\beta_0}{4} \|U\|^2_{DG,t}.
\end{aligned}
$$

Using assumption (3.17) for the constant c_W implies that

$$
\begin{aligned}
&a_h(U,U,t) + \beta_0\, J_h(U,U,t) \qquad (3.34)\\
&\geq \beta_0 \|U\|^2_{DG,t} - \frac{\beta_0}{4} \sum_{K\in\mathcal{T}_{h,t}} |U|^2_{H^1(K)} - \frac{\beta_0}{4} J_h(U,U,t)\\
&\quad -\frac{\beta_0}{8} \|u_D\|^2_{DGB,t} - \frac{\beta_0}{4} \|U\|^2_{DG,t}\\
&\geq \frac{\beta_0}{2} \|U\|^2_{DG,t} - \frac{\beta_0}{2} \|u_D\|^2_{DGB,t}.
\end{aligned}
$$

Finally, integrating over the interval I_m, we get (3.18). □

3.2.2 Estimates of the convective terms

Further, we estimate the convective terms $b_h(U,U,t)$ and $d_h(U,U,t)$.

Lemma 2. *For each $k_1 > 0$ there exists a constant $c_b > 0$ such that for the approximate solution U of problem (2.11)–(2.13) we have the inequality*

$$
\int_{I_m} |b_h(U,U,t)| \mathrm{d}t \leq \frac{\beta_0}{k_1} \int_{I_m} \|U\|^2_{DG,t} \mathrm{d}t + c_b \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t. \qquad (3.35)
$$

(The constant c_b depends on k_1, namely, $c_b = c_1^2 \frac{k_1}{\beta_0}$, where $c_1 > 0$ is independent of k_1.)

Proof. By (2.28),

$$b_h(U,U,t) = \underbrace{-\sum_{K\in\mathcal{T}_{h,t}}\int_K \sum_{s=1}^d f_s(U)\frac{\partial U}{\partial x_s}\,\mathrm{d}x}_{:=\sigma_1} \tag{3.36}$$

$$\underbrace{+\sum_{\Gamma\in\mathcal{F}^I_{h,t}}\int_\Gamma H(U_\Gamma^{(L)},U_\Gamma^{(R)},\mathbf{n}_\Gamma)\,[U]_\Gamma\,\mathrm{d}S + \sum_{\Gamma\in\mathcal{F}^B_{h,t}}\int_\Gamma H(U_\Gamma^{(L)},U_\Gamma^{(L)},\mathbf{n}_\Gamma)\,U|_\Gamma\,\mathrm{d}S.}_{:=\sigma_2}$$

Then from the Lipschitz-continuity of the functions f_s, $s=1,\dots,d$, with the modul $L_f>0$, assumption that $f_s(0)=0$ and the Cauchy inequality, we obtain

$$\begin{aligned} |\sigma_1| &\le \sum_{K\in\mathcal{T}_{h,t}}\int_K\sum_{s=1}^d |f_s(U)-f_s(0)|\left|\frac{\partial U}{\partial x_s}\right|\mathrm{d}x \\ &\le L_f\sum_{K\in\mathcal{T}_{h,t}}\int_K\sum_{s=1}^d |U|\left|\frac{\partial U}{\partial x_s}\right|\mathrm{d}x \le L_f\sqrt{d}\,\|U\|_{L^2(\Omega)}|U|_{H^1(\Omega_t,\mathcal{T}_{h,t})}. \end{aligned} \tag{3.37}$$

Now we shall estimate σ_2. From the relation $f_s(0)=0$, $s=1,\dots,d$, and the consistency of property **(H2)** of the numerical flux H we have $H(0,0,\mathbf{n}_\Gamma)=0$. Then we can use the Lipschitz-continuity of H and get

$$\begin{aligned} |\sigma_2| \le\ & L_H\sum_{\Gamma\in\mathcal{F}^I_{h,t}}\int_\Gamma(|U_\Gamma^{(L)}|+|U_\Gamma^{(R)}|)\,|[U]_\Gamma|\,\mathrm{d}S \\ &+ L_H\sum_{\Gamma\in\mathcal{F}^B_{h,t}}\int_\Gamma(|U_\Gamma^{(L)}|+|U_\Gamma^{(L)}|)\,|U|_\Gamma^{(L)}|\,\mathrm{d}S. \end{aligned}$$

Using that $U_\Gamma^{(R)}=U_\Gamma^{(L)}$ for $\Gamma\in\mathcal{F}^B_{h,t}$, Cauchy inequality, and the relation $h(\Gamma)\le h_K$, if $\Gamma\subset\partial K$, we obtain

$$\begin{aligned} |\sigma_2| &\le L_H\sum_{\Gamma\in\mathcal{F}^I_{h,t}}\int_\Gamma(|U_\Gamma^{(L)}|+|U_\Gamma^{(R)}|)\,|U_\Gamma^{(L)}|\,\mathrm{d}S \\ &\quad +L_H\sum_{\Gamma\in\mathcal{F}^B_{h,t}}\int_\Gamma(|U_\Gamma^{(L)}|+|U_\Gamma^{(R)}|)\,|U|_\Gamma^{(L)}|\,\mathrm{d}S \\ &\le \frac{L_H}{\sqrt{c_W}}\left(c_W\sum_{\Gamma\in\mathcal{F}^I_{h,t}}\int_\Gamma\frac{|U_\Gamma^{(L)}|^2}{h(\Gamma)}\,\mathrm{d}S + c_W\sum_{\Gamma\in\mathcal{F}^B_{h,t}}\int_\Gamma\frac{|U_\Gamma^{(L)}|^2}{h(\Gamma)}\,\mathrm{d}S\right)^{1/2} \\ &\quad\times\left(\sum_{\Gamma\in\mathcal{F}_{h,t}}h(\Gamma)\int_\Gamma(|U_\Gamma^{(L)}|+|U_\Gamma^{(R)}|)^2\,\mathrm{d}S\right)^{1/2} \end{aligned} \tag{3.38}$$

$$
\begin{aligned}
&\le \frac{L_H}{\sqrt{c_W}}\, J_h(U,U,t)^{1/2} \left(\sum_{\Gamma\in\mathcal{F}_{h,t}} 2h(\Gamma) \int_\Gamma (|U_\Gamma^{(L)}|^2 + |U_\Gamma^{(R)}|^2)\,\mathrm{d}S \right)^{1/2} \\
&\le L_H\, J_h(U,U,t)^{1/2} \\
&\quad\times \left(\sum_{\Gamma\in\mathcal{F}_{h,t}} h_{K_\Gamma^{(L)}} \int_{\partial K_\Gamma^{(L)}\cap\Gamma} |U_\Gamma^{(L)}|^2 dS + h_{K_\Gamma^{(R)}} \int_{\partial K_\Gamma^{(R)}\cap\Gamma} |U_\Gamma^{(R)}|^2\,\mathrm{d}S \right)^{1/2} \\
&\le L_H\, J_h(U,U,t)^{1/2} \left(\sum_{K\in\mathcal{T}_{h,t}} \int_{\partial K} h_K |U|^2\,\mathrm{d}S \right)^{1/2} \\
&= L_H\, J_h(U,U,t)^{1/2} \left(\sum_{K\in\mathcal{T}_{h,t}} h_K \|U\|^2_{L^2(\partial K)} \right)^{1/2}.
\end{aligned}
$$

Substituting (3.37) and (3.38) into (3.36), using the Cauchy inequality and the definition of the $\|\cdot\|_{DG,t}$-norm, we find that

$$
\begin{aligned}
&|b_h(U,U,t)| \\
&\le L_f\sqrt{d}\,\|U\|\,|U|_{H^1(\Omega_t,\mathcal{T}_{h,t})} + L_H\, J_h(U,U,t)^{1/2} \left(\sum_{K\in\mathcal{T}_{h,t}} h_K \|U\|^2_{L^2(\partial K)} \right)^{1/2} \\
&\le \left(L_f^2 d\,\|U\|^2 + L_H^2 \sum_{K\in\mathcal{T}_{h,t}} h_K \|U\|^2_{L^2(\partial K)} \right)^{1/2} \left(|U|^2_{H^1(\Omega_t,\mathcal{T}_{h,t})} + J_h(U,U,t) \right)^{1/2} \\
&\le c\,\|U\|_{DG,t} \left(\|U\| + \left(\sum_{K\in\mathcal{T}_{h,t}} h_K \|U\|^2_{L^2(\partial K)} \right)^{1/2} \right),
\end{aligned}
$$

where $c = \left(\max\{L_f^2\, d,\, L_H^2\}\right)^{1/2}$. Furthermore, the multiplicative trace inequality and the inverse inequality imply that

$$
\begin{aligned}
\sum_{K\in\mathcal{T}_{h,t}} h_K \|U\|^2_{L^2(\partial K)} &\le c_M \sum_{K\in\mathcal{T}_{h,t}} h_K \left(\|U\|_{L^2(K)} |U|_{H^1(K)} + h_K^{-1} \|U\|^2_{L^2(K)} \right) \\
&\le c_M(c_I+1) \sum_{K\in\mathcal{T}_{h,t}} \|U\|^2_{L^2(K)} = c_M(c_I+1)\|U\|^2.
\end{aligned}
$$

Hence, from this relation and Young's inequality we get

$$
\begin{aligned}
&|b_h(U,U,t)| \le c\,\|U\|_{DG,t} \left(\|U\| + \left(\sum_{K\in\mathcal{T}_{h,t}} h_K \|U\|^2_{L^2(\partial K)} \right)^{1/2} \right) \\
&\le c_1\,\|U\|_{DG,t}\,\|U\| \le \frac{\beta_0}{k_1} \|U\|^2_{DG,t} + c_1^2 \frac{k_1}{\beta_0} \|U\|^2 = \frac{\beta_0}{k_1}\,\|U\|^2_{DG,t} + c_b\,\|U\|^2,
\end{aligned}
$$

where $c_1 = c(1+\sqrt{c_M(c_I+1)})$, $k_1 > 0$ and $c_b = c_1^2 \frac{k_1}{\beta_0}$. Integrating over the interval I_m, we finally have (3.35). □

Lemma 3. *For every $k_2 > 0$ there exists a constant $c_d > 0$ such that for the approximate solution U of problem (2.11)–(2.13) we have the inequality*

$$
\int_{I_m} |d_h(U,U,t)|\,\mathrm{d}t \le \frac{\beta_0}{2k_2} \int_{I_m} \|U\|^2_{DG,t}\,\mathrm{d}t + \frac{c_d}{2\beta_0} \|U\|^2_{\Omega_t}\,\mathrm{d}t. \qquad (3.39)
$$

Proof. By (2.29), (3.10) and the Cauchy and Young's inequalities,

$$\begin{aligned}\int_{I_m} |d_h(U,U,t)|\,\mathrm{d}t &\le c_z \int_{I_m} \sum_{K\in\mathcal{T}_{h,t}} \int_K \sum_{s=1}^{d} |U||\frac{\partial U}{\partial x_s}|\,\mathrm{d}x\,\mathrm{d}t \\ &\le c_z \int_{I_m} \|U\|_{\Omega_t} |U|_{H^1(\Omega_t,\mathcal{T}_{h,t})}\,\mathrm{d}t \\ &\le c_z \int_{I_m} \|U\|_{\Omega_t} \|U\|_{DG,t}\,\mathrm{d}t \\ &\le \frac{\beta_0}{2k_2}\int_{I_m} \|U\|^2_{DG,t}\,\mathrm{d}t + \frac{c_z^2 k_2}{2\beta_0}\int_{I_m} \|U\|^2_{\Omega_t}\,\mathrm{d}t,\end{aligned}$$

which is (3.39) with $c_d = c_z^2 k_2$. □

3.2.3 Estimate of the right-hand side form

Now we continue with the estimate of the right-hand side form $l_h(U,t)$.

Lemma 4. *For the approximate solution U of problem (2.11)–(2.13) and any $k_3 > 0$ we have*

$$\begin{aligned}&\int_{I_m} |l_h(U,t)|\,\mathrm{d}t \qquad (3.40)\\ &\le \frac{1}{2}\int_{I_m}\left(\|g\|^2_{\Omega_t} + \|U\|^2_{\Omega_t}\right)\,\mathrm{d}t + \beta_0 k_3 \int_{I_m} \|u_D\|^2_{DGB,t}\,\mathrm{d}t + \frac{\beta_0}{k_3}\int_{I_m} \|U\|^2_{DG,t}\,\mathrm{d}t.\end{aligned}$$

Proof. It follows from (2.30) that

$$|l_h(U,t)| = |(g,U) + \beta_0\, c_W \sum_{\Gamma\in\mathcal{F}^B_{h,t}} h(\Gamma)^{-1} \int_\Gamma u_D|_\Gamma\, U|_\Gamma\,\mathrm{d}S|.$$

After using the Cauchy inequality for the first term on the right-hand side and applying Young's inequality with $k_3 > 0$ for the second term, we find that

$$\begin{aligned}&|(g,U) + \beta_0\, c_W \sum_{\Gamma\in\mathcal{F}^B_{h,t}} h(\Gamma)^{-1} \int_\Gamma u_D|_\Gamma\, U|_\Gamma\,\mathrm{d}S| \\ &\le \frac{1}{2}(\|g\|^2_{\Omega_t} + \|U\|^2_{\Omega_t}) + \beta_0 k_3\, \underbrace{c_W \sum_{\Gamma\in\mathcal{F}^B_{h,t}} h^{-1}_{K_\Gamma^{(L)}} \int_\Gamma |u_D|_\Gamma|^2\,\mathrm{d}S}_{=\|u_D\|^2_{DGB,t}} \\ &\quad + \frac{\beta_0}{k_3}\, \underbrace{c_W \sum_{\Gamma\in\mathcal{F}^B_{h,t}} h^{-1}_{K_\Gamma^{(L)}} \int_\Gamma |U|_\Gamma|^2\,\mathrm{d}S}_{\le J_h(U,U,t)\le \|U\|^2_{DG,t}}.\end{aligned}$$

Hence,

$$|l_h(U,t)| \le \frac{1}{2}(\|g\|^2_{\Omega_t} + \|U\|^2_{\Omega_t}) + \beta_0 k_3 \|u_D\|^2_{DGB,t} + \frac{\beta_0}{k_3}\|U\|^2_{DG,t},$$

from which we get (3.40) by integrating both sides over the interval I_m. □

3.2.4 Estimates of the ALE derivative

Finally we need to estimate the term with the ALE derivative. In the proof we will use the Reynolds transport theorem.

Lemma 5. *(Reynolds transport theorem) Let $\mathcal{A} \in W^{1,\infty}(0,T;W^{1,\infty}(\Omega_t))$ be an ALE-mapping with domain velocity $\boldsymbol{z}(x,t)$ defined in* (2.8). *For any $t \in [0,T]$ and $v(x,t) \in W^{1,\infty}(\Omega_t)$ it holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} v(x,t)\mathrm{d}x = \int_{\Omega_t} \left(D_t v(x,t) + v(x,t)\ \mathrm{div}\,\boldsymbol{z}(x,t)\right)\mathrm{d}x, \tag{3.41}$$

where the derivative $D_t v$ is defined in (2.9).

Proof. See, e.g. [41] or [1]. □

Lemma 6. *It holds that*

$$\int_{I_m} (D_t U, U)_{\Omega_t}\,\mathrm{d}t \geq \frac{1}{2}\left(\|U_m^-\|^2_{\Omega_{t_m}} - \|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} - c_z \int_{I_m} \|U\|^2_{\Omega_t}\,\mathrm{d}t\right), \tag{3.42}$$

$$\begin{aligned}&\left(\{U\}_{m-1}, U_{m-1}^+\right)_{\Omega_{t_{m-1}}} \\ &\quad = \frac{1}{2}\left(\|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} + \|\{U\}_{m-1}\|^2_{\Omega_{t_{m-1}}} - \|U_{m-1}^-\|^2_{\Omega_{t_{m-1}}}\right),\end{aligned} \tag{3.43}$$

$$\begin{aligned}&\int_{I_m} (D_t U, U)_{\Omega_t}\,\mathrm{d}t + \left(\{U\}_{m-1}, U_{m-1}^+\right)_{\Omega_{t_{m-1}}} \\ &\quad \geq \frac{1}{2}\|U_m^-\|^2_{\Omega_{t_m}} + \frac{1}{2}\|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} - \frac{c_z}{2}\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t - \left(U_{m-1}^-, U_{m-1}^+\right)_{\Omega_{t_{m-1}}}.\end{aligned} \tag{3.44}$$

Proof. We start with the first inequality. We have

$$\int_{I_m} (D_t U, U)_{\Omega_t}\,\mathrm{d}t = \int_{I_m} \sum_{K \in \mathcal{T}_{h,t}} (D_t U, U)_K\,\mathrm{d}t. \tag{3.45}$$

By virtue of relation (2.15), the Reynolds transport theorem (3.41) and relation (2.10), we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_K U^2(x,t)\,\mathrm{d}x \\ &= \int_K \left(\frac{\partial U^2(x,t)}{\partial t} + \boldsymbol{z}(x,t)\cdot\nabla(U^2(x,t)) + U^2(x,t)\mathrm{div}\,\boldsymbol{z}(x,t)\right)\mathrm{d}x \\ &= \int_K \left(2U(x,t)\left(\frac{\partial U(x,t)}{\partial t} + \boldsymbol{z}(x,t)\cdot\nabla U(x,t)\right) + U^2(x,t)\mathrm{div}\,\boldsymbol{z}(x,t)\right)\mathrm{d}x \\ &= 2(D_t U, U)_K + (U^2, \mathrm{div}\,\boldsymbol{z})_K.\end{aligned} \tag{3.46}$$

Expressing $(D_t U, U)_K$, summing over $K \in \mathcal{T}_{h,t}$ and integrating over I_m together with assumption (3.10) and using Fubini's theorem yield

$$\begin{aligned}\int_{I_m} (D_t U, U)_{\Omega_t}\,\mathrm{d}t &= \frac{1}{2}\int_{I_m}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} U^2\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{2}\int_{I_m}(U^2, \mathrm{div}\,\boldsymbol{z})_{\Omega_t}\,\mathrm{d}t \\ &= \frac{1}{2}\|U_m^-\|^2_{\Omega_{t_m}} - \frac{1}{2}\|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} - \frac{1}{2}\int_{I_m}(U^2, \mathrm{div}\,\boldsymbol{z})_{\Omega_t}\,\mathrm{d}t \\ &\geq \frac{1}{2}\|U_m^-\|^2_{\Omega_{t_m}} - \frac{1}{2}\|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} - \frac{c_z}{2}\int_{I_m}\|U\|^2_{\Omega_t}\,\mathrm{d}t,\end{aligned} \tag{3.47}$$

which gives (3.42).

Further, we find that

$$
\begin{aligned}
&2(U^+_{m-1} - U^-_{m-1}, U^+_{m-1})_{\Omega_{t_{m-1}}} \\
&= (U^+_{m-1} - U^-_{m-1}, U^+_{m-1})_{\Omega_{t_{m-1}}} + (U^+_{m-1} - U^-_{m-1}, U^+_{m-1})_{\Omega_{t_{m-1}}} \\
&= \|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}} - (U^-_{m-1}, U^+_{m-1})_{\Omega_{t_{m-1}}} + (U^+_{m-1} - U^-_{m-1}, U^+_{m-1})_{\Omega_{t_{m-1}}} \\
&= \|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}} - (U^-_{m-1}, U^+_{m-1})_{\Omega_{t_{m-1}}} + (U^+_{m-1} - U^-_{m-1}, U^+_{m-1} - U^-_{m-1})_{\Omega_{t_{m-1}}} \\
&\qquad\qquad + (U^+_{m-1} - U^-_{m-1}, U^-_{m-1})_{\Omega_{t_{m-1}}} \\
&= \|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}} - (U^-_{m-1}, U^+_{m-1})_{\Omega_{t_{m-1}}} + \|\{U\}_{m-1}\|^2_{\Omega_{t_{m-1}}} \\
&\qquad\qquad + (U^+_{m-1}, U^-_{m-1})_{\Omega_{t_{m-1}}} - \|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}} \\
&= \|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}} + \|\{U\}_{m-1}\|^2_{\Omega_{t_{m-1}}} - \|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}},
\end{aligned}
$$

which immediately implies (3.43).

Concerning inequality (3.44), from (3.47) we get

$$
\begin{aligned}
&\int_{I_m} (D_t U, U)_{\Omega_t} \mathrm{d}t + \Big(\{U\}_{m-1}, U^+_{m-1}\Big)_{\Omega_{t_{m-1}}} \\
&= \frac{1}{2} \int_{I_m} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_t} U^2 \, \mathrm{d}x \, \mathrm{d}t - \frac{1}{2} \int_{I_m} (U^2, \operatorname{div} \boldsymbol{z})_{\Omega_t} \mathrm{d}t + \Big(U^+_{m-1} - U^-_{m-1}, U^+_{m-1}\Big)_{\Omega_{t_{m-1}}} \\
&= \frac{1}{2}\|U^-_m\|^2_{\Omega_{t_m}} - \frac{1}{2}\|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}} - \frac{1}{2} \int_{I_m} (U^2, \operatorname{div} \boldsymbol{z})_{\Omega_t} \mathrm{d}t \\
&\qquad\qquad + \|U^+_{m-1}\|_{\Omega_{t_{m-1}}} - (U^-_{m-1}, U^+_{m-1})_{\Omega_{t_{m-1}}} \\
&\geq \frac{1}{2} \left(\|U^-_m\|^2_{\Omega_{t_m}} + \|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}} - c_z \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t \right) - \Big(U^-_{m-1}, U^+_{m-1}\Big)_{\Omega_{t_{m-1}}},
\end{aligned}
$$

which proves the lemma. □

3.3 Discrete characteristic function

In our further considerations, the concept of a discrete characteristic function generalized to time-dependent domains will play an important role. The discrete characteristic function was introduced in [23] in the framework of the time discontinuous Galerkin method combined with conforming finite elements applied to a linear parabolic problem. The discrete characteristic function was generalized in connection with the STDGM for nonlinear parabolic problems in [9], [20], [31]. Here it is generalized to time-dependent domains.

3.3.1 Definition of the discrete characteristic function

For $m = 1, \ldots, M$ we use the following notation:

$$U = U(x, t), \quad x \in \Omega_t, \; t \in I_m \tag{3.48}$$

will denote the approximate solution in Ω_t, and

$$\tilde{U} = \tilde{U}(X, t) = U(\mathcal{A}_t(X), t), \quad X \in \Omega_{t_{m-1}}, \; t \in I_m \tag{3.49}$$

denotes the approximate solution transformed to the reference domain $\Omega_{t_{m-1}}$.

For $s \in I_m$ by $\tilde{\mathcal{U}}_s = \tilde{\mathcal{U}}_s(X,t)$, $X \in \Omega_{t_{m-1}}$, $t \in I_m$, we denote the discrete characteristic function to $\tilde{U}$ at a point $s \in I_m$. It is defined as $\tilde{\mathcal{U}}_s \in P^q(I_m; S_h^{p,m-1})$ such that

$$\begin{aligned} \int_{I_m} (\tilde{\mathcal{U}}_s, \varphi)_{\Omega_{t_{m-1}}} \,\mathrm{d}t &= \int_{t_{m-1}}^{s} (\tilde{U}, \varphi)_{\Omega_{t_{m-1}}} \,\mathrm{d}t \quad \forall \varphi \in P^{q-1}(I_m; S_h^{p,m-1}), && (3.50)\\ \tilde{\mathcal{U}}_s(X, t_{m-1}+) &= \tilde{U}(X, t_{m-1}+),\ X \in \Omega_{t_{m-1}}. && (3.51) \end{aligned}$$

The existence and uniqueness of the discrete characteristic function satisfying (3.50)–(3.51) is proved in the monograph [31].

Further, we introduce the discrete characteristic function $\mathcal{U}_s = \mathcal{U}_s(x,t)$, $x \in \Omega_t, t \in I_m$ to $U \in S_{h,\tau}^{p,q}$ at a point $s \in I_m$:

$$\mathcal{U}_s(x,t) = \tilde{\mathcal{U}}_s(\mathcal{A}_t^{-1}(x), t),\ x \in \Omega_t,\ t \in I_m. \tag{3.52}$$

Hence, in view of (2.20), $\mathcal{U}_s \in S_{h,\tau}^{p,q}$ and for $X \in \Omega_{t_{m-1}}$ we have

$$\mathcal{U}_s(X, t_{m-1}+) = U(X, t_{m-1}+). \tag{3.53}$$

3.3.2 Continuity of the discrete characteristic function

In what follows, we prove some important properties of the discrete characteristic function. Namely, we prove that the discrete characteristic function mapping $U \to \mathcal{U}_s$ is continuous with respect to the norms $\|\cdot\|_{L^2(\Omega_t)}$ and $\|\cdot\|_{DG,t}$. In the proof we use a result from [9] for the discrete characteristic function on a reference domain:

There exists a constant $\tilde{c}_{CH}^{(1)} > 0$ depending on q only such that

$$\int_{I_m} \|\tilde{\mathcal{U}}_s\|^2_{\Omega_{t_{m-1}}} \,\mathrm{d}t \;\le\; \tilde{c}_{CH}^{(1)} \int_{I_m} \|\tilde{U}\|^2_{\Omega_{t_{m-1}}} \,\mathrm{d}t, \tag{3.54}$$

for all $m = 1, \ldots, M$ and $h \in (0, \overline{h})$.

Lemma 7. *There exist constants $C_{L7}^{*},\ C_{L7}^{**} > 0$ such that*

$$C_{L7}^{*}\, h(\hat{\Gamma})^{-1} \le h(\Gamma)^{-1} \le C_{L7}^{**}\, h(\hat{\Gamma})^{-1} \tag{3.55}$$

for all $\hat{\Gamma} \in \mathcal{F}_{h,t_{m-1}}, \Gamma = \mathcal{A}_t(\hat{\Gamma}) \in \mathcal{F}_{h,t}$ and all $t \in \overline{I}_m,\ m = 1, \ldots, M,\ h \in (0, \overline{h})$.

Proof. We use the relation between Γ and $\hat{\Gamma}$ and the properties (3.7) and (3.8) of the mappings $\mathcal{A}_t$ and $\mathcal{A}_t^{-1}$. We also take into account that $\hat{\Gamma} \subset \hat{K}$ for some $\hat{K} \in \hat{\mathcal{T}}_{h,t_{m-1}}$, $\Gamma \subset K = \mathcal{A}_t(\hat{K}) \in \mathcal{T}_{h,t}$ and that the Jacobian matrices $\frac{d\mathcal{A}_t}{\mathrm{d}X}$ and $\frac{d\mathcal{A}_t^{-1}}{\mathrm{d}x}$ are constant on $\hat{K}$ and K, respectively. Then we can write

$$\begin{aligned} h(\Gamma) &= \mathrm{diam}(\Gamma) = \max_{x,x^* \in \Gamma} |x - x^*| = \max_{X,X^* \in \hat{\Gamma}} |\mathcal{A}_t(X) - \mathcal{A}_t(X^*)| \\ &\le \max_{X \in \hat{\Gamma}} \left\| \frac{\mathrm{d}\mathcal{A}_t(X)}{\mathrm{d}X} \right\| \max_{X,X^* \in \hat{\Gamma}} |X - X^*| \le C_A^{+} \max_{X,X^* \in \hat{\Gamma}} |X - X^*| = C_A^{+}\, h(\hat{\Gamma}). \end{aligned}$$

Similarly, we get

$$\begin{aligned} h(\hat\Gamma) &= \mathrm{diam}(\hat\Gamma) = \max_{X,X^*\in\hat\Gamma} |X - X^*| = \max_{x,x^*\in\Gamma} |\mathcal{A}_t^{-1}(x) - \mathcal{A}_t^{-1}(x^*)| \\ &\le \max_{x\in\Gamma} \left\| \frac{\mathrm{d}\mathcal{A}_t^{-1}(x)}{\mathrm{d}x} \right\| \max_{x,x^*\in\Gamma} |x - x^*| \le C_A^- \max_{x,x^*\in\Gamma} |x - x^*| = C_A^- \, h(\Gamma). \end{aligned}$$

These inequalities immediately imply (3.55) with constants $C_{L7}^* = (C_A^+)^{-1}$ and $C_{L7}^{**} = C_A^-$. □

Theorem 1. *There exist constants $c_{CH}^{(1)}, c_{CH}^{(2)} > 0$, such that*

$$\int_{I_m} \|\mathcal{U}_s\|_{\Omega_t}^2 \, \mathrm{d}t \le c_{CH}^{(1)} \int_{I_m} \|U\|_{\Omega_t}^2 \, \mathrm{d}t \tag{3.56}$$

$$\int_{I_m} \|\mathcal{U}_s\|_{DG,t}^2 \, \mathrm{d}t \le c_{CH}^{(2)} \int_{I_m} \|U\|_{DG,t}^2 \, \mathrm{d}t \tag{3.57}$$

for all $s \in I_m$, $m = 1, \ldots, M$ and $h \in (0, \overline{h})$.

Proof. We begin with the proof of the first inequality. We have

$$\begin{aligned} &\|\mathcal{U}_s(t)\|_{\Omega_t}^2 = \int_{\Omega_t} |\mathcal{U}_s(x,t)|^2 \, \mathrm{d}x = \int_{\Omega_t} |\tilde{\mathcal{U}}_s(\mathcal{A}_t^{-1}(x), t)|^2 \, \mathrm{d}x \\ &= \int_{\Omega_{t_{m-1}}} |\tilde{\mathcal{U}}_s(X,t)|^2 J(X,t) \, \mathrm{d}X \le C_J^+ \int_{\Omega_{t_{m-1}}} |\tilde{\mathcal{U}}_s(X,t)|^2 \, \mathrm{d}X \\ &= C_J^+ \|\tilde{\mathcal{U}}_s(t)\|_{\Omega_{t_{m-1}}}^2 \end{aligned}$$

Integrating over I_m and using (3.54), (3.49) and (3.6), we obtain

$$\begin{aligned} \int_{I_m} \|\mathcal{U}_s(t)\|_{\Omega_t}^2 \, \mathrm{d}t &\le C_J^+ \int_{I_m} \|\tilde{\mathcal{U}}_s(t)\|_{\Omega_{t_{m-1}}}^2 \, \mathrm{d}t \\ &\le C_J^+ \tilde{c}_{CH}^{(1)} \int_{I_m} \|\tilde{U}(t)\|_{\Omega_{t_{m-1}}}^2 \, \mathrm{d}t \\ &= C_J^+ \tilde{c}_{CH}^{(1)} \int_{I_m} \left(\int_{\Omega_{t_{m-1}}} |\tilde{U}(X,t)|^2 \, \mathrm{d}X \right) \mathrm{d}t \\ &= C_J^+ \tilde{c}_{CH}^{(1)} \int_{I_m} \left(\int_{\Omega_{t_{m-1}}} |U(\mathcal{A}_t(X),t)|^2 \, \mathrm{d}X \right) \mathrm{d}t \\ &= C_J^+ \tilde{c}_{CH}^{(1)} \int_{I_m} \left(\int_{\Omega_t} |U(x,t)|^2 J^{-1}(x,t) \, \mathrm{d}x \right) \mathrm{d}t \\ &\le C_J^+ \tilde{c}_{CH}^{(1)} C_J^- \int_{I_m} \left(\int_{\Omega_t} |U(x,t)|^2 \, \mathrm{d}x \right) \mathrm{d}t \\ &= C_J^+ \tilde{c}_{CH}^{(1)} C_J^- \int_{I_m} \|U(t)\|_{\Omega_t}^2 \, \mathrm{d}t. \end{aligned} \tag{3.58}$$

Setting $c_{CH}^{(1)} = C_J^+ \tilde{c}_{CH}^{(1)} C_J^-$, we get (3.56).

Now we pay our attention to the proof of the second inequality in the theorem. From the definition of the DG-norm we have

$$\begin{aligned} &\int_{I_m} ||\mathcal{U}_s||_{DG,t}^2 \, \mathrm{d}t \\ &= \int_{I_m} \sum_{K\in\mathcal{T}_{h,t}} |\mathcal{U}_s|_{H^1(K)}^2 \, \mathrm{d}t + \int_{I_m} \left(\sum_{\Gamma\in\mathcal{F}_{h,t}^I} \frac{c_W}{h(\Gamma)} \int_\Gamma [\mathcal{U}_s]^2 \, \mathrm{d}S \right) \mathrm{d}t \\ &\quad + \int_{I_m} \left(\sum_{\Gamma\in\mathcal{F}_{h,t}^B} \frac{c_W}{h(\Gamma)} \int_\Gamma |\mathcal{U}_s|^2 \, \mathrm{d}S \right) \mathrm{d}t, \end{aligned} \tag{3.59}$$

where $\mathcal{F}^I_{h,t} = \{\mathcal{A}^{m-1}_{h,t}(\hat{\Gamma});\ \hat{\Gamma} \in \mathcal{F}^I_{h,t_{m-1}}\}$ and similarly $\mathcal{F}^B_{h,t} = \{\mathcal{A}^{m-1}_{h,t}(\hat{\Gamma});\ \hat{\Gamma} \in \mathcal{F}^B_{h,t_{m-1}}\}$.

Further, we estimate each term on the right-hand side of (3.59). From [31], relation (6.161), it follows that

$$\sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \int_{I_m} |\tilde{\mathcal{U}}_s(t)|^2_{H^1(\hat{K})}\,\mathrm{d}t \le \tilde{c}^{(2)}_{CH} \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \int_{I_m} |\tilde{U}(t)|^2_{H^1(\hat{K})}\,\mathrm{d}t, \tag{3.60}$$

with a constant $\tilde{c}^{(2)}_{CH} > 0$ depending on q only. For simplicity let us denote

$$B_t = B_t(X) = \frac{\mathrm{d}\mathcal{A}^{m-1}_{h,t}(X)}{\mathrm{d}X}, \quad B_t^{-1} = B_t^{-1}(x) = \frac{\mathrm{d}(\mathcal{A}^{m-1}_{h,t})^{-1}(x)}{\mathrm{d}x}. \tag{3.61}$$

Then it follows from (3.7) and (3.8) that $\|B_t\| \le C^+_A$ and $\|B_t^{-1}\| \le C^-_A$.
Now, for $K \in \mathcal{T}_{h,t}$, $K = \mathcal{A}_t(\hat{K})$ with $\hat{K} \in \hat{\mathcal{T}}_{h,t_{m-1}}$, using that $\|B_t|_{\hat{K}}\|$ and $\|B_t^{-1}|_K\|$ are constant, we have

$$\begin{aligned}
&\sum_{K\in\mathcal{T}_{h,t}} |\mathcal{U}_s(t)|^2_{H^1(K)} = \sum_{K\in\mathcal{T}_{h,t}} \int_K |\nabla\mathcal{U}_s(x,t)|^2\,\mathrm{d}x \\
&= \int_K \sum_{K\in\mathcal{T}_{h,t}} \left|\nabla(\tilde{\mathcal{U}}_s(\mathcal{A}_t^{-1}(x),t))\right|^2 \mathrm{d}x \\
&\le \int_{\hat{K}} \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \|B_t^{-1}|_K\|^2 \left|\nabla\tilde{\mathcal{U}}_s(X,t)\right|^2 J(X,t)\,\mathrm{d}X \\
&\le (C^-_A)^2 C^+_J \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} |\tilde{\mathcal{U}}_s(t)|^2_{H^1(\hat{K})}.
\end{aligned} \tag{3.62}$$

Integrating over I_m, using (3.60), (3.49), (3.6), Fubini's and the substitution theorem we find that

$$
\begin{aligned}
&\int_{I_m} \sum_{K\in\mathcal{T}_{h,t}} |\mathcal{U}_s(t)|^2_{H^1(K)}\,\mathrm{d}t \qquad (3.63)\\
&\le (C_A^-)^2 C_J^+ \int_{I_m} \sum_{\hat K\in\hat{\mathcal{T}}_{h,t_{m-1}}} |\tilde{\mathcal{U}}_s(t)|^2_{H^1(\hat K)}\,\mathrm{d}t\\
&\le (C_A^-)^2 C_J^+ \tilde c^{(2)}_{CH} \int_{I_m} \sum_{\hat K\in\hat{\mathcal{T}}_{h,t_{m-1}}} |\tilde U(t)|^2_{H^1(\hat K)}\,\mathrm{d}t\\
&= (C_A^-)^2 C_J^+ \tilde c^{(2)}_{CH} \int_{I_m} \left(\sum_{\hat K\in\hat{\mathcal{T}}_{h,t_{m-1}}} \int_{\hat K} |\nabla \tilde U(X,t)|^2\,\mathrm{d}X \right) \mathrm{d}t\\
&= (C_A^-)^2 C_J^+ \tilde c^{(2)}_{CH} \int_{I_m} \left(\sum_{\hat K\in\hat{\mathcal{T}}_{h,t_{m-1}}} \int_{\hat K} |\nabla (U(\mathcal{A}_t(X),t))|^2\,\mathrm{d}X \right) \mathrm{d}t\\
&\le (C_A^-)^2 C_J^+ \tilde c^{(2)}_{CH} \int_{I_m} \left(\sum_{K\in\mathcal{T}_{h,t}} \int_{K} \|B_t|_K\|^2 |\nabla U(x,t)|^2 J_K^{-1}(x,t)\,\mathrm{d}x \right) \mathrm{d}t\\
&\le (C_A^-)^2 C_J^+ \tilde c^{(2)}_{CH} (C_A^+)^2 (C_J^-)^{-1} \int_{I_m} \left(\sum_{K\in\mathcal{T}_{h,t}} \int_{K} |\nabla U(x,t)|^2\,\mathrm{d}x \right) \mathrm{d}t\\
&\le (C_A^-)^2 C_J^+ \tilde c^{(2)}_{CH} (C_A^+)^2 (C_J^-)^{-1} \int_{I_m} \sum_{K\in\mathcal{T}_{h,t}} |U(t)|^2_{H^1(K)}\,\mathrm{d}t\\
&= C^{(a)}_{CH} \int_{I_m} |U(t)|^2_{H^1(\Omega_t,\mathcal{T}_{h,t})}\,\mathrm{d}t,
\end{aligned}
$$

where $C^{(a)}_{CH} := (C_A^-)^2 C_J^+ \tilde c^{(2)}_{CH} (C_A^+)^2 (C_J^-)^{-1}$.

Now we turn our attention to the term

$$
\int_{I_m} \left(\sum_{\Gamma\in\mathcal{F}^I_{h,t}} \frac{c_W}{h(\Gamma)} \int_\Gamma [\mathcal{U}_s]^2\,\mathrm{d}S \right) \mathrm{d}t.
$$

First, we assume that $d = 2$. We will use the analogy to estimate (6.162) from [31]. This implies that

$$
\begin{aligned}
&\int_{I_m} \left(\sum_{\hat\Gamma\in\mathcal{F}^I_{h,t_{m-1}}} \frac{c_W}{h(\hat\Gamma)} \int_{\hat\Gamma} [\tilde{\mathcal{U}}_s]^2\,\mathrm{d}S^{\hat\Gamma} \right) \mathrm{d}t \qquad (3.64)\\
&\qquad \le \tilde c^{(3)}_{CH} \int_{I_m} \left(\sum_{\hat\Gamma\in\mathcal{F}^I_{h,t_{m-1}}} \frac{c_W}{h(\hat\Gamma)} \int_{\hat\Gamma} [\tilde U]^2\,\mathrm{d}S^{\hat\Gamma} \right) \mathrm{d}t.
\end{aligned}
$$

Here $\mathrm{d}S^{\hat\Gamma}$ denotes the element of the arc $\hat\Gamma$. Similarly we use the notation $\mathrm{d}S^{\Gamma}$.

Now we consider the relation $\Gamma = \mathcal{A}_t(\hat\Gamma)$, $\hat\Gamma \in \mathcal{F}^I_{h,t_{m-1}}$, and introduce a parametrization of $\hat\Gamma$:

$$
\hat\Gamma = \mathcal{B}^{\hat\Gamma}_{m-1}([0,1]) = \{X = \mathcal{B}^{\hat\Gamma}_{m-1}(\upsilon); \upsilon \in [0,1]\}.
$$

Then an element of $\hat{\Gamma}$ can be expressed as

$$\mathrm{d}S^{\hat{\Gamma}} = |(\mathcal{B}^{\hat{\Gamma}}_{m-1})'(v)|\,\mathrm{d}v, \quad v \in [0,1].$$

These relations imply that

$$\begin{aligned} \Gamma &= \{x = \mathcal{A}_t(\mathcal{B}^{\hat{\Gamma}}_{m-1}(v)); v \in [0,1]\} \\ \mathrm{d}S^{\Gamma} &= \left| \frac{\mathrm{d}\mathcal{A}_t}{\mathrm{d}X}(\mathcal{B}^{\hat{\Gamma}}_{m-1}(v))(\mathcal{B}^{\hat{\Gamma}}_{m-1})'(v) \right| \mathrm{d}v, \quad v \in [0,1]. \end{aligned}$$

The term $(\mathcal{B}^{\hat{\Gamma}}_{m-1})'(v)$ is a tangent vector to $\hat{\Gamma}$ at the point $\mathcal{B}^{\hat{\Gamma}}_{m-1}(v)$. It follows from the properties of the mapping $\mathcal{A}_t$ that the values of

$$\frac{\mathrm{d}\mathcal{A}_t}{\mathrm{d}X}(\mathcal{B}^{\hat{\Gamma}}_{m-1}(v))(\mathcal{B}^{\hat{\Gamma}}_{m-1})'(v)$$

are identical from the sides of both elements $K^{(L)}_{\hat{\Gamma}}$ and $K^{(R)}_{\hat{\Gamma}}$ adjacent to $\hat{\Gamma}$. Then we can use the above relations, inequalities (3.55), (3.7), and write

$$\begin{aligned} &\int_{\Gamma} \frac{1}{h(\Gamma)} [\mathcal{U}_s]^2 \mathrm{d}S^{\Gamma} \qquad (3.65) \\ &= \int_0^1 \frac{1}{h(\Gamma)} [\mathcal{U}_s(\mathcal{A}_t(\mathcal{B}^{\hat{\Gamma}}_{m-1}(v)))]^2 \left| \frac{\mathrm{d}\mathcal{A}_t}{\mathrm{d}X}(\mathcal{B}^{\hat{\Gamma}}_{m-1}(v))(\mathcal{B}^{\hat{\Gamma}}_{m-1})'(v) \right| \mathrm{d}v \\ &\leq \int_0^1 \frac{1}{h(\Gamma)} [\tilde{\mathcal{U}}_s(\mathcal{B}^{\hat{\Gamma}}_{m-1}(v))]^2 \underbrace{\left\| \frac{\mathrm{d}\mathcal{A}_t}{\mathrm{d}X}(\mathcal{B}^{\hat{\Gamma}}_{m-1}(v)) \right\|}_{\leq C_A^+} \left| (\mathcal{B}^{\hat{\Gamma}}_{m-1})'(v) \right| \mathrm{d}v \\ &\leq C_A^+ \int_{\hat{\Gamma}} \frac{C^{**}_{L7}}{h(\hat{\Gamma})} [\tilde{\mathcal{U}}_s]^2 \mathrm{d}S^{\hat{\Gamma}}. \end{aligned}$$

From (3.64) and (3.65) we get

$$\begin{aligned} &\int_{I_m} \left(\sum_{\Gamma \in \mathcal{F}^I_{h,t}} \frac{c_W}{h(\Gamma)} \int_{\Gamma} [\mathcal{U}_s]^2 \,\mathrm{d}S^{\Gamma} \right) \mathrm{d}t \qquad (3.66) \\ &\leq \tilde{c}^{(3)}_{CH} C_A^+ C^{**}_{L7} \int_{I_m} \left(\sum_{\hat{\Gamma} \in \mathcal{F}^I_{h,t_{m-1}}} \frac{c_W}{h(\hat{\Gamma})} \int_{\hat{\Gamma}} [\tilde{U}]^2 \,\mathrm{d}S^{\hat{\Gamma}} \right) \mathrm{d}t. \end{aligned}$$

Further, for $\Gamma = \mathcal{A}_t(\hat{\Gamma})$, where $\hat{\Gamma} \in \mathcal{F}^I_{h,t_{m-1}}$, we consider the parametrization

$$\begin{aligned} &\Gamma = \{x = \mathcal{B}^{\Gamma}_t(v); v \in [0,1]\}, \\ &\mathrm{d}S^{\Gamma} = \left| (\mathcal{B}^{\Gamma}_t)' \right| \mathrm{d}v, \\ &\hat{\Gamma} = \{X = \mathcal{A}_t^{-1}(\mathcal{B}^{\Gamma}_t(v)); v \in [0,1]\}, \\ &\mathrm{d}S^{\hat{\Gamma}} = \left| \frac{\mathrm{d}\mathcal{A}_t^{-1}}{\mathrm{d}x}(\mathcal{B}^{\Gamma}_t(v))(\mathcal{B}^{\Gamma}_t)'(v) \right| \mathrm{d}v. \end{aligned}$$

Then, by (3.8),

$$\begin{aligned}
\int_{\hat\Gamma}[\tilde U]^2\,\mathrm{d}S^{\hat\Gamma} &= \int_0^1 \underbrace{[\tilde U(\mathcal{A}_t^{-1}(\mathcal{B}_t^\Gamma(v)))]^2}_{[U(\mathcal{B}_t^\Gamma(v))]^2}\left|\frac{\mathrm{d}\mathcal{A}_t^{-1}}{\mathrm{d}x}(\mathcal{B}_t^\Gamma(v))(\mathcal{B}_t^\Gamma)'(v)\right|\,\mathrm{d}v\\
&\le \int_0^1 [U(\mathcal{B}_t^\Gamma(v))]^2 \underbrace{\left\|\frac{\mathrm{d}\mathcal{A}_t^{-1}}{\mathrm{d}x}(\mathcal{B}_t^\Gamma(v))\right\|}_{\le C_A^-}\left|(\mathcal{B}_t^\Gamma)'(v)\right|\,\mathrm{d}v\\
&\le C_A^-\int_0^1 [U(\mathcal{B}_t^\Gamma(v))]^2|(\mathcal{B}_t^\Gamma)'(v)|\,\mathrm{d}v\\
&= C_A^-\int_\Gamma [U]^2\,\mathrm{d}S^\Gamma.
\end{aligned}$$

Substituting back to (3.66) and using (3.55), we find that

$$\begin{aligned}
&\int_{I_m}\left(\sum_{\Gamma\in\mathcal{F}_{h,t}^I}\frac{c_W}{h(\Gamma)}\int_\Gamma[\mathcal{U}_s]^2\,\mathrm{d}S^\Gamma\right)\,\mathrm{d}t \qquad (3.67)\\
&\le \tilde c_{CH}^{(3)}C_A^+C_{L7}^{**}(C_{L7}^*)^{-1}C_A^-\int_{I_m}\left(\sum_{\Gamma\in\mathcal{F}_{h,t}^I}\frac{c_W}{h(\Gamma)}\int_\Gamma[U]^2\,\mathrm{d}S\right)\,\mathrm{d}t\\
&= C_{CH}^{(b)}\int_{I_m}\left(\sum_{\Gamma\in\mathcal{F}_{h,t}^I}\frac{c_W}{h(\Gamma)}\int_\Gamma[U]^2\,\mathrm{d}S\right)\,\mathrm{d}t,
\end{aligned}$$

where $C_{CH}^{(b)}=\tilde c_{CH}^{(3)}C_A^+C_{L7}^{**}(C_{L7}^*)^{-1}C_A^-$.

Similarly we can prove the inequality

$$\int_{I_m}\left(\sum_{\Gamma\in\mathcal{F}_{h,t}^B}\frac{c_W}{h(\Gamma)}\int_\Gamma|\mathcal{U}_s|^2\,\mathrm{d}S^\Gamma\right)\,\mathrm{d}t \le C_{CH}^{(c)}\int_{I_m}\left(\sum_{\Gamma\in\mathcal{F}_{h,t}^B}\frac{c_W}{h(\Gamma)}\int_\Gamma|U|^2\,\mathrm{d}S\right)\,\mathrm{d}t. \quad (3.68)$$

Finally, (3.63), (3.67) and (3.68) imply (3.57) with $c_{CH}^{(2)}=\max\{C_{CH}^{(a)},C_{CH}^{(b)},C_{CH}^{(c)}\}$.

The proof of (3.67) and (3.68) in case of $d=3$ is much more complicated. I am grateful to Zuzana Vlasáková for her advice.

We introduce a parametrization of $\hat\Gamma$. Let Δ^2 be a reference simplex in $\mathbb{R}^2$ (with one vertex being the origin and all of the other vertices have only one non-zero coordinate equal to 1). Now

$$\begin{aligned}
&\Gamma=\mathcal{A}_t(\hat\Gamma),\quad \hat\Gamma\in\mathcal{F}_{h,t_{m-1}}^I,\\
&\hat\Gamma=\mathcal{B}_{m-1}^{\hat\Gamma}(\Delta^2)=\{X=\mathcal{B}_{m-1}^{\hat\Gamma}(v);\ v\in\Delta^2\},
\end{aligned}$$

$$\begin{aligned}
\mathrm{d}S^{\hat\Gamma} &= \left\|\frac{\partial\mathcal{B}_{m-1}^{\hat\Gamma}}{\partial x^1}(v)\times\frac{\partial\mathcal{B}_{m-1}^{\hat\Gamma}}{\partial x^2}(v)\right\|\,\mathrm{d}x^1\,\mathrm{d}x^2,\quad v\in\Delta^2,\\
\Gamma &= \{x=\mathcal{A}_t(\mathcal{B}_{m-1}^{\hat\Gamma}(v));\ v\in\Delta^2\},\\
\mathrm{d}S^\Gamma &= \left\|\frac{\mathrm{d}\mathcal{A}_t}{\mathrm{d}X}(\mathcal{B}_{m-1}^{\hat\Gamma}(v))\frac{\partial\mathcal{B}_{m-1}^{\hat\Gamma}}{\partial x^1}(v)\times\frac{\mathrm{d}\mathcal{A}_t}{\mathrm{d}X}(\mathcal{B}_{m-1}^{\hat\Gamma}(v))\frac{\partial\mathcal{B}_{m-1}^{\hat\Gamma}}{\partial x^2}(v)\right\|\,\mathrm{d}x^1\,\mathrm{d}x^2,\ v\in\Delta^2.
\end{aligned}$$

By the symbol $\times$ we denote the vector product. The terms $\frac{\partial \mathcal{B}^{\hat\Gamma}_{m-1}}{\partial x^i}(v)$ are tangent vectors to $\hat\Gamma$ at the point $\mathcal{B}^{\hat\Gamma}_{m-1}(v)$. It follows from the properties of the mapping $\mathcal{A}_t$ that the values of $\frac{\mathrm{d}\mathcal{A}_t}{\mathrm{d}X}(\mathcal{B}^{\hat\Gamma}_{m-1}(v))\frac{\partial \mathcal{B}^{\hat\Gamma}_{m-1}}{\partial x^i}(v)$ are identical from the sides of both elements $\hat K^{\hat\Gamma}_L$ and $\hat K^{\hat\Gamma}_R$ adjacent to $\hat\Gamma$.

Then we can write

$$\begin{aligned}
&\int_\Gamma \frac{1}{h(\Gamma)}[U_s]^2 \,\mathrm{d}S^\Gamma \\
&= \int_{\Delta^2} \frac{1}{h(\Gamma)}[U_s(\mathcal{A}_t(\mathcal{B}^{\hat\Gamma}_{m-1}(v)))]^2 \\
&\qquad \left\| \frac{\mathrm{d}\mathcal{A}_t}{\mathrm{d}X}(\mathcal{B}^{\hat\Gamma}_{m-1}(v))\frac{\partial \mathcal{B}^{\hat\Gamma}_{m-1}}{\partial x^1}(v) \times \frac{\mathrm{d}\mathcal{A}_t}{\mathrm{d}X}(\mathcal{B}^{\hat\Gamma}_{m-1}(v))\frac{\partial \mathcal{B}^{\hat\Gamma}_{m-1}}{\partial x^2}(v)\right\| \,\mathrm{d}x^1\,\mathrm{d}x^2 \\
&\le \int_{\Delta^2} \frac{1}{h(\Gamma)}[\tilde U_s(\mathcal{B}^{\hat\Gamma}_{m-1}(v))]^2 \left\|\frac{\mathrm{d}\mathcal{A}_t}{\mathrm{d}X}(\mathcal{B}^{\hat\Gamma}_{m-1}(v))\right\|^2 \left\|\frac{\partial \mathcal{B}^{\hat\Gamma}_{m-1}}{\partial x^1}(v) \times \frac{\partial \mathcal{B}^{\hat\Gamma}_{m-1}}{\partial x^2}(v)\right\| \,\mathrm{d}x^1\,\mathrm{d}x^2 \\
&\le (C_A^+)^2 \int_{\hat\Gamma} \frac{C_{L6}^{**}}{h(\hat\Gamma)}[\tilde U_s]^2 \,\mathrm{d}S^{\hat\Gamma}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_{I_m}\left(\sum_{\Gamma\in\mathcal{F}^I_{h,t}} \frac{c_W}{h(\Gamma)}\int_\Gamma [U_s]^2\,\mathrm{d}S^\Gamma\right)\mathrm{d}t \qquad (3.69)\\
&\qquad \le \tilde c^{(3)}_{CH}(C_A^+)^2 C_{L6}^{**}\int_{I_m}\left(\sum_{\hat\Gamma\in\mathcal{F}^I_{h,t_{m-1}}}\frac{c_W}{h(\hat\Gamma)}\int_{\hat\Gamma}[\tilde U]^2\,\mathrm{d}S^{\hat\Gamma}\right)\mathrm{d}t.
\end{aligned}$$

Further for $\Gamma = \mathcal{A}_t(\hat\Gamma)$, $\hat\Gamma\in\mathcal{F}^I_{h,t_{m-1}}$, we consider the parametrization

$$\begin{aligned}
&\Gamma = \{x = \mathcal{B}^\Gamma_t(v);\ v\in\Delta^2\},\\
&\hat\Gamma = \{X = \mathcal{A}_t^{-1}(\mathcal{B}^\Gamma_t(v));\ v\in\Delta^2\},\\
&\mathrm{d}S^\Gamma = \left\|\frac{\partial \mathcal{B}^\Gamma_{m-1}}{\partial x^1}(v)\times\frac{\partial \mathcal{B}^\Gamma_{m-1}}{\partial x^2}(v)\right\|\,\mathrm{d}v,\quad v\in\Delta^2\\
&\mathrm{d}S^{\hat\Gamma} = \left\|\frac{\mathrm{d}\mathcal{A}_t^{-1}}{\mathrm{d}x}(\mathcal{B}^\Gamma_t(v))\frac{\partial \mathcal{B}^\Gamma_t}{\partial x^1}(v)\times\frac{\mathrm{d}\mathcal{A}_t^{-1}}{\mathrm{d}x}(\mathcal{B}^\Gamma_t(v))\frac{\partial \mathcal{B}^\Gamma_t}{\partial x^2}(v)\right\|\,\mathrm{d}v,\quad v\in\Delta^2.
\end{aligned}$$

Then

$$\begin{aligned}
&\int_{\hat\Gamma}[\tilde U]^2\,\mathrm{d}S^{\hat\Gamma} \qquad (3.70)\\
&= \int_{\Delta^2}[\tilde U(\mathcal{A}_t^{-1}(\mathcal{B}^\Gamma_t(v)))]^2\left\|\frac{\mathrm{d}\mathcal{A}_t^{-1}}{\mathrm{d}x}(\mathcal{B}^\Gamma_t(v))\frac{\partial \mathcal{B}^\Gamma_t}{\partial x^1}(v)\times\frac{\mathrm{d}\mathcal{A}_t^{-1}}{\mathrm{d}x}(\mathcal{B}^\Gamma_t(v))\frac{\partial \mathcal{B}^\Gamma_t}{\partial x^2}(v)\right\|\,\mathrm{d}x^1\,\mathrm{d}x^2\\
&\le \int_{\Delta^2}[U(\mathcal{B}^\Gamma_t(v))]^2\left\|\frac{\mathrm{d}\mathcal{A}_t^{-1}}{\mathrm{d}x}(\mathcal{B}^\Gamma_t(v))\right\|^2\left\|\frac{\partial \mathcal{B}^\Gamma_{m-1}}{\partial x^1}(v)\times\frac{\partial \mathcal{B}^\Gamma_{m-1}}{\partial x^2}(v)\right\|\,\mathrm{d}x^1\,\mathrm{d}x^2\\
&\le (C_A^-)^2\int_{\Delta^2}[U]^2\,\mathrm{d}S^\Gamma.
\end{aligned}$$

Together we get

$$\int_{I_m}\left(\sum_{\Gamma\in\mathcal{F}^I_{h,t}}\frac{c_W}{h(\Gamma)}\int_\Gamma [U_s]^2\,\mathrm{d}S^\Gamma\right)\,\mathrm{d}t$$

$$\le \tilde{c}^{(3)}_{CH}(C^+_A)^2 C^{**}_{L6}(C^*_{L6})^{-1}(C^-_A)^2\int_{I_m}\left(\sum_{\Gamma\in\mathcal{F}^I_{h,t}}\frac{c_W}{h(\Gamma)}\int_\Gamma [U]^2\,\mathrm{d}S^\Gamma\right)\,\mathrm{d}t,$$

which is the 3D version of (3.67). Similarly we prove (3.68) in the 3D case. □

3.4 Proof of the unconditional stability

Now we can apply estimates from Section 3.2 to the basic identity (3.14). These estimates, apart from another, produce a problematic term $\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t$, which we will need to estimate in terms of data. To overcome this difficulty we use the generalized discrete characteristic function in time-dependent domains.

3.4.1 Estimates of the basic idientity

Theorem 2. *There exists a constant $C_{T2}>0$ such that*

$$\|U^-_m\|^2_{\Omega_{t_m}} - \|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}} + \|\{U\}_{m-1}\|^2_{\Omega_{t_{m-1}}} + \frac{\beta_0}{2}\int_{I_m}\|U\|^2_{DG,t}\mathrm{d}t \quad (3.71)$$

$$\le C_{T2}\left(\int_{I_m}\|g\|^2_{\Omega_t}\mathrm{d}t + \int_{I_m}\|u_D\|^2_{DGB,t}\mathrm{d}t + \int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t\right).$$

Proof. From (3.14) we get

$$\underbrace{\int_{I_m}(D_tU,U)_{\Omega_t}\mathrm{d}t}_{\sigma_1} + \underbrace{\int_{I_m}A_h(U,U,t)\,\mathrm{d}t}_{\sigma_2} + \underbrace{\int_{I_m}b_h(U,U,t)\,\mathrm{d}t}_{\sigma_3}$$

$$+\underbrace{\int_{I_m}d_h(U,U,t)\,\mathrm{d}t}_{\sigma_4} + \underbrace{\left(\{U\}_{m-1},U^+_{m-1}\right)_{\Omega_{t_{m-1}}}}_{\sigma_5} = \underbrace{\int_{I_m}l_h(U,t)\,\mathrm{d}t}_{\sigma_6}.$$

By virtue of (3.42), (3.18), (3.35), (3.39), (3.43) and (3.40), we find that

$$\begin{aligned}
\sigma_1 &\ge \frac{1}{2}\|U^-_m\|^2_{\Omega_{t_m}} - \frac{1}{2}\|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}} - \frac{c_z}{2}\int_{I_m}\|U\|^2_{\Omega_t}\,\mathrm{d}t,\\
\sigma_2 &\ge \frac{\beta_0}{2}\int_{I_m}\|U\|^2_{DG,t}\mathrm{d}t - \frac{\beta_0}{2}\int_{I_m}\|u_D\|_{DGB,t}\,\mathrm{d}t\\
\sigma_3 &\le \frac{\beta_0}{2k_1}\int_{I_m}\|U\|^2_{DG,t}\mathrm{d}t + c_b\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t\\
\sigma_4 &\le \frac{\beta_0}{2k_2}\int_{I_m}\|U\|^2_{DG,t}\mathrm{d}t + \frac{c_d}{2\beta_0}\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t,\\
\sigma_5 &= \frac{1}{2}\left(\|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}} + \|\{U\}_{m-1}\|^2_{\Omega_{t_{m-1}}} - \|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}}\right)\\
\sigma_6 &\le \frac{1}{2}\int_{I_m}\left(\|g\|^2_{\Omega_t} + \|U\|^2_{\Omega_t}\right)\mathrm{d}t + \frac{\beta_0 k_3}{2}\int_{I_m}\|u_D\|^2_{DGB,t}\mathrm{d}t + \frac{\beta_0}{2k_3}\int_{I_m}\|U\|^2_{DG,t}\mathrm{d}t.
\end{aligned}$$

The above relations imply that

$$
\begin{aligned}
&\frac{1}{2}\|U_m^-\|^2_{\Omega_{t_m}} - \frac{1}{2}\|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} - \frac{c_z}{2}\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t \\
&+\frac{\beta_0}{2}\int_{I_m}\|U\|^2_{DG,t}\mathrm{d}t - \frac{\beta_0}{2}\int_{I_m}\|u_D\|^2_{DGB,t}\mathrm{d}t \\
&+\frac{1}{2}\|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} + \frac{1}{2}\|\{U\}_{m-1}\|^2_{\Omega_{t_{m-1}}} - \frac{1}{2}\|U_{m-1}^-\|^2_{\Omega_{t_{m-1}}} \\
&\leq \frac{\beta_0}{2k_1}\int_{I_m}\|U\|^2_{DG,t}\mathrm{d}t + c_b\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t + \frac{\beta_0}{2k_2}\int_{I_m}\|U\|^2_{DG,t}\mathrm{d}t \\
&+\frac{c_d}{2\beta_0}\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t + \frac{1}{2}\int_{I_m}\|g\|^2_{\Omega_t}\mathrm{d}t + \frac{1}{2}\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t \\
&+\frac{\beta_0 k_3}{2}\int_{I_m}\|u_D\|^2_{DGB,t}\mathrm{d}t + \frac{\beta_0}{2k_3}\int_{I_m}\|U\|^2_{DG,t}\mathrm{d}t.
\end{aligned}
$$

After multiplying by two and rearranging, we get

$$
\begin{aligned}
&\|U_m^-\|^2_{\Omega_{t_m}} - \|U_{m-1}^-\|^2_{\Omega_{t_{m-1}}} + \|\{U\}_{m-1}\|^2_{\Omega_{t_{m-1}}} \\
&\quad+\beta_0\left(1-\frac{1}{k_1}-\frac{1}{k_2}-\frac{1}{k_3}\right)\int_{I_m}\|U\|^2_{DG,t}\mathrm{d}t \\
&\leq \int_{I_m}\|g\|^2_{\Omega_t}\mathrm{d}t + \beta_0(1+k_3)\int_{I_m}\|u_D\|^2_{DGB,t}\mathrm{d}t + \\
&\left(c_z+1+\frac{c_d}{\beta_0}+2c_b\right)\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t.
\end{aligned}
$$

Hence, choosing $k_1 = k_2 = k_3 = 6$, we get (3.71) with $C_{T2} = \max\{1, 7\beta_0, c_z + 1 + c_d/\beta_0 + 2c_b\}$. ☐

Theorem 3. *There exist constants $C_{T3}, C^*_{T3} > 0$ such that for any $\delta_1 > 0$ we have*

$$
\begin{aligned}
&\|U_m^-\|_{\Omega^2_{t_m}} + \|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} + \frac{\beta_0}{2}\int_{I_m}\|U\|^2_{DG,t}\mathrm{d}t \\
&\leq C_{T3}\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t + C^*_{T3}\int_{I_m}\left(\|g\|^2_{\Omega_t} + \|u_D\|^2_{DGB,t}\right)\mathrm{d}t \\
&+\frac{2}{\delta_1}\|U_{m-1}^-\|^2_{\Omega_{t_{m-1}}} + 4\delta_1\|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}}.
\end{aligned}
\tag{3.72}
$$

Proof. From (3.14), by virtue of (3.44), (3.18), (3.35), (3.39) and (3.40), we get

$$
\begin{aligned}
&\underbrace{\int_{I_m}(D_tU, U)_{\Omega_t}\mathrm{d}t + \left(\{U\}_{m-1}, U_{m-1}^+\right)_{\Omega_{t_{m-1}}}}_{\sigma_1} + \underbrace{\int_{I_m}A_h(U,U,t)\,\mathrm{d}t}_{\sigma_2} \\
&+\underbrace{\int_{I_m}b_h(U,U,t)\,\mathrm{d}t}_{\sigma_3} + \underbrace{\int_{I_m}d_h(U,U,t)\,\mathrm{d}t}_{\sigma_4} = \underbrace{\int_{I_m}l_h(U,t)\,\mathrm{d}t}_{\sigma_5},
\end{aligned}
$$

where

$$
\begin{aligned}
\sigma_1 &\geq \frac{1}{2}\|U_m^-\|^2_{\Omega_{t_m}} + \frac{1}{2}\|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} - \frac{c_z}{2}\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t - \Big(U_{m-1}^-, U_{m-1}^+\Big)_{\Omega_{t_{m-1}}} \\
\sigma_2 &\geq \frac{\beta_0}{2}\int_{I_m}\|U\|^2_{DG,t}\mathrm{d}t - \frac{\beta_0}{2}\int_{I_m}\|u_D\|^2_{DGB,t}\mathrm{d}t \\
\sigma_3 &\leq \frac{\beta_0}{2k_1}\int_{I_m}\|U\|^2_{DG,t}\mathrm{d}t + c_b\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t \\
\sigma_4 &\leq \frac{\beta_0}{2k_2}\int_{I_m}\|U\|^2_{DG,t}\mathrm{d}t + \frac{c_d}{2\beta_0}\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t \\
\sigma_5 &\leq \frac{1}{2}\int_{I_m}\Big(\|g\|^2_{\Omega_t} + \|U\|^2_{\Omega_t}\Big)\mathrm{d}t + \frac{\beta_0 k_3}{2}\int_{I_m}\|u_D\|^2_{DGB,t}\mathrm{d}t + \frac{\beta_0}{2k_3}\int_{I_m}\|U\|^2_{DG,t}\mathrm{d}t,
\end{aligned}
$$

We remind that $c_b = c_1^2 k_1/\beta_0$ and $c_d = c_z^2 k_2$.

After rearranging we have

$$
\begin{aligned}
&\frac{1}{2}\|U_m^-\|^2_{\Omega_{t_m}} + \frac{1}{2}\|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} - \frac{c_z}{2}\|U\|^2_{\Omega_t}\mathrm{d}t - \Big(U_{m-1}^-, U_{m-1}^+\Big)_{\Omega_{t_{m-1}}} \\
&\quad + \frac{\beta_0}{2}\int_{I_m}\|U\|^2_{DG,t}\mathrm{d}t - \frac{\beta_0}{2}\int_{I_m}\|u_D\|^2_{DGB,t}\mathrm{d}t \\
&\leq \frac{\beta_0}{2k_1}\int_{I_m}\|U\|^2_{DG,t}\mathrm{d}t + c_b\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t + \frac{\beta_0}{2k_2}\int_{I_m}\|U\|^2_{DG,t}\mathrm{d}t \\
&\quad + \frac{c_d}{2\beta_0}\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t + \frac{1}{2}\int_{I_m}\|g\|^2_{\Omega_t}\mathrm{d}t + \frac{1}{2}\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t \\
&\quad + \frac{\beta_0 k_3}{2}\int_{I_m}\|u_D\|^2_{DGB,t}\mathrm{d}t + \frac{\beta_0}{2k_3}\int_{I_m}\|U\|^2_{DG,t}\mathrm{d}t.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\|U_m^-\|^2_{\Omega_{t_m}} + \|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} + \beta_0\left(1 - \frac{1}{k_1} - \frac{1}{k_2} - \frac{1}{k_3}\right)\int_{I_m}\|U\|^2_{DG,t}\mathrm{d}t \\
&\leq \int_{I_m}\|g\|^2_{\Omega_t}\mathrm{d}t + \beta_0(1+k_3)\int_{I_m}\|u_D\|^2_{DGB,t}\mathrm{d}t \\
&\quad + \left(1 + c_z + 2c_b + \frac{c_d}{\beta_0}\right)\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t + 2\Big(U_{m-1}^-, U_{m-1}^+\Big)_{\Omega_{t_{m-1}}}.
\end{aligned}
$$

Using Young's inequality for the term $2(U_{m-1}^-, U_{m-1}^+)$ and setting $k_1 = k_2 = k_3 = 6$, we get (3.72), where $C_{T3} = 1 + c_z + 2c_b + c_d/\beta_0$ and $C_{T3}^* = \max\{1, 7\beta_0\}$. □

3.4.2 Estimates with the discrete characteristic function

In this section we derive key estimates, which will help us to estimate the problematic term $\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t$ in terms of data. In Lemmas 9 - 13 we will estimate similar terms as in Section 3.2, but the test function (second variable) will be replaced by the generalized discrete characteristic function.

We introduce the following notation:

$$
\begin{aligned}
t_{m-1+l/q} &= t_{m-1} + \tau_m\frac{l}{q}, \\
U_{m-1+l/q} &= U(t_{m-1+l/q}), \quad l = 0, \ldots, q.
\end{aligned}
$$

Lemma 8. *There exist constants $L_q^*, M_q^* > 0$ such that for $m = 1, \dots, M$ we have*

$$\sum_{l=0}^{q} \|U_{m-1+l/q}\|^2_{\Omega_{t_{m-1+l/q}}} \geq \frac{L_q^*}{\tau_m} \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t, \tag{3.73}$$

$$\|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} \leq \frac{M_q^*}{\tau_m} \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t. \tag{3.74}$$

Proof. Using the equivalence of norms in the space of polynomials of degree $\leq q$, for $p(t) = \tilde{U}(X, t),\ t \in I_m$, and any fixed $X \in \Omega_{t_{m-1}}$, we have

$$\sum_{l=0}^{q} \left|\tilde{U}(X, t_{m-1+l/q})\right|^2 \geq \frac{L_q}{\tau_m} \int_{I_m} |\tilde{U}(X,t)|^2 \, \mathrm{d}t,$$

$$\left|\tilde{U}(X, t_{m-1}^+)\right|^2 \leq \frac{M_q}{\tau_m} \int_{I_m} |\tilde{U}(X,t)|^2 \, \mathrm{d}t,$$

where the constants L_q, $M_q > 0$ were introduced in [31], Section 6.2.3.2. Integrating over $\Omega_{t_{m-1}}$ and using Fubini's theorem, we get

$$\begin{aligned}\sum_{l=0}^{q} \int_{\Omega_{t_{m-1}}} |\tilde{U}\left(X, t_{m-1+l/q}\right)|^2 \mathrm{d}X &\geq \frac{L_q}{\tau_m} \int_{\Omega_{t_{m-1}}} \left(\int_{I_m} |\tilde{U}(X,t)|^2 \mathrm{d}t \right) \mathrm{d}X \\ &= \frac{L_q}{\tau_m} \int_{I_m} \left(\int_{\Omega_{t_{m-1}}} |\tilde{U}(X,t)|^2 \mathrm{d}X \right) \mathrm{d}t.\end{aligned}$$

Analogously we find that

$$\int_{\Omega_{t_{m-1}}} |\tilde{U}\left(X, t_{m-1}^+\right)|^2 \, \mathrm{d}X \leq \frac{M_q}{\tau_m} \int_{I_m} \left(\int_{\Omega_{t_{m-1}}} |\tilde{U}(X,t)|^2 \mathrm{d}X \right) \mathrm{d}t.$$

Now the substitution $X = \mathcal{A}_t^{-1}(x)$, where $X \in \Omega_{t_{m-1}}$, $x \in \Omega_t$, the relation $\tilde{U}(\mathcal{A}_t^{-1}(x), t) = U(x,t)$ and (3.6) imply that

$$\begin{aligned}&\sum_{l=0}^{q} \|U_{m-1+l/q}\|^2_{\Omega_{t_{m-1+l/q}}} \\ &\geq C_J^- \sum_{l=0}^{q} \int_{\Omega_{t_{m-1+l/q}}} |U(x, t_{m-1+l/q})|^2 J^{-1}(x, t_{m-1+l/q}) \, \mathrm{d}x \\ &= C_J^- \sum_{l=0}^{q} \int_{\Omega_{t_{m-1}}} |\tilde{U}(X, t_{m-1+l/q})|^2 \mathrm{d}X \\ &\geq \frac{L_q}{\tau_m} C_J^- \int_{I_m} \left(\int_{\Omega_{t_{m-1}}} |\tilde{U}(X,t)|^2 \mathrm{d}X \right) \mathrm{d}t \\ &= \frac{L_q}{\tau_m} C_J^- \int_{I_m} \left(\int_{\Omega_t} |\tilde{U}(\mathcal{A}_t^{-1}(x), t)|^2 J^{-1}(x,t) \, \mathrm{d}x \right) \mathrm{d}t \\ &\geq \frac{L_q}{\tau_m} (C_J^+)^{-1} C_J^- \int_{I_m} \left(\int_{\Omega_t} |U(x,t)|^2 \, \mathrm{d}x \right) \mathrm{d}t \\ &= \frac{L_q}{\tau_m} (C_J^+)^{-1} C_J^- \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t.\end{aligned}$$

Hence, we get (3.73) with $L_q^* = L_q (C_J^+)^{-1} C_J^-$.

Further, since $x = \mathcal{A}_{t_{m-1}}(X) = X$ for $X \in \Omega_{t_{m-1}}$ and, thus, $\tilde{U}(X, t_{m-1}+) = U(x, t_{m-1}+)$, using the substitution theorem and (3.6), we obtain

$$\begin{aligned}
\|U_{m-1}+\|^2_{\Omega_{t_{m-1}}} &= \int_{\Omega_{t_{m-1}}} |\tilde{U}\big(X, t_{m-1}+\big)|^2 \mathrm{d}X \\
&\le \frac{M_q}{\tau_m} \int_{I_m} \left(\int_{\Omega_{t_{m-1}}} |\tilde{U}(X,t)|^2 \mathrm{d}X \right) \mathrm{d}t \\
&= \frac{M_q}{\tau_m} \int_{I_m} \left(\int_{\Omega_t} |\tilde{U}(\mathcal{A}_t^{-1}(x), t)|^2 J^{-1}(x,t)\, \mathrm{d}x \right) \mathrm{d}t \\
&\le \frac{M_q}{\tau_m} (C_J^-)^{-1} \int_{I_m} \left(\int_{\Omega_t} |U(x,t)|^2\, \mathrm{d}x \right) \mathrm{d}t \\
&\le \frac{M_q^*}{\tau_m} \int_{I_m} \|U\|^2_{\Omega_t}\, \mathrm{d}t,
\end{aligned}$$

where $M_q^* = M_q (C_J^-)^{-1}$. □

In what follows, because of simplicity, we use the notation $\tilde{U}' = \frac{\partial \tilde{U}}{\partial t}$ and do not write the arguments X and t in integrals.

Lemma 9. *There exists a constant $C_{L9} > 0$ such that*

$$\begin{aligned}
&\int_{I_m} (D_t U, \mathcal{U}_s)_{\Omega_t}\, \mathrm{d}t + (\{U\}_{m-1}, \mathcal{U}_s(t_{m-1}^+))_{\Omega_{t_{m-1}}} \\
&\ge \frac{1}{2} \left(\|U(s-)\|^2_{\Omega_s} + \|U(t_{m-1}^+)\|^2_{\Omega_{t_{m-1}}} \right) \\
&\quad - C_{L9} \int_{I_m} \|U\|^2_{\Omega_t}\, \mathrm{d}t - (U_{m-1}^+, U_{m-1}^-)_{\Omega_{t_{m-1}}}.
\end{aligned} \tag{3.75}$$

for any $s \in I_m,\ m = 1, \ldots, M$ and $h \in (0, \overline{h})$.

Proof. By virtue of the definition of the ALE derivative (2.9), the definitions (3.49), (3.50)-(3.51), (3.52) of functions $\tilde{U}, \tilde{\mathcal{U}}_s, \mathcal{U}_s$, the fact that $\tilde{U}'$ is a polynomial of degree $\le q-1$ in time and the substitution theorem for any $s \in \overline{I}_m$ we can write

$$\begin{aligned}
&\int_{I_m} (D_t U, \mathcal{U}_s)_{\Omega_t} \mathrm{d}t = \int_{I_m} \left(\tilde{U}', \tilde{\mathcal{U}}_s J \right)_{\Omega_{t_{m-1}}} \mathrm{d}t \\
&= \int_{I_m} \left(\tilde{U}', \tilde{\mathcal{U}}_s \right)_{\Omega_{t_{m-1}}} \mathrm{d}t + \int_{I_m} \left(\tilde{U}', \tilde{\mathcal{U}}_s (J-1) \right)_{\Omega_{t_{m-1}}} \mathrm{d}t \\
&= \int_{t_{m-1}}^s \left(\tilde{U}', \tilde{U} \right)_{\Omega_{t_{m-1}}} \mathrm{d}t + \int_{I_m} \left(\tilde{U}', \tilde{\mathcal{U}}_s (J-1) \right)_{\Omega_{t_{m-1}}} \mathrm{d}t \\
&= \int_{t_{m-1}}^s \left(\tilde{U}', \tilde{U} J \right)_{\Omega_{t_{m-1}}} \mathrm{d}t + \int_{t_{m-1}}^s \left(\tilde{U}', \tilde{U}(1-J) \right)_{\Omega_{t_{m-1}}} \mathrm{d}t \\
&\qquad + \int_{I_m} \left(\tilde{U}', \tilde{\mathcal{U}}_s (J-1) \right)_{\Omega_{t_{m-1}}} \mathrm{d}t \\
&= \int_{t_{m-1}}^s (D_t U, U)_{\Omega_t} \mathrm{d}t + \int_{t_{m-1}}^s \left(\tilde{U}', \tilde{U}(1-J) \right)_{\Omega_{t_{m-1}}} \mathrm{d}t \\
&\qquad + \int_{I_m} \left(\tilde{U}', \tilde{\mathcal{U}}_s (J-1) \right)_{\Omega_{t_{m-1}}} \mathrm{d}t.
\end{aligned} \tag{3.76}$$

Now we estimate the second and third term on the right-hand side. We begin with the third term. The fact that J is constant on each $\hat{K} \in \hat{\mathcal{T}}_{h,t_{m-1}}$ and the substitution theorem imply that

$$\left|\int_{I_m} \left(\tilde{U}', \tilde{\mathcal{U}}_s(J-1)\right)_{\Omega_{t_{m-1}}} \mathrm{d}t\right| = \left|\sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \int_{I_m} (J_{\hat{K}}-1)\left(\int_{\hat{K}} \tilde{U}'\tilde{\mathcal{U}}_s \,\mathrm{d}X\right) \mathrm{d}t\right|$$
$$\leq \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \max_{t\in I_m} |J_{\hat{K}}-1| \int_{I_m} \left(\int_{\hat{K}} |\tilde{U}'\tilde{\mathcal{U}}_s| \,\mathrm{d}X\right) \mathrm{d}t.$$

Using the relation $J_{\hat{K}}(t_{m-1}) = 1$, we have

$$\max_{t\in I_m} |J_{\hat{K}}-1| \leq \int_{t_{m-1}}^{t_m} |J'_{\hat{K}}| \,\mathrm{d}t \leq c_J \tau_m,$$

where $c_J > 0$ is a constant independent of h, τ_m, m. Then we find that

$$\sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \max_{t\in I_m} |J_{\hat{K}}-1| \int_{I_m} \int_{\hat{K}} |\tilde{U}'\tilde{\mathcal{U}}_s| \,\mathrm{d}X\mathrm{d}t$$
$$\leq c_J \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \tau_m \int_{I_m} \left(\int_{\hat{K}} |\tilde{U}'\tilde{\mathcal{U}}_s| \,\mathrm{d}X\right) \mathrm{d}t$$
$$= c_J \tau_m \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \int_{\hat{K}} \left(\int_{I_m} |\tilde{U}'\tilde{\mathcal{U}}_s| \,\mathrm{d}t\right) \mathrm{d}X$$
$$\leq c_J \tau_m \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \int_{\hat{K}} \left(\left(\int_{I_m} |\tilde{U}'|^2 \,\mathrm{d}t\right)^{1/2} \left(\int_{I_m} |\tilde{\mathcal{U}}_s|^2 \,\mathrm{d}t\right)^{1/2}\right) \mathrm{d}X.$$

Now we apply the inverse inequality in time: There exists a constant $\hat{c}_I$ such that

$$\left(\int_{I_m} |\tilde{U}'(X,t)|^2 \,\mathrm{d}t\right)^{1/2} \leq \frac{\hat{c}_I}{\tau_m} \left(\int_{I_m} |\tilde{U}(X,t)|^2 \,\mathrm{d}t\right)^{1/2} \tag{3.77}$$

holds for every $X \in \Omega_{t_{m-1}}$, $\tau_m \in (0, \overline{\tau})$ and $m = 1, \ldots, M$.

This inequality, Young's inequality, Fubini's theorem, inequality (3.54), the substitution theorem and assumption (3.6) imply that

$$\begin{aligned}
&\tau_m \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \int_{\hat{K}} \left(\left(\int_{I_m} |\tilde{U}'|^2 \,\mathrm{d}t \right)^{1/2} \left(\int_{I_m} |\tilde{\mathcal{U}}_s|^2 \,\mathrm{d}t \right)^{1/2} \right) \mathrm{d}X \\
&\le \hat{c}_I \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \int_{\hat{K}} \left(\int_{I_m} |\tilde{U}|^2 \mathrm{d}t \right)^{1/2} \left(\int_{I_m} |\tilde{\mathcal{U}}_s|^2 \mathrm{d}t \right)^{1/2} \mathrm{d}X \\
&\le \frac{\hat{c}_I}{2} \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \int_{\hat{K}} \left(\int_{I_m} \left(|\tilde{U}|^2 + |\tilde{\mathcal{U}}_s|^2 \right) \mathrm{d}t \right) \mathrm{d}X \\
&= \frac{\hat{c}_I}{2} \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \int_{I_m} \left(\int_{\hat{K}} \left(|\tilde{U}|^2 + |\tilde{\mathcal{U}}_s|^2 \right) \mathrm{d}X \right) \mathrm{d}t \\
&= \frac{\hat{c}_I}{2} \left(\int_{I_m} \|\tilde{U}\|^2_{\Omega_{t_{m-1}}} \mathrm{d}t + \int_{I_m} \|\tilde{\mathcal{U}}_s\|^2_{\Omega_{t_{m-1}}} \mathrm{d}t \right) \\
&\le \frac{\hat{c}_I}{2} (1 + \tilde{c}^{(1)}_{CH}) \int_{I_m} \|\tilde{U}\|^2_{\Omega_{t_{m-1}}} \mathrm{d}t \\
&= \frac{\hat{c}_I}{2} (1 + \tilde{c}^{(1)}_{CH}) \int_{I_m} \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \int_{\hat{K}} \left(|\tilde{U}|^2 \mathrm{d}X \right) \mathrm{d}t \\
&= \frac{\hat{c}_I}{2} (1 + \tilde{c}^{(1)}_{CH}) \int_{I_m} \left(\int_{\Omega_t} |U|^2 J^{-1} \mathrm{d}x \right) \mathrm{d}t \\
&\le c^* \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t,
\end{aligned}$$

where $c^* = (C_J^-)^{-1} \hat{c}_I (1 + \tilde{c}^{(1)}_{CH})/2$. Summarizing the obtained results, we see that we have proved the inequality

$$\left| \int_{I_m} \left(\tilde{U}', \tilde{\mathcal{U}}_s (J-1) \right)_{\Omega_{t_{m-1}}} \mathrm{d}t \right| \le c^* c_J \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t. \tag{3.78}$$

Similarly as above we can estimate the second term on the right-hand side of (3.76):

$$\begin{aligned}
&\left| \int_{t_{m-1}}^{s} \left(\tilde{U}', \tilde{U}(1-J) \right)_{\Omega_{t_{m-1}}} \mathrm{d}t \right| \\
&\le \left| \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \int_{I_m} (1-J) \left(\int_{\hat{K}} \tilde{U}' \tilde{U} \,\mathrm{d}X \right) \mathrm{d}t \right| \\
&\le \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \max_{t\in I_m} |1 - J_{\hat{K}}| \int_{I_m} \int_{\hat{K}} |\tilde{U}' \tilde{U}| \,\mathrm{d}X \mathrm{d}t \\
&\le c_J \tau_m \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \int_{I_m} \int_{\hat{K}} |\tilde{U}' \tilde{U}| \,\mathrm{d}X \mathrm{d}t \\
&= c_J \tau_m \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \int_{\hat{K}} \left(\int_{I_m} |\tilde{U}' \tilde{U}| \mathrm{d}t \right) \mathrm{d}X \\
&\le c_J \tau_m \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \int_{\hat{K}} \left(\left(\int_{I_m} |\tilde{U}'|^2 \,\mathrm{d}t \right)^{1/2} \left(\int_{I_m} |\tilde{U}|^2 \,\mathrm{d}t \right)^{1/2} \right) \mathrm{d}X =: \mathrm{RHS}.
\end{aligned}$$

Now the inverse inequality in time, Young's inequality, Fubini's theorem and (3.6) yield the inequality

$$
\begin{aligned}
\text{RHS} &\leq c^* \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t}} \int_{\hat{K}} \left(\int_{I_m} |\tilde{U}|^2 \mathrm{d}t\right)^{1/2} \left(\int_{I_m} |\tilde{U}|^2 \mathrm{d}t\right)^{1/2} \mathrm{d}X \\
&\leq c^{**} \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t}} \int_{\hat{K}} \left(\int_{I_m} |\tilde{U}|^2 + |\tilde{U}|^2 \mathrm{d}t\right) \mathrm{d}X \\
&\leq c \int_{I_m} \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t}} \int_{\hat{K}} |\tilde{U}|^2 \mathrm{d}X\, \mathrm{d}t \leq c \int_{I_m} \sum_{K\in\mathcal{T}_{h,t}} \int_K |U|^2 J^{-1}\, \mathrm{d}x\, \mathrm{d}t \\
&\leq c \int_{I_m} \left(\int_{\Omega_t} |U|^2 \mathrm{d}X\right) \mathrm{d}t = c \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t.
\end{aligned}
$$

From above estimates we find that

$$
\left| \int_{t_{m-1}}^{s} \left(\tilde{U}', \tilde{U}(1-J)\right)_{\Omega_{t_{m-1}}} \mathrm{d}t \right| \leq c_1 \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t, \tag{3.79}
$$

where $c_1 = c_J (C_J^-)^{-1} \hat{c}_I / 2$.

Finally, from (3.76), (3.78), (3.79) and analogy to (3.44), (3.53) putting $c_2 = c^* c_J + c_1$ we find that

$$
\begin{aligned}
&\int_{I_m} (D_t U, \mathcal{U}_s)_{\Omega_t} \mathrm{d}t + (\{U\}_{m-1}, \mathcal{U}_s(t_{m-1}+))_{\Omega_{t_{m-1}}} \\
&\geq \int_{t_{m-1}}^{s} (D_t U, U)_{\Omega_t} \mathrm{d}t + \|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}} - (U^-_{m-1}, U^+_{m-1})_{\Omega_{t_{m-1}}} - c_2 \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t \\
&= \frac{1}{2} \int_{t_{m-1}}^{s} \left(\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_t} U^2(x,t) \mathrm{d}x \right) \mathrm{d}t - \frac{1}{2} \int_{t_{m-1}}^{s} \left(U^2 \mathrm{div}, \boldsymbol{z} \right)_{\Omega_t} \mathrm{d}t \\
&\quad + \|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}} - (U^-_{m-1}, U^+_{m-1})_{\Omega_{t_{m-1}}} - c_2 \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t \\
&= \frac{1}{2} \left(\|U(s-)\|^2_{\Omega_s} + \|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}} \right) - \frac{c_z}{2} \int_{t_{m-1}}^{s} \|U\|_{\Omega_t} \mathrm{d}t \\
&\quad - c_2 \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t - (U^-_{m-1}, U^+_{m-1})_{\Omega_{t_{m-1}}},
\end{aligned}
$$

which implies (3.75) with $C_{L9} = c_z/2 + c_2$. □

In the following lemmas, for simplicity we use the notation $\mathcal{U}_l^*$ and $\tilde{\mathcal{U}}_l^*$ for the discrete characteristic functions to U and $\tilde{U}$, respectively at the time instant $t_{m-1+l/q}$, $l = 0, \ldots, q$.

Lemma 10. *There exists a constant $C_{L10} > 0$ such that*

$$
|a_h(U, \mathcal{U}_l^*, t) + \beta_0 J_h(U, \mathcal{U}_l^*, t)| \leq C_{L10} \left(\|U\|^2_{DG,t} + \|\mathcal{U}_l^*\|^2_{DG,t} + \|u_D\|^2_{DGB,t} \right) \tag{3.80}
$$

for all $t \in I_m$, $m = 1, \ldots, M$, $l = 0, \ldots, q$, $h \in (0, \overline{h})$.

Proof. Using the definition of the form a_h, the property of the function β, the

Cauchy inequality and Young's inequality, we get

$$
\begin{aligned}
&|a_h(U,\mathcal{U}_l^*,t)| \qquad (3.81)\\
&\le \beta_1 \sum_{K\in\mathcal{T}_{h,t}} \int_K \left(|\nabla U|^2 + |\nabla \mathcal{U}_l^*|^2\right) \mathrm{d}x\\
&+\beta_1 \sum_{\Gamma\in\mathcal{F}_{h,t}^I} \int_\Gamma \left(\frac{h(\Gamma)}{c_W}\left(|\nabla U_\Gamma^{(L)}|^2 + |\nabla U_\Gamma^{(R)}|^2\right) + \frac{c_W}{h(\Gamma)}\,[\mathcal{U}_l^*]^2 \right) \mathrm{d}S\\
&+\beta_1 \sum_{\Gamma\in\mathcal{F}_{h,t}^I} \int_\Gamma \left(\frac{h(\Gamma)}{c_W}\left(|\nabla (\mathcal{U}_l^*)_\Gamma^{(L)}|^2 + |\nabla (\mathcal{U}_l^*)_\Gamma^{(R)}|^2\right) + \frac{c_W}{h(\Gamma)}\,[U]^2 \right) \mathrm{d}S\\
&+\beta_1 \sum_{\Gamma\in\mathcal{F}_{h,t}^B} \int_\Gamma \left(\frac{h(\Gamma)}{c_W}|\nabla U|^2 + \frac{c_W}{h(\Gamma)}\,|\mathcal{U}_l^*|^2 \right) \mathrm{d}S\\
&+\beta_1 \sum_{\Gamma\in\mathcal{F}_{h,t}^B} \int_\Gamma \left(\frac{h(\Gamma)}{c_W}|\nabla \mathcal{U}_l^*|^2 + \frac{c_W}{h(\Gamma)}\,|U|^2 \right) \mathrm{d}S\\
&+\beta_1 \sum_{\Gamma\in\mathcal{F}_{h,t}^B} \int_\Gamma |\nabla \mathcal{U}_l^*|\,|u_D|\,\mathrm{d}S.
\end{aligned}
$$

The last term can be estimated using Young's inequality and the relation $h(\Gamma) \le h_{K_\Gamma^{(L)}}$, for each $\varepsilon > 0$ in the following way:

$$
\begin{aligned}
&\beta_1 \sum_{\Gamma\in\mathcal{F}_{h,t}^B} \int_\Gamma |\nabla \mathcal{U}_l^*|\; |u_D|\; \mathrm{d}S\\
&\le \frac{\beta_1\varepsilon}{2} \sum_{\Gamma\in\mathcal{F}_{h,t}^B} \int_\Gamma h(\Gamma)^{-1}\,|u_D|^2\;\mathrm{d}S + \frac{\beta_1}{2\varepsilon} \sum_{\Gamma\in\mathcal{F}_{h,t}^B} \int_\Gamma h_{K_\Gamma^{(L)}}\,|\nabla \mathcal{U}_l^*|^2\;\mathrm{d}S\\
&\le \frac{\beta_1\varepsilon}{2c_W} J_h^B(u_D,u_D) + \frac{\beta_1}{2\varepsilon} \sum_{\Gamma\in\mathcal{F}_{h,t}^B} \int_{\partial K_\Gamma^{(L)}} h_{K_\Gamma^{(L)}}\,|\nabla \mathcal{U}_l^*|^2\;\mathrm{d}S.
\end{aligned}
$$

Now we express the first term on the right-hand side of this inequality with the aid of the definition of the $\|\cdot\|_{DGB,t}$-norm (3.2) and to the second term we apply the multiplicative trace inequality (3.12) and the inverse inequality (3.13). We get

$$
\beta_1 \sum_{\Gamma\in\mathcal{F}_{h,t}^B} \int_\Gamma |\nabla \mathcal{U}_l^*|\; |u_D|\;\mathrm{d}S \le \frac{\beta_1\varepsilon}{2c_W}\|u_D\|_{DGB,t}^2 + \frac{\beta_1}{2\varepsilon}c_M(c_I+1)\|\mathcal{U}_l^*\|_{DG,t}^2. \quad (3.82)
$$

Setting $\varepsilon := \frac{\beta_1}{\beta_0} c_M (c_I + 1)$ in (3.82) and substituting back to (3.81) we get

$$
\begin{aligned}
|a_h(U, \mathcal{U}_l^*, t)| &\leq \beta_1 \sum_{K \in \mathcal{T}_{h,t}} \int_K \left(|\nabla U|^2 + |\nabla \mathcal{U}_l^*|^2 \right) \mathrm{d}x \\
&+ \beta_1 \sum_{\Gamma \in \mathcal{F}_{h,t}^I} \int_\Gamma \frac{h(\Gamma)}{c_W} \left(|\nabla U_\Gamma^{(L)}|^2 + |\nabla U_\Gamma^{(R)}|^2 \right) \mathrm{d}S + \beta_1 \sum_{\Gamma \in \mathcal{F}_{h,t}^B} \int_\Gamma \frac{h(\Gamma)}{c_W} |\nabla U|^2 \, \mathrm{d}S \\
&+ \beta_1 \sum_{\Gamma \in \mathcal{F}_{h,t}^I} \int_\Gamma \frac{h(\Gamma)}{c_W} \left(|\nabla (\mathcal{U}_l^*)_\Gamma^{(L)}|^2 + |\nabla (\mathcal{U}_l^*)_\Gamma^{(R)}|^2 \right) \mathrm{d}S \\
&+ \beta_1 \sum_{\Gamma \in \mathcal{F}_{h,t}^B} \int_\Gamma \frac{h(\Gamma)}{c_W} |\nabla \mathcal{U}_l^*|^2 \, \mathrm{d}S + \frac{\beta_1^2}{2\beta_0 c_W} c_M (c_I + 1) \|u_D\|_{DGB,t}^2 \\
&+ \frac{\beta_0}{2} \|\mathcal{U}_l^*\|_{DG,t}^2.
\end{aligned}
$$

From the definition (2.25) and Young's inequality we find that

$$
J_{h,}(U, \mathcal{U}_l^*, t) \leq J_h(U, U, t) + J_h(\mathcal{U}_l^*, \mathcal{U}_l^*, t). \tag{3.83}
$$

Using the inequality $h(\Gamma) \leq h_K$ for $\Gamma \subset \partial K$ and (3.83), we have

$$
\begin{aligned}
&|a_h(U, \mathcal{U}_l^*, t)| \qquad (3.84) \\
&\leq \beta_1 \sum_{K \in \mathcal{T}_{h,t}} \int_K \left(|\nabla U|^2 + |\nabla \mathcal{U}_l^*|^2 \right) \mathrm{d}x \\
&\quad + \frac{\beta_1}{c_W} \sum_{\Gamma \in \mathcal{F}_{h,t}^I} \int_\Gamma \left(h_{K_\Gamma^{(L)}} |\nabla U_\Gamma^{(L)}|^2 + h_{K_\Gamma^{(R)}} |\nabla U_\Gamma^{(R)}|^2 \right) \mathrm{d}S \\
&\quad + \frac{\beta_1}{c_W} \sum_{\Gamma \in \mathcal{F}_{h,t}^B} \int_\Gamma h_{K_\Gamma^{(L)}} |\nabla U|^2 \, \mathrm{d}S \\
&\quad + \frac{\beta_1}{c_W} \sum_{\Gamma \in \mathcal{F}_{h,t}^I} \int_\Gamma \left(h_{K_\Gamma^{(L)}} |\nabla (\mathcal{U}_l^*)_\Gamma^{(L)}|^2 + h_{K_\Gamma^{(R)}} |\nabla (\mathcal{U}_l^*)_\Gamma^{(R)}|^2 \right) \mathrm{d}S \\
&\quad + \frac{\beta_1}{c_W} \sum_{\Gamma \in \mathcal{F}_{h,t}^B} \int_\Gamma h_{K_\Gamma^{(L)}} |\nabla \mathcal{U}_l^*|^2 \, \mathrm{d}S + \frac{\beta_1^2}{2\beta_0 c_W} c_M (c_I + 1) \|u_D\|_{DGB,t}^2 \\
&\quad + \frac{\beta_0}{2} \|\mathcal{U}_l^*\|_{DG,t}^2 + \beta_1 \, J_h(\mathcal{U}_l^*, \mathcal{U}_l^*, t) + \beta_1 \, J_h(U, U, t) \\
&\leq \beta_1 \sum_{K \in \mathcal{T}_{h,t}} \int_K \left(|\nabla U|^2 + |\nabla \mathcal{U}_l^*|^2 \right) \mathrm{d}x \\
&\quad + \frac{\beta_1}{c_W} \sum_{K \in \mathcal{T}_{h,t}} \int_{\partial K} h_K \left(|\nabla U|^2 + |\nabla \mathcal{U}_l^*|^2 \right) \mathrm{d}S \\
&\quad + \frac{\beta_1^2}{2\beta_0 c_W} c_M (c_I + 1) \|u_D\|_{DGB,t}^2 + \frac{\beta_0}{2} \|\mathcal{U}_l^*\|_{DG,t}^2.
\end{aligned}
$$

Now, applying the multiplicative inequality and the inverse inequality, we can obtain the estimate

$$\sum_{K\in\mathcal{T}_{h,t}} \int_{\partial K} h_K \left(|\nabla U|^2 + |\nabla \mathcal{U}_l^*|^2\right) \mathrm{d}S \tag{3.85}$$
$$= \sum_{K\in\mathcal{T}_{h,t}} h_K \left(\|\nabla U\|^2_{L^2(\partial K)} + \|\nabla \mathcal{U}_l^*\|^2_{L^2(\partial K)}\right)$$
$$\leq c_M \sum_{K\in\mathcal{T}_{h,t}} h_K (\|\nabla U\|_{L^2(K)} \underbrace{|\nabla U|_{H^1(K)}}_{\leq c_I h_K^{-1}\|\nabla U\|_{L^2(K)}} + h_K^{-1}\|\nabla U\|^2_{L^2(K)})$$
$$+ c_M \sum_{K\in\mathcal{T}_{h,t}} h_K (\|\nabla \mathcal{U}_l^*\|_{L^2(K)} \underbrace{|\nabla \mathcal{U}_l^*|_{H^1(K)}}_{\leq c_I h_K^{-1}\|\nabla \mathcal{U}_l^*\|_{L^2(K)}} + h_K^{-1}\|\nabla \mathcal{U}_l^*\|^2_{L^2(K)})$$
$$\leq c_M(c_I+1) \sum_{K\in\mathcal{T}_{h,t}} \left(\|\nabla U\|^2_{L^2(K)} + \|\nabla \mathcal{U}_l^*\|^2_{L^2(K)}\right)$$
$$\leq c_M(c_I+1) \sum_{K\in\mathcal{T}_{h,t}} \left(|U|^2_{H^1(\Omega)} + |\mathcal{U}_l^*|^2_{H^1(\Omega)}\right).$$

From (3.84), (3.85), the definition of the $\|\cdot\|_{DG,t}$-norm, using the inequality (3.83) and putting $C_{L10} = \max\{\beta_0 + \beta_1 + \beta_1 c_M(c_I+1)/c_W, \beta_1^2 c_M(c_I+1)/(2\beta_0 c_W)\}$, we finally get

$$|a_h(U,\mathcal{U}_l^*,t) + \beta_0\, J_h(U,\mathcal{U}_l^*,t)| \leq \left(\beta_1 + \frac{\beta_1}{c_W} c_M(c_I+1)\right) |U|^2_{H^1(\Omega_t,\mathcal{T}_{h,t})}$$
$$+(\beta_0+\beta_1) J_h(U,U,t) + \left(\beta_1 + \frac{\beta_0}{2} + \frac{\beta_1}{c_W} c_M(c_I+1)\right) |\mathcal{U}_l^*|^2_{H^1(\Omega_t,\mathcal{T}_{h,t})}$$
$$+(\beta_0+\beta_1) J_h(\mathcal{U}_l^*,\mathcal{U}_l^*,t) + \frac{\beta_1^2}{2\beta_0 c_W} c_M(c_I+1)\|u_D\|^2_{DGB,t}$$
$$\leq C_{L10}\left(\|U\|^2_{DG,t} + \|\mathcal{U}_l^*\|^2_{DG,t} + \|u_D\|^2_{DGB,t}\right).$$

□

Lemma 11. *For each $k_1^* > 0$ there exists a constant $c_b^* > 0$ such that for the approximate solution U and the discrete characteristic function $\mathcal{U}_l^*$ we have the inequality*

$$\int_{I_m} |b_h(U,\mathcal{U}_l^*,t)|\mathrm{d}t \leq \frac{\beta_0}{2k_1^*} \int_{I_m} \|\mathcal{U}_l^*\|^2_{DG,t}\mathrm{d}t + c_b^* \int_{I_m} \|U\|^2_{\Omega_t}\mathrm{d}t. \tag{3.86}$$

Proof. By (2.28),

$$b_h(U,\mathcal{U}_l^*,t) = \underbrace{-\sum_{K\in\mathcal{T}_{h,t}} \int_K \sum_{s=1}^d f_s(U) \frac{\partial \mathcal{U}_l^*}{\partial x_s}\,\mathrm{d}x}_{:=\sigma_1} \tag{3.87}$$
$$+ \underbrace{\sum_{\Gamma\in\mathcal{F}^I_{h,t}} \int_\Gamma H(U_\Gamma^{(L)}, U_\Gamma^{(R)}, \mathbf{n}_\Gamma)\,[\mathcal{U}_l^*]_\Gamma\,\mathrm{d}S + \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma H(U_\Gamma^{(L)}, U_\Gamma^{(L)}, \mathbf{n}_\Gamma)\,\mathcal{U}_l^*\,\mathrm{d}S.}_{:=\sigma_2}$$

Then from the Lipschitz-continuity of the functions f_s, $s = 1,\dots,d$, with the modul $L_f > 0$, assumption that $f_s(0) = 0$ and the Cauchy inequality, we obtain

$$
\begin{aligned}
|\sigma_1| &\leq \sum_{K\in\mathcal{T}_{h,t}} \int_K \sum_{s=1}^{d} |f_s(U) - f_s(0)| \left| \frac{\partial \mathcal{U}_l^*}{\partial x_s} \right| \mathrm{d}x \\
&\leq L_f \sum_{K\in\mathcal{T}_{h,t}} \int_K \sum_{s=1}^{d} |U| \left| \frac{\partial \mathcal{U}_l^*}{\partial x_s} \right| \mathrm{d}x \leq L_f \sqrt{d}\, \|U\|_{\Omega_t} |\mathcal{U}_l^*|_{H^1(\Omega_t,\mathcal{T}_{h,t})}.
\end{aligned} \tag{3.88}
$$

Now we shall estimate σ_2. From the relation $f_s(0) = 0$, $s = 1, \dots, d$, and the consistency property **(H2)** of the numerical flux H we have $H(0, 0, \mathbf{n}_\Gamma) = 0$. Then we can use the Lipschitz-continuity of H and get

$$
\begin{aligned}
|\sigma_2| \;\leq\; & L_H \sum_{\Gamma\in\mathcal{F}^I_{h,t}} \int_\Gamma (|U_\Gamma^{(L)}| + |U_\Gamma^{(R)}|)\, [\mathcal{U}_l^*]\, \mathrm{d}S \\
& + L_H \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma (|U_\Gamma^{(L)}| + |U_\Gamma^{(L)}|)\, |(\mathcal{U}_l^*)_\Gamma^{(L)}|\, \mathrm{d}S.
\end{aligned}
$$

Using the fact that $U_\Gamma^{(R)} = U_\Gamma^{(L)}$ for $\Gamma \in \mathcal{F}^B_{h,t}$, the Cauchy inequality and the relation $h(\Gamma) \leq h_K$, if $\Gamma \subset \partial K$, we obtain

$$
\begin{aligned}
&|\sigma_2| \\
&\leq L_H \sum_{\Gamma\in\mathcal{F}^I_{h,t}} \int_\Gamma (|U_\Gamma^{(L)}| + |U_\Gamma^{(R)}|)\, |[\mathcal{U}_l^*]|\, \mathrm{d}S \\
&\quad + L_H \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma (|U_\Gamma^{(L)}| + |U_\Gamma^{(R)}|)\, |(\mathcal{U}_l^*)_\Gamma^{(L)}|\, \mathrm{d}S \\
&\leq \frac{L_H}{\sqrt{c_W}} \left(c_W \sum_{\Gamma\in\mathcal{F}^I_{h,t}} \int_\Gamma \frac{[\mathcal{U}_l^*]^2}{h(\Gamma)} \mathrm{d}S + c_W \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma \frac{\left((\mathcal{U}_l^*)_\Gamma^{(L)}\right)^2}{h(\Gamma)} \mathrm{d}S \right)^{1/2} \\
&\quad \times \left(\sum_{\Gamma\in\mathcal{F}_{h,t}} h(\Gamma) \int_\Gamma (|U_\Gamma^{(L)}| + |U_\Gamma^{(R)}|)^2 \, \mathrm{d}S \right)^{1/2} \\
&\leq \frac{L_H}{\sqrt{c_W}} J_h(\mathcal{U}_l^*, \mathcal{U}_l^*, t)^{1/2} \left(\sum_{\Gamma\in\mathcal{F}_{h,t}} h(\Gamma) \int_\Gamma \left(|U_\Gamma^{(L)}|^2 + |U_\Gamma^{(R)}|^2 \right) \mathrm{d}S \right)^{1/2} \\
&\leq \frac{L_H}{\sqrt{c_W}} J_h(\mathcal{U}_l^*, \mathcal{U}_l^*, t)^{1/2} \\
&\quad \times \left(\sum_{\Gamma\in\mathcal{F}_{h,t}} h_{K_\Gamma^{(L)}} \int_{\partial K_\Gamma^{(L)}\cap\Gamma} |U_\Gamma^{(L)}|^2 \mathrm{d}S + h_{K_\Gamma^{(R)}} \int_{\partial K_\Gamma^{(R)}\cap\Gamma} |U_\Gamma^{(R)}|^2 \, \mathrm{d}S \right)^{1/2} \\
&\leq \frac{L_H}{\sqrt{c_W}} J_h(\mathcal{U}_l^*, \mathcal{U}_l^*, t)^{1/2} \left(\sum_{K\in\mathcal{T}_{h,t}} \int_{\partial K} h_K |U|^2 \, \mathrm{d}S \right)^{1/2} \\
&= \frac{L_H}{\sqrt{c_W}} J_h(\mathcal{U}_l^*, \mathcal{U}_l^*, t)^{1/2} \left(\sum_{K\in\mathcal{T}_{h,t}} h_K \|U\|^2_{L^2(\partial K)} \right)^{1/2}.
\end{aligned} \tag{3.89}
$$

Substituting (3.88) and (3.89) into (3.87), using the Cauchy inequality and the definition of the $\|\cdot\|_{DG,t}$-norm, we find that

$$\begin{aligned}
|b_h(U,\mathcal{U}_l^*,t)| &\leq L_f\sqrt{d}\,\|U\|_{\Omega_t}\,|\mathcal{U}_l^*|_{H^1(\Omega_t,\mathcal{T}_{h,t})} \qquad (3.90)\\
&\quad + \frac{L_H}{\sqrt{c_W}}\,J_h(\mathcal{U}_l^*,\mathcal{U}_l^*,t)^{1/2}\left(\sum_{K\in\mathcal{T}_{h,t}} h_K\|U\|^2_{L^2(\partial K)}\right)^{1/2}\\
&\leq \left(L_f^2 d\,\|U\|^2_{\Omega_t} + \frac{L_H^2}{c_W}\sum_{K\in\mathcal{T}_{h,t}} h_K\|U\|^2_{L^2(\partial K)}\right)^{1/2}\\
&\quad \times\left(|\mathcal{U}_l^*|^2_{H^1(\Omega_t,\mathcal{T}_{h,t})} + J_h(\mathcal{U}_l^*,\mathcal{U}_l^*,t)\right)^{1/2}\\
&\leq c\,\|\mathcal{U}_l^*\|_{DG,t}\left(\|U\|_{\Omega_t} + \left(\sum_{K\in\mathcal{T}_{h,t}} h_K\|U\|^2_{L^2(\partial K)}\right)^{1/2}\right),
\end{aligned}$$

where $c = \left(\max\{L_f^2\, d,\, L_H^2/c_W\}\right)^{1/2}$. Furthermore, the multiplicative trace inequality and the inverse inequality imply that

$$\begin{aligned}
\sum_{K\in\mathcal{T}_{h,t}} h_K\|U\|^2_{L^2(\partial K)} &\leq c_M\sum_{K\in\mathcal{T}_{h,t}} h_K\left(\|U\|_{L^2(K)}|U|_{H^1(K)} + h_K^{-1}\|U\|^2_{L^2(K)}\right)\\
&\leq c_M(c_I+1)\sum_{K\in\mathcal{T}_{h,t}}\|U\|^2_{L^2(K)} = c_M(c_I+1)\|U\|^2_{\Omega_t}.
\end{aligned}$$

Hence, from this relation, (3.90) and Young's inequality we get

$$\begin{aligned}
|b_h(U,\mathcal{U}_l^*,t)| &\leq c_1\,\|\mathcal{U}_l^*\|_{DG,t}\,\|U\|_{\Omega_t} \leq \frac{\beta_0}{2k_1^*}\|\mathcal{U}_l^*\|^2_{DG,t} + c_1^2\frac{k_1^*}{2\beta_0}\|U\|^2_{\Omega_t}\\
&= \frac{\beta_0}{2k_1^*}\,\|\mathcal{U}_l^*\|^2_{DG,t} + c_b^*\,\|U\|^2_{\Omega_t},
\end{aligned}$$

where $c_1 = c(1+\sqrt{c_M(c_I+1)})$, $k_1^* > 0$ and $c_b^* = c_1^2 k_1^*/\beta_0$. Integrating over the interval I_m, we finally have (3.86). □

Lemma 12. *For each $k_2^* > 0$ there exists a constant $c_d^* > 0$ such that the approximate solution U and the discrete characteristic function $\mathcal{U}_l^*$ satisfy the inequality*

$$\int_{I_m}|d_h(U,\mathcal{U}_l^*,t)|\,\mathrm{d}t \leq \frac{\beta_0}{2k_2^*}\int_{I_m}\|U\|^2_{DG,t}\,\mathrm{d}t + \frac{c_d^*}{2\beta_0}\int_{I_m}\|\mathcal{U}_l^*\|^2_{\Omega_t}\,\mathrm{d}t. \qquad (3.91)$$

Proof. By (2.29), (3.10) and the Cauchy and Young's inequalities,

$$\begin{aligned}
\int_{I_m}|d_h(U,\mathcal{U}_l^*,t)|\,\mathrm{d}t &\leq c_z\int_{I_m}\sum_{K\in\mathcal{T}_{h,t}}\int_K\sum_{s=1}^d|\mathcal{U}_l^*||\frac{\partial U}{\partial x_s}|\,\mathrm{d}x\,\mathrm{d}t\\
&\leq c_z\int_{I_m}\|\mathcal{U}_l^*\|_{\Omega_t}|U|_{H^1(\Omega_t,\mathcal{T}_{h,t})}\,\mathrm{d}t\\
&\leq c_z\int_{I_m}\|\mathcal{U}_l^*\|_{\Omega_t}\|U\|_{DG,t}\,\mathrm{d}t\\
&\leq \frac{\beta_0}{2k_2^*}\int_{I_m}\|U\|^2_{DG,t}\,\mathrm{d}t + \frac{c_z^2k_2^*}{2\beta_0}\int_{I_m}\|\mathcal{U}_l^*\|^2_{\Omega_t}\,\mathrm{d}t,
\end{aligned}$$

which is (3.91) with $c_d^* = c_z^2 k_2^*$. □

Lemma 13. *For the approximate solution U, the discrete characteristic function $\mathcal{U}_l^*$ and any $k_3 > 0$ we have*

$$\begin{aligned}\int_{I_m} |l_h(\mathcal{U}_l^*, t)| \, \mathrm{d}t \;\; \leq \;\; & \frac{1}{2} \int_{I_m} \left(\|g\|_{\Omega_t}^2 + \|\mathcal{U}_l^*\|_{\Omega_t}^2 \right) \, \mathrm{d}t \\ & + \frac{\beta_0 k_3}{2} \int_{I_m} \|u_D\|_{DGB,t}^2 \, \mathrm{d}t + \frac{\beta_0}{2k_3} \int_{I_m} \|\mathcal{U}_l^*\|_{DG,t}^2 \, \mathrm{d}t.\end{aligned} \tag{3.92}$$

Proof. From (2.30), using the Cauchy and Young's inequality with $k_3 > 0$, we find that

$$\begin{aligned} & |(g, \mathcal{U}_l^*) + \beta_0 \, c_W \sum_{\Gamma \in \mathcal{F}_{h,t}^B} h(\Gamma)^{-1} \int_\Gamma u_D \, \mathcal{U}_l^* \, \mathrm{d}S| \\ & \leq \frac{1}{2}(\|g\|_{\Omega_t}^2 + \|\mathcal{U}_l^*\|_{\Omega_t}^2) + \frac{\beta_0 k_3}{2} \underbrace{c_W \sum_{\Gamma \in \mathcal{F}_{h,t}^B} h(\Gamma)^{-1} \int_\Gamma |u_D|^2 \, \mathrm{d}S}_{= \|u_D\|_{DGB,t}^2} \\ & + \frac{\beta_0}{2k_3} \underbrace{c_W \sum_{\Gamma \in \mathcal{F}_{h,t}^B} h(\Gamma)^{-1} \int_\Gamma |\mathcal{U}_l^*|^2 \, \mathrm{d}S}_{\leq J_h(\mathcal{U}_l^*, \mathcal{U}_l^*, t) \leq \|\mathcal{U}_l^*\|_{DG,t}^2}, \end{aligned}$$

from which we get (3.92) by integrating both sides over the interval I_m. □

3.4.3 Estimate of the term $\int_{I_m} \|U\|_{\Omega_t}^2 \mathrm{d}t$

Using Lemmas 9 - 13 and properties of the discrete characteristic function proved in Theorem 1, we can finally estimate the problematic term $\int_{I_m} \|U\|_{\Omega_t}^2 \mathrm{d}t$ in terms of data.

Theorem 4. *There exist constants $C_{T4}, C_{T4}^* > 0$ such that*

$$\int_{I_m} \|U\|_{\Omega_t}^2 \mathrm{d}t \leq C_{T4} \tau_m \left(\|U_{m-1}^-\|_{\Omega_{t_{m-1}}}^2 + \int_{I_m} \left(\|g\|_{\Omega_t}^2 + \|u_D\|_{DGB,t}^2 \right) \mathrm{d}t \right) \tag{3.93}$$

provided $0 < \tau_m < C_{T4}^*$.

Proof. For $q = 1$, the proof is contained in [6]. Let us assume that $q \geq 2$, $l \in \{1, \ldots, q-1\}$.

From the definition of the approximate solution (2.32)–(2.33) for $\varphi := \mathcal{U}_l^*$ we get

$$\begin{aligned} & \int_{I_m} (D_t U, \mathcal{U}_l^*)_{\Omega_t} \, \mathrm{d}t + \left(\{U\}_{m-1}, \mathcal{U}_l^* \right)_{\Omega_{t_{m-1}}} \\ & = \int_{I_m} (-a_h(U, \mathcal{U}_l^*, t) - \beta_0 J_h(U, \mathcal{U}_l^*, t) - b_h(U, \mathcal{U}_l^*, t)) \, \mathrm{d}t \\ & \quad + \int_{I_m} (-d_h(U, \mathcal{U}_l^*, t) + l_h(\mathcal{U}_l^*, t)) \, \mathrm{d}t. \end{aligned} \tag{3.94}$$

This relation and Lemma 9 imply that

$$\frac{1}{2}\left(\left\|U^-_{m-1+l/q}\right\|^2_{\Omega_{t_{m-1+l/q}}} + \|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}}\right) \tag{3.95}$$
$$\leq \int_{I_m} |a_h(U,\mathcal{U}^*_l,t) + \beta_0 J_h(U,\mathcal{U}^*_l,t)|\,\mathrm{d}t + \int_{I_m} |b_h(U,\mathcal{U}^*_l,t)|\,\mathrm{d}t$$
$$+\int_{I_m} |d_h(U,\mathcal{U}^*_l,t)|\,\mathrm{d}t + \int_{I_m} |l_h(\mathcal{U}^*_l,t)|\,\mathrm{d}t$$
$$+\left(U^-_{m-1}, U^+_{m-1}\right)_{\Omega_{t_{m-1}}} + C_{L9}\int_{I_m} \|U\|^2_{\Omega_t}\,\mathrm{d}t \equiv \mathrm{RHS}.$$

Now we need to estimate the right-hand side of (3.95) from above. Using (3.80), (3.86), (3.91), (3.92) with $k_1 = k_2 = k_3 = 1$, (3.75) and Young's inequality with any $\delta_2 > 0$, we get

$$\begin{aligned}
\mathrm{RHS} \;\leq\; & C_{L10}\int_{I_m}\left(\|U\|^2_{DG,t} + \|\mathcal{U}^*_l\|^2_{DG,t} + \|u_D\|^2_{DGB,t}\right)\mathrm{d}t \\
&+\frac{\beta_0}{2}\int_{I_m}\|\mathcal{U}^*_l\|^2_{DG,t}\mathrm{d}t + c_b\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t + \frac{\beta_0}{2}\int_{I_m}\|U\|^2_{DG,t}\mathrm{d}t \\
&+\frac{c_d}{2\beta_0}\int_{I_m}\|\mathcal{U}^*_l\|^2_{\Omega_t}\mathrm{d}t + \frac{1}{2}\int_{I_m}\left(\|g\|^2_{\Omega_t} + \|\mathcal{U}^*_l\|^2_{\Omega_t}\right)\mathrm{d}t \\
&+\frac{\beta_0}{2}\int_{I_m}\|u_D\|^2_{DGB,t}\mathrm{d}t + \frac{\beta_0}{2}\int_{I_m}\|\mathcal{U}^*_l\|^2_{DG,t} \\
&+\frac{\|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}}}{\delta_2} + \delta_2\|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}} + C_{L9}\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathrm{RHS} \;\leq\; & c_1\int_{I_m}\left(\|U\|^2_{DG,t} + \|\mathcal{U}^*_l\|^2_{DG,t} + \|\mathcal{U}^*_l\|^2_{\Omega_t} + \|U\|^2_{\Omega_t} + \|g\|^2_{\Omega_t} + \|u_D\|^2_{DGB,t}\right)\mathrm{d}t \\
&+\frac{\|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}}}{\delta_2} + \delta_2\|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}},
\end{aligned}$$

where $c_1 = \max\{C_{L10} + \beta_0 + c_d/(2\beta_0) + 1/2, c_b + C_{L9}\}$. Now we apply Theorem 1 on the continuity of the discrete characteristic function:

$$\int_{I_m}\|\mathcal{U}^*_l\|^2_{\Omega_t}\mathrm{d}t \leq c^{(1)}_{\mathrm{CH}}\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t, \quad \int_{I_m}\|\mathcal{U}^*_l\|^2_{DG,t}\mathrm{d}t \leq c^{(2)}_{\mathrm{CH}}\int_{I_m}\|U\|^2_{DG,t}\mathrm{d}t.$$

Hence,

$$\begin{aligned}
\mathrm{RHS} \;\leq\; & c_2\int_{I_m}\left(\|U\|^2_{DG,t} + \|U\|^2_{\Omega_t} + \|g\|^2_{\Omega_t} + \|u_D\|^2_{DGB,t}\right)\mathrm{d}t \\
&+\frac{\|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}}}{\delta_2} + \delta_2\|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}},
\end{aligned}$$

with $c_2 = c_1\max\{1 + c^{(1)}_{CH}, 1 + c^{(2)}_{CH}\}$. Then it follows from (3.95) that

$$\frac{1}{2}\left(\left\|U^-_{m-1+l/q}\right\|^2_{\Omega_{t_{m-1+l/q}}} + \|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}}\right) \tag{3.96}$$
$$\leq c_2\int_{I_m}\left(\|U\|^2_{DG,t} + \|U\|^2_{\Omega_t} + \|g\|^2_{\Omega_t} + \|u_D\|^2_{DGB,t}\right)\mathrm{d}t$$
$$+\frac{\|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}}}{\delta_2} + \delta_2\|U^+_{m-1}\|_{\Omega_{t_{m-1}}}.$$

Further, multiplying (3.96) by $\frac{\beta_0}{4c_2(q-1)}$, summing over $l = 1, \ldots, q-1$ and adding to (3.72), we find that

$$
\begin{aligned}
&\|U_m^-\|_{\Omega_{t_m}}^2 + \frac{\beta_0}{8c_2(q-1)} \sum_{l=1}^{q-1} \|U\|_{\Omega_{t_{m-1+l/q}}}^2 \\
&\quad + \left(\frac{\beta_0}{8c_2} + 1\right) \|U_{m-1}^+\|_{\Omega_{t_{m-1}}}^2 + \frac{\beta_0}{2} \int_{I_m} \|U\|_{DG,t}^2 \mathrm{d}t \\
&\le \frac{\beta_0}{4} \int_{I_m} \|U\|_{DG,t}^2 \mathrm{d}t + \left(\frac{\beta_0}{4} + C_{T3}\right) \int_{I_m} \|U\|_{\Omega_t}^2 \mathrm{d}t \\
&\quad + \left(\frac{\beta_0}{4} + C_{T3}^*\right) \int_{I_m} \left(\|g\|_{\Omega_t}^2 + \|u_D\|_{DGB,t}^2\right) \mathrm{d}t \\
&\quad + \left(\frac{\beta_0}{4c_2\delta_2} + \frac{2}{\delta_1}\right) \|U_{m-1}^-\|_{\Omega_{t_{m-1}}}^2 + \left(\frac{\beta_0\delta_2}{4c_2} + 4\delta_1\right) \|U_{m-1}^+\|_{\Omega_{t_{m-1}}}^2.
\end{aligned}
$$

Setting $c_3 := \min\left\{\frac{\beta_0}{8c_2(q-1)}, \frac{\beta_0}{8c_2} + 1\right\}$ and rearranging, we get

$$
\begin{aligned}
&c_3 \Big(\underbrace{\|U_m^-\|_{\Omega_{t_m}^2} + \sum_{l=1}^{q-1} \|U_{m-1+l/q}^2\|_{\Omega_{t_{m-1+l/q}}}^2 + \|U_{m-1}^+\|_{\Omega_{t_{m-1}}}^2}_{=\sum_{l=0}^{q} \|U_{m-1+l/q}\|_{\Omega_{t_{m-1+l/q}}}^2} \Big) + \frac{\beta_0}{4} \int_{I_m} \|U\|_{DG,t}^2 \mathrm{d}t \\
&\le \left(\frac{\beta_0}{4} + C_{T3}\right) \int_{I_m} \|U\|_{\Omega_t}^2 \mathrm{d}t + \left(\frac{\beta_0}{4} + C_{T3}^*\right) \int_{I_m} \left(\|g\|_{\Omega_t}^2 + \|u_D\|_{DGB,t}^2\right) \mathrm{d}t \\
&\quad + \left(\frac{\beta_0}{4c_2\delta_2} + \frac{2}{\delta_1}\right) \|U_{m-1}^-\|_{\Omega_{t_{m-1}}}^2 + \left(\frac{\beta_0\delta_2}{4c_2} + 4\delta_1\right) \|U_{m-1}^+\|_{\Omega_{t_{m-1}}}^2.
\end{aligned}
$$

It follows from inequalities (3.73) and (3.74) that

$$
\begin{aligned}
&\frac{c_3 L_q^*}{\tau_m} \int_{I_m} \|U\|_{\Omega_t}^2 \mathrm{d}t + \frac{\beta_0}{4} \int_{I_m} \|U\|_{DG,t}^2 \mathrm{d}t \\
&\le \left(\frac{\beta_0\delta_2 M_q^*}{4c_2\tau_m} + \frac{4\delta_1 M_q^*}{\tau_m} + \frac{\beta_0}{4} + C_{T3}\right) \int_{I_m} \|U\|_{\Omega_t}^2 \mathrm{d}t \\
&\quad + \left(\frac{\beta_0}{4} + C_{T3}^*\right) \int_{I_m} \left(\|g\|_{\Omega_t}^2 + \|u_D\|_{DG,t}^2\right) \mathrm{d}t + \left(\frac{\beta_0}{4c_2\delta_2} + \frac{2}{\delta_1}\right) \|U_{m-1}^-\|_{\Omega_{t_{m-1}}}^2.
\end{aligned}
$$

Setting $\delta_1 = \frac{c_3 L_q^*}{16 M_q^*}$, $\delta_2 = \frac{c_3 c_2 L_q^*}{\beta_0 M_q^*}$, $c_4 := \frac{\beta_0}{4c_2\delta_2} + \frac{2}{\delta_1}$, $c_5 := \frac{\beta_0}{4} + C_{T3}^*$ we get

$$
\begin{aligned}
&\left(\frac{c_3 L_q^*}{2\tau_m} - \frac{\beta_0}{4} - C_{T3}\right) \int_{I_m} \|U\|_{\Omega_t}^2 \mathrm{d}t + \frac{\beta_0}{4} \int_{I_m} \|U\|_{DG,t}^2 \mathrm{d}t \\
&\le c_5 \int_{I_m} \left(\|g\|_{\Omega_t}^2 + \|u_D\|_{DGB,t}^2\right) \mathrm{d}t + c_4 \|U_{m-1}^-\|_{\Omega_{t_{m-1}}}^2.
\end{aligned} \tag{3.97}
$$

If the condition $0 < \tau_m \le C_{T4}^* := \frac{c_3 L_q^*}{4(\frac{\beta_0}{4} + C_{T3}^*)}$ is satisfied, then

$$
\frac{c_3 L_q^*}{2\tau_m} - \left(\frac{\beta_0}{4} + C_{T3}\right) \ge \frac{c_3 L_q^*}{4\tau_m}
$$

and from (3.97) we obtain the estimate

$$\frac{c_3 L_q^*}{4\tau_m}\int_{I_m}\|U\|_{\Omega_t}^2 \mathrm{d}t + \frac{\beta_0}{4}\int_{I_m}\|U\|_{DG,t}^2\,\mathrm{d}t \\ \le c_5\int_{I_m}\left(\|g\|_{\Omega_t}^2 + \|u_D\|_{DGB,t}^2\right)\mathrm{d}t + c_4\|U_{m-1}^-\|_{\Omega_{t_{m-1}}}^2,$$

which implies (3.93). □

3.4.4 Main theorem

The stability analysis will be finished by the application of the following auxiliary lemma.

Lemma 14. *(Discrete Gronwall inequality) Let x_m, a_m, b_m and y_m, where $m = 1, 2, \ldots$, be non-negative sequences and let the sequence a_m be nondecreasing. Then, if*

$$x_0 + y_0 \le a_0, \\ x_m + y_m \le a_m + \sum_{j=0}^{m-1} b_j x_j \quad \text{for} \quad m \ge 1,$$

we have

$$x_m + y_m \le a_m \prod_{j=0}^{m-1}(1 + b_j) \quad \text{for} \quad m \ge 0.$$

The proof can be carried out by induction, see [31].

Now, if (3.93) is substituted into (3.71), an inequality is obtained, which is a basis of the proof of our main result about the stability:

$$\|U_m^-\|_{\Omega_{t_m}}^2 - \|U_{m-1}^-\|_{\Omega_{t_{m-1}}}^2 + \|\{U\}_{m-1}\|_{\Omega_{t_{m-1}}}^2 + \frac{\beta_0}{2}\int_{I_m}\|U\|_{DG,t}^2\,\mathrm{d}t \\ \le (C_{T2} + C_{T4}\,\tau_m)\int_{I_m}(\|g\|_{\Omega_t}^2 + \|u_D\|_{DGB,t}^2)\,\mathrm{d}t + C_{T2}C_{T4}\,\tau_m\|U_{m-1}^-\|_{\Omega_{t_{m-1}}}^2. \tag{3.98}$$

Theorem 5. *Let $0 < \tau_m \le C_{T4}^*$ for $m = 1, \ldots, M$. Then there exists a constant $C_{T5} > 0$ such that*

$$\|U_m^-\|_{\Omega_{t_m}}^2 + \sum_{j=1}^{m}\|\{U\}_{j-1}\|_{\Omega_{t_{j-1}}}^2 + \frac{\beta_0}{2}\sum_{j=1}^{m}\int_{I_j}\|U\|_{DG,t}^2\,\mathrm{d}t \\ \le C_{T5}\left(\|U_0^-\|_{\Omega_{t_0}}^2 + \sum_{j=1}^{m}\int_{I_j} R_{t,j}\,\mathrm{d}t\right), \quad m = 1, \ldots, M,\ h \in (0, \overline{h}), \tag{3.99}$$

where $R_{t,j} = (C_{T2} + C_{T4}\,\tau_j)\,(\|g\|_{\Omega_t}^2 + \|u_D\|_{DGB,t}^2)$ for $t \in I_j$.

Proof. Writing j instead of m in (3.98), we obtain

$$\|U_j^-\|_{\Omega_{t_j}}^2 - \|U_{j-1}^-\|_{\Omega_{t_{j-1}}} + \|\{U\}_{j-1}\|_{\Omega_{t_{m-1}}}^2 + \frac{\beta_0}{2}\int_{I_j}\|U\|_{DG,t}^2\,\mathrm{d}t \\ \le \int_{I_j} R_{t,j}\,\mathrm{d}t + C_{T2}C_{T4}\,\tau_j\|U_{j-1}^-\|_{\Omega_{t_{j-1}}}^2.$$

Let $m \geq 1$. The summation over all $j = 1, \dots, m$ yields the inequality

$$\begin{aligned} &\|U_m^-\|^2_{\Omega_{t_m}} + \sum_{j=1}^{m} \|\{U\}_{j-1}\|^2_{\Omega_{t_{j-1}}} + \frac{\beta_0}{2} \sum_{j=1}^{m} \int_{I_j} \|U\|^2_{DG,t} \, \mathrm{d}t \\ &\leq \|U_0^-\|^2_{\Omega_0} + C_{T2} C_{T4} \sum_{j=0}^{m-1} \tau_{j+1} \|U_j^-\|^2_{\Omega_{t_j}} + \sum_{j=1}^{m} \int_{I_j} R_{t,j} \, \mathrm{d}t. \end{aligned}$$

The use of the discrete Gronwall inequality form Lemma 14 with setting

$$\begin{aligned} x_0 &= a_0 = \|U_0^-\|^2_{\Omega_{t_0}}, \quad y_0 = 0, \\ x_m &= \|U_m^-\|^2_{\Omega_{t_m}}, \\ y_m &= \sum_{j=1}^{m} \|\{U\}_{j-1}\|^2_{\Omega_{t_{j-1}}} + \frac{\beta_0}{2} \sum_{j=1}^{m} \int_{I_j} \|U\|^2_{DG,t} \, \mathrm{d}t, \\ a_m &= \|U_0^-\|^2_{\Omega_{t_0}} + \sum_{j=1}^{m} \int_{I_j} R_{t,j} \, \mathrm{d}t, \\ b_j &= C_{T2} C_{T4} \, \tau_{j+1}, \quad j = 0, 1, \dots, m-1, \end{aligned}$$

yields

$$\begin{aligned} &\|U_m^-\|^2_{\Omega_{t_m}} + \sum_{j=1}^{m} \|\{U\}_{j-1}\|^2_{\Omega_{t_{j-1}}} + \frac{\beta_0}{2} \sum_{j=1}^{m} \int_{I_j} \|U\|^2_{DG,t} \, \mathrm{d}t \\ &\leq \left(\|U_0^-\|^2_{\Omega_{t_0}} + \sum_{j=1}^{m} \int_{j} R_{t,j} \, \mathrm{d}t \right) \prod_{j=0}^{m-1} \Big(1 + C_{T2} C_{T4} \, \tau_{j+1} \Big). \end{aligned} \tag{3.100}$$

Finally (3.100) and the inequality $1+\sigma < \exp(\sigma)$ valid for any $\sigma > 0$ immediately yield (3.99) with the constant $C_{T5} := \exp(C_{T2} C_{T4} T)$. □

In virtue of Theorem 5, the approximate solution obtained by the ALE-STDGM (2.32) - (2.33) is bounded by a constant depending on data of the problem, namely the functions from the initial and boundary conditions and right-hand side of the solved differential equation (2.1). This constant is independent of the time step $\tau_m \leq T$, which means that the method is unconditionally stable.

4. Error estimation

In what follows we shall be concerned with the analysis of error estimates for the ALE-STDGM. To this end, we consider the form a_h defined by (2.24) as depending on four variables, namely v, w, φ, t:

$$
\begin{aligned}
a_h(v,w,\varphi,t) = & \sum_{K\in\mathcal{T}_{h,t}} \int_K \beta(v)\nabla w\cdot\nabla\varphi\,\mathrm{d}x \qquad (4.1)\\
& - \sum_{\Gamma\in\mathcal{F}^I_{h,t}} \int_\Gamma \left(\langle\beta(v)\nabla w\rangle\cdot\boldsymbol{n}_\Gamma\,[\varphi] + \Theta\,\langle\beta(v)\nabla\varphi\rangle\cdot\boldsymbol{n}_\Gamma\,[w]\right)\mathrm{d}S\\
& - \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma \left(\beta(v)\nabla w\cdot\boldsymbol{n}_\Gamma\,\varphi + \Theta\beta(v)\nabla\varphi\cdot\boldsymbol{n}_\Gamma\,w - \Theta\beta(v)\nabla\varphi\cdot\boldsymbol{n}_\Gamma\,u_D\right)\mathrm{d}S.
\end{aligned}
$$

Then the approximate solution of problem (2.11)–(2.13) is defined as $U\in S^{p,q}_{h,\tau}$ satisfying

$$
\begin{aligned}
&\int_{I_m}\Big((D_tU,\varphi)_{\Omega_t} + a_h(U,U,\varphi,t) + \beta_0 J_h(U,\varphi,t) + b_h(U,\varphi,t) + d_h(U,\varphi,t)\Big)\,\mathrm{d}t \qquad (4.2)\\
&\quad + (\{U\}_{m-1},\varphi^+_{m-1})_{\Omega_{t_{m-1}}} = \int_{I_m} l_h(\varphi,t)\,\mathrm{d}t \quad \forall\varphi\in S^{p,q}_{h,\tau},\quad m=1,\dots,M,\\
&\quad U_0^-\in S_h^{p,0},\quad (U_0^- - u^0, v_h)_{\Omega_0} = 0 \quad \forall v_h\in S_h^{p,0}. \qquad (4.3)
\end{aligned}
$$

The regular exact solution u of problem (2.11)–(2.13) satifies the identity

$$
\begin{aligned}
&\int_{I_m}\Big((D_tu,\varphi)_{\Omega_t} + a_h(u,u,\varphi,t) + \beta_0 J_h(u,\varphi,t) + b_h(u,\varphi,t) + d_h(u,\varphi,t)\Big)\,\mathrm{d}t \qquad (4.4)\\
&\quad + (\{u\}_{m-1},\varphi^+_{m-1})_{\Omega_{t_{m-1}}} = \int_{I_m} l_h(\varphi,t)\,\mathrm{d}t \quad \forall\varphi\in S^{p,q}_{h,\tau},\quad m=1,\dots,M.
\end{aligned}
$$

Of course, $(\{u\}_{m-1},\varphi^+_{m-1})_{\Omega_{t_{m-1}}} = 0$.

In the further sections, if it is not mentioned we consider $m=1,\dots,M$.

4.1 Important estimates

We are interested in the estimation of the error $e = U - u$. It will be expressed in the form

$$
e = \xi + \eta, \quad \text{where } \xi = U - \pi u \in S^{p,q}_{h,\tau} \quad \text{and } \eta = \pi u - u. \qquad (4.5)
$$

Here π is a projection into the space $S^{p,q}_{h,\tau}$. It will be defined later in Section 4.3.

Subtracting (4.4) from (4.2), for every $\varphi\in S^{p,q}_{h,\tau}$, we find that

$$
\begin{aligned}
&\int_{I_m}\big((D_t\xi,\varphi)_{\Omega_t} + a_h(U,U,\varphi,t) - a_h(u,u,\varphi,t) + \beta_0 J_h(\xi,\varphi,t) \qquad (4.6)\\
&\quad + d_h(\xi,\varphi,t)\big)\,\mathrm{d}t + \Big(\{\xi\}_{m-1},\varphi^+_{m-1}\Big)_{\Omega_{t_{m-1}}}\\
&= \int_{I_m}\left(b_h(u,\varphi,t) - b_h(U,\varphi,t)\right)\mathrm{d}t - \int_{I_m}(D_t\eta,\varphi)_{\Omega_t}\;\mathrm{d}t - \beta_0\int_{I_m} J_h(\eta,\varphi,t)\;\mathrm{d}t\\
&\quad - \int_{I_m} d_h(\eta,\varphi,t)\mathrm{d}t - \Big(\{\eta\}_{m-1},\varphi^+_{m-1}\Big)_{\Omega_{t_{m-1}}}.
\end{aligned}
$$

Using the identity

$$
\begin{aligned}
& a_h(U,U,\varphi,t) - a_h(u,u,\varphi,t) \\
& \quad = a_h(U,U,\varphi,t) - a_h(U,\pi u,\varphi,t) + a_h(U,\pi u,\varphi,t) \\
& \quad - a_h(u,\pi u,\varphi,t) + a_h(u,\pi u,\varphi,t) - a_h(u,u,\varphi,t)
\end{aligned}
\tag{4.7}
$$

and setting $\varphi := \xi$ in (4.6) we get

$$
\begin{aligned}
& \int_{I_m} \left((D_t\xi,\xi)_{\Omega_t} + a_h(U,U,\xi,t) - a_h(U,\pi u,\xi,t) \right) \,\mathrm{d}t \\
& \quad + \int_{I_m} \left(\beta_0 J_h(\xi,\xi,t) + d_h(\xi,\xi,t) \right) \,\mathrm{d}t + \left(\{\xi\}_{m-1}, \xi_{m-1}^{+} \right)_{\Omega_{t_{m-1}}} \\
& = \int_{I_m} \left(-a_h(U,\pi u,\xi,t) + a_h(u,\pi u,\xi,t) - a_h(u,\pi u,\xi,t) + a_h(u,u,\xi,t) \right) \,\mathrm{d}t \\
& \quad + \int_{I_m} \left(b_h(u,\xi,t) - b_h(U,\xi,t) - \beta_0 J_h(\eta,\xi,t) - d_h(\eta,\xi,t) \right) \,\mathrm{d}t \\
& \quad - \int_{I_m} (D_t\eta,\xi)_{\Omega_t} \,\mathrm{d}t - \left(\{\eta\}_{m-1}, \xi_{m-1}^{+} \right)_{\Omega_{t_{m-1}}}.
\end{aligned}
\tag{4.8}
$$

In what follows, we need to estimate all of the individual terms appearing in (4.8). We shall use the following notation:

$$
R_t(\eta) = \|\eta\|_{DG,t}^2 + \|\eta\|_{\Omega_t}^2 + \sum_{K\in\mathcal{T}_{h,t}} \left(|\eta|_{H^1(K)}^2 + h_K^2 |\eta|_{H^2(K)}^2 \right), \tag{4.9}
$$

$$
R_t^*(\eta) = \|\eta\|_{DG,t}^2 + \sum_{K\in\mathcal{T}_{h,t}} \left(h_K^2 |\eta|_{H^2(K)}^2 \right) \le R_t(\eta). \tag{4.10}
$$

4.1.1 Estimates of the diffusion and penalty term

Lemma 15. *Let*

$$
c_W > 0, \qquad \text{for } \Theta = -1 \quad \textit{(NIPG)}, \tag{4.11}
$$

$$
c_W \ge \left(\frac{4\beta_1}{\beta_0} \right)^2 C_{MI} \quad \text{for } \Theta = 1 \quad \textit{(SIPG)}, \tag{4.12}
$$

$$
c_W \ge 2 \left(\frac{2\beta_1}{\beta_0} \right)^2 C_{MI} \quad \text{for } \Theta = 0 \quad \textit{(IIPG)}, \tag{4.13}
$$

where $C_{MI} = c_M(c_I + 1)$*. Then, for* $t \in I_m$*,* $m = 1, \dots, M$*,*

$$
a_h(U,U,\xi,t) - a_h(U,\pi u,\xi,t) + \beta_0 J_h(\xi,\xi,t) \ge \frac{\beta_0}{2} \|\xi\|_{DG,t}^2. \tag{4.14}
$$

Proof. For the proof see Lemma 6.37 in the monograph [31]. □

Lemma 16. *There exists a constant* $C_a > 0$ *independent of* $U, u, h, \tau, t \in I_m, M$ *and* m *such that*

$$
a_h(U,U,\varphi,t) - a_h(U,\pi u,\varphi,t) + \beta_0 J_h(\xi,\varphi,t) \le C_a (\|\xi\|_{DG,t}^2 + \|\varphi\|_{DG,t}^2) \tag{4.15}
$$

for any $\varphi \in S_{h,\tau}^{p,q}$ *and* $m = 1, \dots, M$*.*

Proof. For the proof see Lemma 6.39 in [31]. □

We continue with other useful estimates of the diffusion form a_h.

Lemma 17. *For arbitrary $k_c, k_d > 0$ there exist constants $C_c = C_c(k_c)$, $C_d = C_d(k_d) > 0$ independent of $U, u, h, \tau, t \in I_m, M$ and m such that for each $\varphi \in S_{h,\tau}^{p,q}$ the following estimates hold:*

$$|a_h(U, \pi u, \varphi, t) - a_h(u, \pi u, \varphi, t)| \le \frac{\beta_0}{k_c} \|\varphi\|_{DG,t}^2 + C_c(\|\xi\|_{\Omega_t}^2 + R_t(\eta)), \tag{4.16}$$

$$|a_h(u, \pi u, \varphi, t) - a_h(u, u, \varphi, t)| \le \frac{\beta_0}{k_d} \|\varphi\|_{DG,t}^2 + C_d R_t^*(\eta), \tag{4.17}$$

where R_t and R_t^* are defined in (4.9) *and* (4.10).

Proof. For the proof see Lemma 6.40 in the monograph [31]. □

Lemma 18. *There exists a constant $\delta > 0$ such that the following inequality holds*

$$\beta_0 \int_{I_m} |J_h(\eta, \xi, t)| \,\mathrm{d}t \;\le\; \frac{\beta_0}{2\delta} \int_{I_m} R_t(\eta) \,\mathrm{d}t + 2\beta_0 \delta \int_{I_m} \|\xi\|_{DG,t}^2 \,\mathrm{d}t. \tag{4.18}$$

Proof. From the definition of the form J_h, using the Cauchy and Young's inequality and the definition of the $\|\cdot\|_{DG,t}$-norm, we get

$$\begin{aligned} |J_h(\eta, \xi, t)| &\le J_h(\eta, \eta, t)^{1/2} J_h(\xi, \xi, t)^{1/2} \\ &\le \frac{1}{2\delta} J_h(\eta, \eta, t) + 2\delta J_h(\xi, \xi, t) \\ &\le \frac{1}{2\delta} R_t(\eta) + 2\delta \|\xi\|_{DG,t}^2. \end{aligned}$$

Multiplying by β_0 and integrating over I_m we get (4.18). □

4.1.2 Estimates of the convective terms

Lemma 19. *For every $k_b > 0$ there exists a constant $C_b > 0$ independent of $U, u, h, \tau, t \in I_m, M$ and m such that for each $\varphi \in S_{h,\tau}^{p,q}$*

$$\begin{aligned} &|b_h(U, \varphi, t) - b_h(u, \varphi, t)| \\ &\quad \le \frac{\beta_0}{k_b} \|\varphi\|_{DG,t}^2 + C_b \left(\|\xi\|_{\Omega_t}^2 + \|\eta\|_{\Omega_t}^2 + \sum_{K \in \mathcal{T}_{h,t}} h_k^2 |\eta|_{H^1(K)}^2 \right). \end{aligned} \tag{4.19}$$

Proof. See Lemma 6.36 in [31]. □

Lemma 20. *For every $k_e > 0$ there exists a constant $c_e > 0$ such that the following inequalities hold*

$$\int_{I_m} |d_h(\xi, \xi, t)| \,\mathrm{d}t \;\le\; \frac{\beta_0}{2k_e} \int_{I_m} \|\xi\|_{DG,t}^2 \,\mathrm{d}t + \frac{c_e}{2\beta_0} \int_{I_m} \|\xi\|_{\Omega_t}^2 \,\mathrm{d}t, \tag{4.20}$$

$$\int_{I_m} |d_h(\eta, \xi, t)| \,\mathrm{d}t \;\le\; \frac{\beta_0}{2k_e} \int_{I_m} \|\eta\|_{DG,t}^2 \,\mathrm{d}t + \frac{c_e}{2\beta_0} \int_{I_m} \|\xi\|_{\Omega_t}^2 \,\mathrm{d}t. \tag{4.21}$$

Proof. It can be proved similarly as Lemma 3 and Lemma 12. □

4.1.3 Estimates of the ALE derivative term

Lemma 21. *Let $\xi \in S_{h,\tau}^{p,q}$. Then we have*

$$\int_{I_m} (D_t\xi, \xi)_{\Omega_t}\,\mathrm{d}t \tag{4.22}$$
$$\geq \frac{1}{2}\left(\|\xi_m^-\|^2_{\Omega_{t_m}} - \|\xi_{m-1}^+\|^2_{\Omega_{t_{m-1}}} - c_z \int_{I_m} \|\xi\|^2_{\Omega_t}\,\mathrm{d}t \right),$$

$$\left(\{\xi\}_{m-1}, \xi_{m-1}^+\right)_{\Omega_{t_{m-1}}} \tag{4.23}$$
$$= \frac{1}{2}\left(\|\xi_{m-1}^+\|^2_{\Omega_{t_{m-1}}} + \|\{\xi\}_{m-1}\|^2_{\Omega_{t_{m-1}}} - \|\xi_{m-1}^-\|^2_{\Omega_{t_{m-1}}} \right),$$

$$\int_{I_m} (D_t\xi, \xi)_{\Omega_t}\,\mathrm{d}t + \left(\{\xi\}_{m-1}, \xi_{m-1}^+\right)_{\Omega_{t_{m-1}}} \tag{4.24}$$
$$\geq \frac{1}{2}\left(\|\xi_m^-\|^2_{\Omega_{t_{m-1}}} + \|\{\xi\}_{m-1}\|^2_{\Omega_{t_{m-1}}} - \|\xi_{m-1}^-\|^2_{\Omega_{t_{m-1}}} \right) - \frac{c_z}{2}\int_{I_m} \|\xi\|^2_{\Omega_t}\,\mathrm{d}t$$

$$\int_{I_m} (D_t\xi, \xi)_{\Omega_t}\,\mathrm{d}t + \left(\{\xi\}_{m-1}, \xi_{m-1}^+\right)_{\Omega_{t_{m-1}}} \tag{4.25}$$
$$\geq \frac{1}{2}\|\xi_m^-\|^2_{\Omega_{t_m}} + \frac{1}{2}\|\xi_{m-1}^+\|^2_{\Omega_{t_{m-1}}} - \frac{c_z}{2}\int_{I_m} \|\xi\|^2_{\Omega_t}\mathrm{d}t - \left(\xi_{m-1}^-, \xi_{m-1}^+\right)_{\Omega_{t_{m-1}}}.$$

Proof. We start with the first inequality. We have

$$\int_{I_m} (D_t\xi, \xi)_{\Omega_t}\,\mathrm{d}t = \sum_{K\in\mathcal{T}_{h,t}} \int_{I_m} (D_t\xi, \xi)_K\,\mathrm{d}t, \tag{4.26}$$

where of course K depends on t. By virtue of assumption (2.15), the Reynolds transport theorem (3.41) and relation (2.10), we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_K \xi^2(x,t)\,\mathrm{d}x \qquad (4.27)\\
&= \int_K \left(\frac{\partial \xi^2(x,t)}{\partial t} + \boldsymbol{z}(x,t)\cdot\nabla(\xi^2(x,t)) + \xi^2(x,t)\mathrm{div}\,\boldsymbol{z}(x,t) \right)\mathrm{d}x\\
&= \int_K \left(2\xi(x,t)\left(\frac{\partial \xi(x,t)}{\partial t} + \boldsymbol{z}(x,t)\cdot\nabla\xi(x,t) \right) + \xi^2(x,t)\mathrm{div}\,\boldsymbol{z}(x,t) \right)\mathrm{d}x\\
&= 2(D_t\xi,\xi)_K + (\xi^2, \mathrm{div}\,\boldsymbol{z})_K.
\end{aligned}$$

Expressing $(D_t\xi,\xi)_K$, summing over $K\in\mathcal{T}_{h,t}$ and integrating over I_m together with assumption (3.10) yield

$$\begin{aligned}
\int_{I_m} (D_t\xi,\xi)_{\Omega_t}\,\mathrm{d}t &= \frac{1}{2}\int_{I_m}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}\xi^2\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{2}\int_{I_m}(\xi^2,\mathrm{div}\,\boldsymbol{z})_{\Omega_t}\,\mathrm{d}t \qquad (4.28)\\
&= \frac{1}{2}\|\xi_m^-\|^2_{\Omega_{t_m}} - \frac{1}{2}\|\xi_{m-1}^+\|^2_{\Omega_{t_{m-1}}} - \frac{1}{2}\int_{I_m}(\xi^2,\mathrm{div}\,\boldsymbol{z})_{\Omega_t}\,\mathrm{d}t\\
&\geq \frac{1}{2}\|\xi_m^-\|^2_{\Omega_{t_m}} - \frac{1}{2}\|\xi_{m-1}^+\|^2_{\Omega_{t_{m-1}}} - \frac{c_z}{2}\int_{I_m}\|\xi\|^2_{\Omega_t}\,\mathrm{d}t,
\end{aligned}$$

which is (4.22).

By a simple manipulation we obtain (4.23), which together with (4.22) implies

(4.24). Concerning inequality (4.25), by virtue of (4.28) we have

$$\begin{aligned}
&\int_{I_m} (D_t\xi, \xi)_{\Omega_t} \mathrm{d}t + \Big(\{\xi\}_{m-1}, \xi^+_{m-1}\Big)_{\Omega_{t_{m-1}}} \\
&= \frac{1}{2}\|\xi^-_m\|^2_{\Omega_{t_m}} - \frac{1}{2}\|\xi^+_{m-1}\|^2_{\Omega_{t_{m-1}}} \\
&\quad -\frac{1}{2}\int_{I_m} (\xi^2, \operatorname{div} \boldsymbol{z})_{\Omega_t} \mathrm{d}t + \|\xi^+_{m-1}\|^2_{\Omega_{t_{m-1}}} - (\xi^-_{m-1}, \xi^+_{m-1})_{\Omega_{t_{m-1}}} \\
&\geq \frac{1}{2}\left(\|\xi^-_m\|^2_{\Omega_{t_m}} + \|\xi^+_{m-1}\|^2_{\Omega_{t_{m-1}}} - c_z \int_{I_m} \|\xi\|^2_{\Omega_t} \mathrm{d}t\right) - \Big(\xi^-_{m-1}, \xi^+_{m-1}\Big)_{\Omega_{t_{m-1}}},
\end{aligned}$$

which proves the lemma. □

Let us consider the functions ξ and η, defined by (4.5), in the interval $[t_{m-1}, t_m]$ and set

$$\begin{gathered}
\tilde{\eta}(X,t) = \eta(\mathcal{A}_t(X), t), \quad \tilde{\xi}(X,t) = \xi(\mathcal{A}_t(X), t), \\
x = \mathcal{A}_t(X) = \mathcal{A}^{m-1}_{h,t}(X),\ X \in \Omega_{t_{m-1}},\ x \in \Omega_t,\ t \in [t_{m-1}, t_m].
\end{gathered} \tag{4.29}$$

Lemma 22. *There exists a constant $C_\eta > 0$ independent of u, U, h, τ, m, M such that*

$$\begin{aligned}
&\left|\int_{I_m} (D_t\eta, \xi)_{\Omega_t} \mathrm{d}t + (\{\eta\}_{m-1}, \xi^+_{m-1})_{\Omega_{t_{m-1}}}\right| \\
&\leq C_\eta \left(\tau_m^2 \int_{I_m} \left\|\frac{\partial \tilde{\eta}}{\partial t}\right\|^2_{\Omega_{t_{m-1}}} \mathrm{d}t + \int_{I_m} \|\xi\|^2_{\Omega_t} \mathrm{d}t\right) \\
&\quad + \delta_0 \|\{\xi\}_{m-1}\|^2_{\Omega_{t_{m-1}}} + \frac{1}{\delta_0}\|\eta^-_{m-1}\|^2_{\Omega_{t_{m-1}}}
\end{aligned} \tag{4.30}$$

holds for every $\delta_0 > 0$.

Proof. By (2.9) and (4.29) we have

$$D_t\eta(x,t) = \frac{\partial}{\partial t}\tilde{\eta}(X,t), \quad x = \mathcal{A}_t(X). \tag{4.31}$$

Using the substitution theorem, we get

$$\begin{aligned}
\int_{I_m} (D_t\eta, \xi)_{\Omega_t} \mathrm{d}t &= \int_{I_m} \left(\int_{\Omega_t} D_t\eta(x,t)\, \xi(x,t)\, \mathrm{d}x\right) \mathrm{d}t \\
&= \int_{I_m} \left(\int_{\Omega_{t_{m-1}}} \frac{\partial \tilde{\eta}(X,t)}{\partial t} \tilde{\xi}(X,t)\, J(X,t)\, \mathrm{d}X\right) \mathrm{d}t.
\end{aligned} \tag{4.32}$$

Here $J(X,t)$ is the Jacobian defined in Section 3.1. Using the fact, that

$$J(X, t_{m-1}) \equiv 1,$$

we shall rewrite (4.32) in the following way:

$$\begin{aligned}
&\int_{I_m} (D_t\eta, \xi)_{\Omega_t} \mathrm{d}t \\
&= \underbrace{\int_{I_m} \left(\int_{\Omega_{t_{m-1}}} \frac{\partial \tilde{\eta}(X,t)}{\partial t} \tilde{\xi}(X,t)\, \mathrm{d}X\right) \mathrm{d}t}_{\gamma_1} \\
&\quad + \underbrace{\int_{I_m} \left(\int_{\Omega_{t_{m-1}}} \frac{\partial \tilde{\eta}(X,t)}{\partial t} \tilde{\xi}(X,t)(J(X,t) - J(X,t_{m-1}))\, \mathrm{d}X\right) \mathrm{d}t.}_{\gamma_2}
\end{aligned} \tag{4.33}$$

By virtue of (3.4), $J \in W^{1,\infty}(I_m; L^\infty(\Omega_{t_{m-1}}))$ and, thus, since J is constant on each K and satisfies (3.9), we have

$$\max_{\substack{X\in\Omega_{t_{m-1}}\\ t\in I_m}} |J(X,t) - J(X,t_{m-1})| \le c_J\,\tau_m. \tag{4.34}$$

From (4.34), Young's inequality and the substitution theorem we get the estimate

$$\begin{aligned} |\gamma_2| &\le c_J^2\,\tau_m^2 \int_{I_m} \left\| \frac{\partial\tilde\eta}{\partial t} \right\|^2_{\Omega_{t_{m-1}}} \mathrm{d}t + \int_{I_m} \|\tilde\xi\|^2_{\Omega_{t_{m-1}}} \mathrm{d}t \qquad (4.35)\\ &= c_J^2\,\tau_m^2 \int_{I_m} \left\| \frac{\partial\tilde\eta}{\partial t} \right\|^2_{\Omega_{t_{m-1}}} \mathrm{d}t + \int_{I_m}\int_{\Omega_{t_{m-1}}} |\tilde\xi(X,t)|^2\,\mathrm{d}X\,\mathrm{d}t\\ &= c_J^2\,\tau_m^2 \int_{I_m} \left\| \frac{\partial\tilde\eta}{\partial t} \right\|^2_{\Omega_{t_{m-1}}} \mathrm{d}t + \int_{I_m}\int_{\Omega_t} |\xi(x,t)|^2 J^{-1}(x,t)\,\mathrm{d}x\,\mathrm{d}t\\ &\le c_J^2\,\tau_m^2 \int_{I_m} \left\| \frac{\partial\tilde\eta}{\partial t} \right\|^2_{\Omega_{t_{m-1}}} \mathrm{d}t + (C_J^-)^{-1}\int_{I_m} \|\xi\|^2_{\Omega_t}\,\mathrm{d}t. \end{aligned}$$

Now we shall estimate expression γ_1. The integration by parts implies that

$$\gamma_1 = (\tilde\eta_m^-, \tilde\xi_m^-)_{\Omega_{t_{m-1}}} - (\tilde\eta_{m-1}^+, \tilde\xi_{m-1}^+)_{\Omega_{t_{m-1}}} - \int_{I_m} \left(\tilde\eta, \frac{\partial\tilde\xi}{\partial t}\right)_{\Omega_{t_{m-1}}} \mathrm{d}t. \tag{4.36}$$

Since $\frac{\partial\tilde\xi}{\partial t}$ is a polynom in t of degree $\le q-1$, by the definition of $\tilde\eta$, the last term on the right-hand side of (4.36) is zero (cf. [31], (6.90)). Moreover, the first term on the right-hand side is also zero. Since $\mathcal{A}_{t_{m-1}^+} = \mathrm{Id}$, we have $\tilde\xi_{m-1}^+ = \xi_{m-1}^+$ and $\tilde\eta_{m-1}^+ = \eta_{m-1}^+$. Hence, taking into account these considerations, we have

$$\gamma_1 = -(\tilde\eta_{m-1}^+, \tilde\xi_{m-1}^+)_{\Omega_{t_{m-1}}} = -(\eta_{m-1}^+, \xi_{m-1}^+)_{\Omega_{t_{m-1}}}. \tag{4.37}$$

To prove (4.30) it remains to estimate

$$\begin{aligned} &\gamma_1 + \left(\{\eta\}_{m-1}, \xi_{m-1}^+\right)_{\Omega_{t_{m-1}}} \qquad (4.38)\\ &= -(\eta_{m-1}^+, \xi_{m-1}^+)_{\Omega_{t_{m-1}}} + \left(\eta_{m-1}^+ - \eta_{m-1}^-, \xi_{m-1}^+\right)_{\Omega_{t_{m-1}}}\\ &= -\left(\eta_{m-1}^-, \xi_{m-1}^+\right)_{\Omega_{t_{m-1}}}. \end{aligned}$$

Similarly as $(\tilde\eta_m^-, \tilde\xi_m^-)_{\Omega_{t_{m-1}}} = 0$, we have $(\tilde\eta_{m-1}^-, \tilde\xi_{m-1}^-)_{\Omega_{t_{m-2}}} = 0$ and we need to prove that

$$(\eta_{m-1}^-, \xi_{m-1}^-)_{\Omega_{t_{m-1}}} = 0. \tag{4.39}$$

If we set $x := \mathcal{A}^{m-2}_{h,t_{m-1}}(X)$ with $X \in \Omega_{t_{m-2}}$ and $x \in \Omega_{t_{m-1}}$, we can write

$$\begin{aligned} &|(\eta_{m-1}^-, \xi_{m-1}^-)_{\Omega_{t_{m-1}}}| = \left| \int_{\Omega_{t_{m-1}}} \eta(x, t_{m-1}-)\xi(x,t_{m-1}-)\,\mathrm{d}x \right| \qquad (4.40)\\ &= \left| \int_{\Omega_{t_{m-2}}} \eta(\mathcal{A}^{m-2}_{h,t_{m-1}}(X), t_{m-1}-)\xi(\mathcal{A}^{m-2}_{h,t_{m-1}-}(X), t_{m-1}-)J(X,t_{m-1}-)\,\mathrm{d}X \right|\\ &\le C_J^+ \left| \int_{\Omega_{t_{m-2}}} \tilde\eta(X,t_{m-1}-)\tilde\xi(X,t_{m-1}-)\,\mathrm{d}X \right|\\ &= C_J^+ (\tilde\eta_{m-1}^-, \tilde\xi_{m-1}^-)_{\Omega_{t_{m-2}}} = 0. \end{aligned}$$

Hence (4.39) holds and thus

$$(\eta_{m-1}^-, \xi_{m-1}^+)_{\Omega_{t_{m-1}}} = (\eta_{m-1}^-, \xi_{m-1}^+ - \xi_{m-1}^-)_{\Omega_{t_{m-1}}} = (\eta_{m-1}^-, \{\xi\}_{m-1})_{\Omega_{t_{m-1}}}. \quad (4.41)$$

Finally, this together with (4.33), (4.35), (4.38) and Young's inequality imply (4.30). □

4.2 Abstract error estimate

Lemma 23. *There exist constants C_1, C_2, $C_3 > 0$ independent of u, U, h, τ, m, M such that*

$$\begin{aligned} &\|\xi_m^-\|_{\Omega_{t_m}}^2 + \frac{1}{2}\|\{\xi\}_{m-1}\|_{\Omega_{t_{m-1}}}^2 - \|\xi_{m-1}^-\|_{\Omega_{t_{m-1}}}^2 + \frac{\beta_0}{2}\int_{I_m}\|\xi\|_{DG,t}^2\,\mathrm{d}t \quad (4.42)\\ &\le C_1\int_{I_m}\|\xi\|_{\Omega_t}^2\,\mathrm{d}t + C_2\int_{I_m}R_t(\eta)\,\mathrm{d}t + C_3\,\tau_m^2\int_{I_m}\left\|\frac{\partial\tilde{\eta}}{\partial t}\right\|_{\Omega_{t_{m-1}}}^2\mathrm{d}t + 4\|\eta_{m-1}^-\|_{\Omega_{t_{m-1}}}^2. \end{aligned}$$

Proof. From (4.8) and Lemmas 15 - 22 we have

$$\begin{aligned} &\underbrace{\int_{I_m}(D_t\xi,\xi)_{\Omega_t}\,\mathrm{d}t + \left(\{\xi\}_{m-1},\xi_{m-1}^+\right)_{\Omega_{t_{m-1}}}}_{\sigma_1}\\ &\quad+\underbrace{\int_{I_m}(a_h(U,U,\xi,t) - a_h(U,\pi u,\xi,t) + \beta_0 J_h(\xi,\xi,t)\,\mathrm{d}t}_{\sigma_2}\\ &= -\underbrace{\int_{I_m}\left(a_h(U,\pi u,\xi,t) - a_h(u,\pi u,\xi,t)\right)\,\mathrm{d}t}_{\sigma_3}\\ &\quad-\underbrace{\int_{I_m}\left(a_h(u,\pi u,\xi,t) - a_h(u,u,\xi,t)\right)\,\mathrm{d}t}_{\sigma_4}\\ &\quad+\underbrace{\int_{I_m}(b_h(u,\xi,t) - b_h(U,\xi,t))\,\mathrm{d}t}_{\sigma_5} - \underbrace{\int_{I_m}\beta_0 J_h(\eta,\xi,t)\,\mathrm{d}t}_{\sigma_6} - \underbrace{\int_{I_m}d_h(\eta,\xi,t)\,\mathrm{d}t}_{\sigma_7}\\ &\quad-\underbrace{\left(\int_{I_m}(D_t\eta,\xi)_{\Omega_t}\,\mathrm{d}t + \left(\{\eta\}_{m-1},\xi_{m-1}^+\right)_{\Omega_{t_{m-1}}}\right)}_{\sigma_8} - \underbrace{\int_{I_m}d_h(\xi,\xi,t)\,\mathrm{d}t}_{\sigma_9}, \end{aligned}$$

where

$$\begin{aligned}
\sigma_1 &\geq \frac{1}{2}\Big(\|\xi_m^-\|^2_{\Omega_{t_{m-1}}} + \|\{\xi\}_{m-1}\|^2_{\Omega_{t_{m-1}}} - \|\xi_{m-1}^-\|^2_{\Omega_{t_{m-1}}}\Big) - \frac{c_z}{2}\int_{I_m} \|\xi\|^2_{\Omega_t}\,\mathrm{d}t,\\
\sigma_2 &\geq \frac{\beta_0}{2}\int_{I_m} \|\xi\|^2_{DG,t}\,\mathrm{d}t,\\
\sigma_3 &\leq \frac{\beta_0}{k_c}\int_{I_m} \|\xi\|^2_{DG,t}\,\mathrm{d}t + C_c\int_{I_m} (\|\xi\|^2_{\Omega_t} + R_t(\eta))\,\mathrm{d}t,\\
\sigma_4 &\leq \frac{\beta_0}{k_d}\int_{I_m} \|\xi\|^2_{DG,t}\,\mathrm{d}t + C_d\int_{I_m} R_t^*(\eta)\,\mathrm{d}t,\\
\sigma_5 &\leq \frac{\beta_0}{k_b}\int_{I_m} \|\xi\|^2_{DG,t}\,\mathrm{d}t + C_b\int_{I_m} (\|\xi\|^2_{\Omega_t} + \|\eta\|^2_{\Omega_t} + \sum_{K\in\mathcal{T}_{h,t}} h_k^2|\eta|^2_{H^1(K)})\,\mathrm{d}t,\\
\sigma_6 &\leq \frac{\beta_0}{2\delta}\int_{I_m} R_t(\eta)\,\mathrm{d}t + 2\beta_0\delta\int_{I_m} \|\xi\|^2_{DG,t}\,\mathrm{d}t,\\
\sigma_7 &\leq \frac{\beta_0}{2k_e}\int_{I_m} \|\eta\|^2_{DG,t}\,\mathrm{d}t + \frac{c_e}{2\beta_0}\int_{I_m} \|\xi\|^2_{\Omega_t}\,\mathrm{d}t,\\
\sigma_8 &\leq C_\eta\Big(\tau_m^2\int_{I_m}\left\|\frac{\partial\tilde\eta}{\partial t}\right\|^2_{\Omega_{t_{m-1}}}\mathrm{d}t + \int_{I_m}\|\xi\|^2_{\Omega_{t_m}}\mathrm{d}t\Big)\\
&\qquad +\delta_0\|\{\xi\}_{m-1}\|^2_{\Omega_{t_{m-1}}} + \frac{1}{\delta_0}\|\eta_{m-1}^-\|^2_{\Omega_{t_{m-1}}},\\
\sigma_9 &\leq \frac{\beta_0}{2k_e}\int_{I_m} \|\xi\|^2_{DG,t}\,\mathrm{d}t + \frac{c_e}{2\beta_0}\int_{I_m} \|\xi\|^2_{\Omega_t}\,\mathrm{d}t.
\end{aligned}$$

Rearranging we get

$$\begin{aligned}
&\frac{1}{2}\Big(\|\xi_m^-\|^2_{\Omega_{t_m}} + \|\{\xi\}_{m-1}\|^2_{\Omega_{t_{m-1}}} - \|\xi_{m-1}^-\|^2_{\Omega_{t_{m-1}}}\Big) + \frac{\beta_0}{2}\int_{I_m}\|\xi\|^2_{DG,t}\,\mathrm{d}t \qquad (4.43)\\
&\leq \int_{I_m}\bigg(\Big(\frac{c_z}{2} + C_c + C_b + C_\eta + \frac{c_e}{\beta_0}\Big)\|\xi\|^2_{\Omega_t}\\
&\quad +\Big(\frac{\beta_0}{k_b} + \frac{\beta_0}{k_c} + \frac{\beta_0}{k_d} + \frac{\beta_0}{2k_e} + 2\beta_0\delta\Big)\|\xi\|^2_{DG,t}\\
&\quad +(C_c + C_b)R_t(\eta) + C_d\,R_t^*(\eta) + \frac{\beta_0}{2\delta}R_t(\eta) + \frac{\beta_0}{2k_e}\|\eta\|^2_{DG,t}\bigg)\,\mathrm{d}t\\
&\quad +C_\eta\tau_m^2\int_{I_m}\left\|\frac{\partial\tilde\eta}{\partial t}\right\|^2_{\Omega_{t_{m-1}}}\mathrm{d}t + \delta_0\|\{\xi\}_{m-1}\|^2_{\Omega_{t_{m-1}}} + \frac{1}{\delta_0}\|\eta_{m-1}^-\|^2_{\Omega_{t_{m-1}}}.
\end{aligned}$$

Now multiplying by two, putting $\delta_0 = 1/4$, $\delta = 1/16$ $k_b = k_c = k_d = 32$, $k_e = 16$ and taking into account that $0 \leq R_t^*(\eta) \leq R_t(\eta)$, we obtain (4.42). □

In our further analysis we need the following modification of Lemma 23.

Lemma 24. *We have*

$$\begin{aligned}
&\|\xi_m^-\|^2_{\Omega_{t_m}} + \|\xi_{m-1}^+\|^2_{\Omega_{t_{m-1}}} + \frac{\beta_0}{2}\int_{I_m}\|\xi\|^2_{DG,t}\,\mathrm{d}t \qquad (4.44)\\
&\leq C_1\int_{I_m}\|\xi\|^2_{\Omega_t}\mathrm{d}t + C_2\int_{I_m} R_t(\eta)\,\mathrm{d}t + C_3\,\tau_m^2\int_{I_m}\left\|\frac{\partial\tilde\eta}{\partial t}\right\|^2_{\Omega_{t_{m-1}}}\mathrm{d}t\\
&\quad +\delta_0\|\{\xi\}_{m-1}\|^2_{\Omega_{t_{m-1}}} + \frac{1}{\delta_0}\|\eta_{m-1}^-\|^2_{\Omega_{t_{m-1}}} + \frac{2}{\delta_1}\|\xi_{m-1}^-\|^2_{\Omega_{t_{m-1}}} + 4\delta_1\|\xi_{m-1}^+\|^2_{\Omega_{t_{m-1}}}
\end{aligned}$$

for arbitrary $\delta_0, \delta_1 > 0$, where C_1, C_2, C_3 are constants from (4.42).

Proof. From (4.8) and Lemmas 15 - 22 we have

$$\underbrace{\int_{I_m} (D_t\xi, \xi)_{\Omega_t}\ \mathrm{d}t + \Big(\{\xi\}_{m-1}, \xi^+_{m-1}\Big)_{\Omega_{t_{m-1}}}}_{\sigma_1}$$

$$+ \underbrace{\int_{I_m} (a_h(U, U, \xi, t) - a_h(U, \pi u, \xi, t) + \beta_0 J_h(\xi, \xi, t)\ \mathrm{d}t}_{\sigma_2}$$

$$= -\underbrace{\int_{I_m} (a_h(U, \pi u, \xi, t) - a_h(u, \pi u, \xi, t))\ \mathrm{d}t}_{\sigma_3}$$

$$- \underbrace{\int_{I_m} (a_h(u, \pi u, \xi, t) - a_h(u, u, \xi, t))\ \mathrm{d}t}_{\sigma_4}$$

$$+ \underbrace{\int_{I_m} (b_h(u, \xi, t) - b_h(U, \xi, t))\ \mathrm{d}t}_{\sigma_5} - \underbrace{\int_{I_m} \beta_0 J_h(\eta, \xi, t)\ \mathrm{d}t}_{\sigma_6} - \underbrace{\int_{I_m} d_h(\eta, \xi, t)\ \mathrm{d}t}_{\sigma_7}$$

$$- \underbrace{\Big(\int_{I_m} (D_t\eta, \xi)_{\Omega_t}\ \mathrm{d}t + \Big(\{\eta\}_{m-1}, \xi^+_{m-1}\Big)_{\Omega_{t_{m-1}}}\Big)}_{\sigma_8} - \underbrace{\int_{I_m} d_h(\xi, \xi, t)\ \mathrm{d}t}_{\sigma_9},$$

where

$$\begin{aligned}
\sigma_1 &\geq \frac{1}{2}\|\xi^-_m\|^2_{\Omega_{t_m}} + \frac{1}{2}\|\xi^+_{m-1}\|^2_{\Omega_{t_{m-1}}} - \frac{c_z}{2}\int_{I_m} \|\xi\|^2_{\Omega_t}\mathrm{d}t - \Big(\xi^-_{m-1}, \xi^+_{m-1}\Big)_{\Omega_{t_{m-1}}}, \\
\sigma_2 &\geq \frac{\beta_0}{2}\int_{I_m} \|\xi\|^2_{DG,t}\ \mathrm{d}t, \\
\sigma_3 &\leq \frac{\beta_0}{k_c}\int_{I_m} \|\xi\|^2_{DG,t}\ \mathrm{d}t + C_c\int_{I_m} (\|\xi\|^2_{\Omega_t} + R_t(\eta))\ \mathrm{d}t, \\
\sigma_4 &\leq \frac{\beta_0}{k_d}\int_{I_m} \|\xi\|^2_{DG,t}\ \mathrm{d}t + C_d\int_{I_m} R^*_t(\eta)\ \mathrm{d}t, \\
\sigma_5 &\leq \frac{\beta_0}{k_b}\int_{I_m} \|\xi\|^2_{DG,t}\ \mathrm{d}t + C_b\int_{I_m} (\|\xi\|^2_{\Omega_t} + \|\eta\|^2_{\Omega_t} + \sum_{K\in\mathcal{T}_{h,t}} h^2_k|\eta|^2_{H^1(K)})\ \mathrm{d}t, \\
\sigma_6 &\leq \frac{\beta_0}{2\delta}\int_{I_m} R_t(\eta)\,\mathrm{d}t + 2\beta_0\delta\int_{I_m} \|\xi\|^2_{DG,t}\,\mathrm{d}t, \\
\sigma_7 &\leq \frac{\beta_0}{2k_e}\int_{I_m} \|\eta\|^2_{DG,t}\,\mathrm{d}t + \frac{c_e}{2\beta_0}\int_{I_m} \|\xi\|^2_{\Omega_t}\,\mathrm{d}t, \\
\sigma_8 &\leq C_\eta\Big(\tau^2_m\int_{I_m} \left\|\frac{\partial\tilde{\eta}}{\partial t}\right\|^2_{\Omega_{t_{m-1}}}\mathrm{d}t + \int_{I_m} \|\xi\|^2_{\Omega_{t_m}}\mathrm{d}t\Big) \\
&\qquad +\delta_0\|\{\xi\}_{m-1}\|^2_{\Omega_{t_{m-1}}} + \frac{1}{\delta_0}\|\eta^-_{m-1}\|^2_{\Omega_{t_{m-1}}}, \\
\sigma_9 &\leq \frac{\beta_0}{2k_e}\int_{I_m} \|\xi\|^2_{DG,t}\,\mathrm{d}t + \frac{c_e}{2\beta_0}\int_{I_m} \|\xi\|^2_{\Omega_t}\,\mathrm{d}t.
\end{aligned}$$

Rearranging we get

$$\frac{1}{2}\|\xi_m^-\|^2_{\Omega_{t_m}} + \frac{1}{2}\|\xi_{m-1}^+\|^2_{\Omega_{t_{m-1}}} + \frac{\beta_0}{2}\int_{I_m}\|\xi\|^2_{DG,t}\,\mathrm{d}t \tag{4.45}$$

$$\le \int_{I_m}\Big(\Big(\frac{c_z}{2} + C_b + C_c + C_\eta + \frac{c_e}{\beta_0}\Big)\|\xi\|^2_{\Omega_t} + \Big(\frac{\beta_0}{k_b} + \frac{\beta_0}{k_c} + \frac{\beta_0}{k_d} + \frac{\beta_0}{2k_e} + 2\beta_0\delta\Big)\|\xi\|^2_{DG,t}$$

$$+\Big(C_c + C_b + \frac{\beta_0}{2\delta}\Big)R_t(\eta) + C_d\,R_t^*(\eta) + \frac{\beta_0}{2k_e}\|\eta\|^2_{DG,t}\Big)\,\mathrm{d}t + C_\eta\tau_m^2\int_{I_m}\left\|\frac{\partial\tilde{\eta}}{\partial t}\right\|^2_{\Omega_{t_{m-1}}}\mathrm{d}t$$

$$+\delta_0\|\{\xi\}_{m-1}\|^2_{\Omega_{t_{m-1}}} + \frac{1}{\delta_0}\|\eta_{m-1}^-\|^2_{\Omega_{t_{m-1}}} + \Big(\xi_{m-1}^-,\xi_{m-1}^+\Big)_{\Omega_{t_{m-1}}}.$$

Now multiplying by two, putting $\delta = 1/16$, $k_b = k_c = k_d = 32$, $k_e = 16$, using Young's inequality to the term $\Big(\xi_{m-1}^-,\xi_{m-1}^+\Big)_{\Omega_{t_{m-1}}}$ with constant $\delta_1 > 0$ and taking into account that $0 \le R_t^*(\eta) \le R_t(\eta)$, we obtain (4.44). □

The further goal is the estimation of expressions containing ξ in terms of η. To this end, it is necessary to estimate the expression $\int_{I_m}\|\xi\|^2_{\Omega_t}\mathrm{d}t$ in a suitable way. The analysis of this problem will be divided into two parts. First we assume that piecewise linear approximation in time is used, i. e. $q = 1$.

4.2.1 Abstract error estimate for linear approximation in time

Similarly as in Lemma 8 for $q = 1$, we find that there exist L_1^*, $M_1^* > 0$ such that for $m = 1, \ldots, M$ we have

$$\left\|\xi_{m-1}^+\right\|^2_{\Omega_{t_{m-1}}} + \|\xi_m^-\|^2_{\Omega_{t_m}} \ge \frac{L_1^*}{\tau_m}\int_{I_m}\|\xi\|^2_{\Omega_t}\mathrm{d}t, \tag{4.46}$$

$$\|\xi_{m-1}^+\|^2_{\Omega_{t_{m-1}}} \le \frac{M_1^*}{\tau_m}\int_{I_m}\|\xi\|^2_{\Omega_t}\mathrm{d}t. \tag{4.47}$$

Theorem 6. *There exist constants $C_4 > 0$ and $C^* > 0$ independent of h, τ_m, m, u, U such that*

$$\int_{I_m}\|\xi\|^2_{\Omega_t}\mathrm{d}t \le C_4\,\tau_m\Big(\int_{I_m}R_t(\eta)\mathrm{d}t + \tau_m^2\int_{I_m}\left\|\frac{\partial\tilde{\eta}}{\partial t}\right\|^2_{\Omega_{t_{m-1}}}\mathrm{d}t \tag{4.48}$$

$$+\delta_0\|\{\xi\}_{m-1}\|^2_{\Omega_{t_{m-1}}} + \frac{1}{\delta_0}\|\eta_{m-1}^-\|^2_{\Omega_{t_{m-1}}} + \|\xi_{m-1}^-\|^2_{\Omega_{t_{m-1}}}\Big),$$

if $0 < \tau_m \le C^*,\ m = 1, \ldots, M$.

Proof. It follows from (4.44), (4.46) and (4.47) that

$$\frac{L_1^*}{\tau_m}\int_{I_m}\|\xi\|^2_{\Omega_t}\mathrm{d}t + \frac{\beta_0}{2}\int_{I_m}\|\xi\|^2_{DG,t}\mathrm{d}t \le \left(C_1 + \frac{4M_1^*\delta_1}{\tau_m}\right)\int_{I_m}\|\xi\|^2_{\Omega_t}\mathrm{d}t$$

$$+C_2\int_{I_m}R_t(\eta)\,\mathrm{d}t + C_3\,\tau_m^2\int_{I_m}\left\|\frac{\partial\tilde{\eta}}{\partial t}\right\|^2_{\Omega_{t_{m-1}}}\mathrm{d}t + \delta_0\|\{\xi\}_{m-1}\|^2_{\Omega_{t_{m-1}}}$$

$$+\frac{1}{\delta_0}\|\eta_{m-1}^-\|^2_{\Omega_{t_{m-1}}} + \frac{2}{\delta_1}\|\xi_{m-1}^-\|^2_{\Omega_{t_{m-1}}},$$

which after neglecting the positive second term on the left-hand side, multiplying by τ_m and rearranging can be written as

$$(L_1^* - C_1\tau_m - 4M_1^*\delta_1)\int_{I_m}\|\xi\|^2_{\Omega_t}\mathrm{d}t \le C_2\tau_m\int_{I_m} R_t(\eta)\,\mathrm{d}t + C_3\,\tau_m^3\int_{I_m}\left\|\frac{\partial\tilde{\eta}}{\partial t}\right\|^2_{\Omega_{t_{m-1}}}\mathrm{d}t$$

$$+\delta_0\tau_m\|\{\xi\}_{m-1}\|^2_{\Omega_{t_{m-1}}} + \frac{\tau_m}{\delta_0}\|\eta^-_{m-1}\|^2_{\Omega_{t_{m-1}}} + \frac{2\tau_m}{\delta_1}\|\xi^-_{m-1}\|^2_{\Omega_{t_{m-1}}},$$

Let $L_1^* - 4M_1^*\delta_1 = \frac{3}{4}L_1^*$ and $C_1\tau_m \le \frac{1}{4}L_1^*$, which means that we set $\delta_1 = \frac{L_1^*}{16M_1^*}$ and assume that

$$0 < \tau_m \le C^* := \frac{L_1^*}{4C_1}. \tag{4.49}$$

Then we find that

$$\frac{L_1^*}{2}\int_{I_m}\|\xi\|^2_{\Omega_t}\mathrm{d}t \le C_2\tau_m\int_{I_m} R_t(\eta)\,\mathrm{d}t + C_3\,\tau_m^3\int_{I_m}\left\|\frac{\partial\tilde{\eta}}{\partial t}\right\|^2_{\Omega_{t_{m-1}}}\mathrm{d}t$$

$$+\delta_0\tau_m\|\{\xi\}_{m-1}\|^2_{\Omega_{t_{m-1}}} + \frac{\tau_m}{\delta_0}\|\eta^-_{m-1}\|^2_{\Omega_{t_{m-1}}} + \frac{32M_1^*\tau_m}{L_1^*}\|\xi^-_{m-1}\|^2_{\Omega_{t_{m-1}}},$$

from which we get estimate (4.48) with

$$C_4 = \max\{2C_2/L_1^*, 2C_3/L_1^*, 2/L_1^*, 64M_1^*C_2/(L_1^*)^2\}.$$

□

Theorem 7. *There exists a constant $C_{AE} > 0$ such that the error $e = U - u$ satisfies the following estimate*

$$\|e^-_m\|^2_{\Omega_{t_m}} + \frac{1}{4}\sum_{j=1}^{m}\|\{e\}_{j-1}\|^2_{\Omega_{t_{j-1}}} + \frac{\beta_0}{2}\sum_{j=1}^{m}\int_{I_j}\|e\|^2_{DG,t}\mathrm{d}t \tag{4.50}$$

$$\le C_{AE}\Big(\|\xi_0^-\|^2_{\Omega_{t_0}} + C_5\Big(\sum_{j=1}^{m}(\tau_j+1)\int_{I_j}R_t(\eta)\,\mathrm{d}t + \sum_{j=1}^{m}\tau_j^2(1+\tau_j)\int_{I_j}\left\|\frac{\partial\tilde{\eta}}{\partial t}\right\|^2_{\Omega_{t_{j-1}}}\mathrm{d}t$$

$$+(T^2+1)\sum_{j=1}^{m}\|\eta^-_{j-1}\|^2_{\Omega_{j-1}}\Big)\Big)$$

$$+2\|\eta^-_m\|^2_{\Omega_{t_m}} + \frac{1}{2}\sum_{j=1}^{m}\|\{\eta\}_{j-1}\|^2_{\Omega_{t_{j-1}}} + \beta_0\sum_{j=1}^{m}\int_{I_j}\|\eta\|^2_{DG,t}\mathrm{d}t,$$

for $m = 1, \ldots, M$.

Proof. Substituting (4.48) into (4.42) we have the estimate

$$\|\xi^-_m\|^2_{\Omega_{t_m}} + \Big(\frac{1}{2} - C_1\,C_4\,\delta_0\tau_m\Big)\|\{\xi\}_{m-1}\|^2_{\Omega_{t_{m-1}}} - \|\xi^-_{m-1}\|^2_{\Omega_{t_{m-1}}} + \frac{\beta_0}{2}\int_{I_m}\|\xi\|^2_{DG,t}\mathrm{d}t$$

$$\le (C_1\,C_4\tau_m + C_2)\int_{I_m}R_t(\eta)\,\mathrm{d}t + (C_4\tau_m^2 + C_3\tau_m^3)\int_{I_m}\left\|\frac{\partial\tilde{\eta}}{\partial t}\right\|^2_{\Omega_{t_{m-1}}}\mathrm{d}t$$

$$+C_4\tau_m\|\xi^-_{m-1}\|^2_{\Omega_{t_{m-1}}} + (C_4\tau_m\frac{1}{\delta_0} + 4)\|\eta^-_{m-1}\|^2_{\Omega_{t_{m-1}}}.$$

Using the estimate $\tau_m \le T$ and choosing δ_0 so that $C_1 C_4 \delta_0 T = \frac{1}{4}$ we find that

$$\|\xi_m^-\|^2_{\Omega_{t_m}} + \frac{1}{4}\|\{\xi\}_{m-1}\|^2_{\Omega_{t_{m-1}}} - \|\xi_{m-1}^-\|^2_{\Omega_{t_{m-1}}} + \frac{\beta_0}{2}\int_{I_m}\|\xi\|^2_{DG,t}\mathrm{d}t \tag{4.51}$$
$$\le (C_1 C_4\tau_m + C_2)\int_{I_m} R_t(\eta)\,\mathrm{d}t + (C_4\tau_m^2 + C_3\tau_m^3)\int_{I_m}\left\|\frac{\partial\tilde{\eta}}{\partial t}\right\|^2_{\Omega_{t_{m-1}}}\mathrm{d}t$$
$$+C_4\tau_m\|\xi_{m-1}^-\|^2_{\Omega_{t_{m-1}}} + (4C_1 C_4^2 T^2 + 4)\|\eta_{m-1}^-\|^2_{\Omega_{t_{m-1}}}.$$

Further, writing j instead of m in (4.51), summing the result over j from 1 to m and setting $C_5 = \max\{\max\{C_1C_4, C_2\}, \max\{C_3, C_4\}, C_4, \max\{4C_1 C_4^2, 4\}\}$ we obtain

$$\|\xi_m^-\|^2_{\Omega_{t_m}} + \frac{1}{4}\sum_{j=1}^{m}\|\{\xi\}_{j-1}\|^2_{\Omega_{t_{j-1}}} + \frac{\beta_0}{2}\sum_{j=1}^{m}\int_{I_j}\|\xi\|^2_{DG,t}\mathrm{d}t \tag{4.52}$$
$$\le C_5\Big(\sum_{j=1}^{m}(\tau_j+1)\int_{I_j} R_t(\eta)\,\mathrm{d}t + \sum_{j=1}^{m}\tau_j^2(1+\tau_j)\int_{I_j}\left\|\frac{\partial\tilde{\eta}}{\partial t}\right\|^2_{\Omega_{t_{j-1}}}\mathrm{d}t$$
$$+\sum_{j=0}^{m-1}\tau_{j+1}\|\xi_j^-\|^2_{\Omega_{t_j}} + (T^2+1)\sum_{j=1}^{m}\|\eta_{j-1}^-\|^2_{\Omega_{j-1}}\Big) + \|\xi_0^-\|^2_{\Omega_{t_0}}.$$

Using the discrete Gronwall inequality from Lemma 14 with the following setting

$$\begin{aligned}
x_0 &= a_0 = \|\xi_0^-\|^2_{\Omega_{t_0}}, \quad y_0 = 0,\\
x_m &= \|\xi_m^-\|^2_{\Omega_{t_m}},\\
y_m &= \frac{1}{4}\sum_{j=1}^{m}\|\{\xi\}_{j-1}\|^2_{\Omega_{t_{j-1}}} + \frac{\beta_0}{2}\sum_{j=1}^{m}\int_{I_j}\|\xi\|^2_{DG,t}\,\mathrm{d}t,\\
a_m &= \|\xi_0^-\|^2_{\Omega_{t_0}} + C_5\Big(\sum_{j=1}^{m}(\tau_j+1)\int_{I_j} R_t(\eta)\,\mathrm{d}t\\
&\quad + \sum_{j=1}^{m}\tau_j^2(1+\tau_j)\int_{I_j}\left\|\frac{\partial\tilde{\eta}}{\partial t}\right\|^2_{\Omega_{t_{j-1}}}\mathrm{d}t + (T^2+1)\sum_{j=1}^{m}\|\eta_{j-1}^-\|^2_{\Omega_{j-1}}\Big),\\
b_j &= C_5\,\tau_{j+1}, \quad j = 0, 1, \ldots, m,
\end{aligned}$$

imply that

$$\|\xi_m^-\|^2_{\Omega_{t_m}} + \frac{1}{4}\sum_{j=1}^{m}\|\{\xi\}_{j-1}\|^2_{\Omega_{t_{j-1}}} + \frac{\beta_0}{2}\sum_{j=1}^{m}\int_{I_j}\|\xi\|^2_{DG,t}\mathrm{d}t \tag{4.53}$$
$$\le \Big(\|\xi_0^-\|^2_{\Omega_{t_0}} + C_5\Big(\sum_{j=1}^{m}(\tau_j+1)\int_{I_j} R_t(\eta)\,\mathrm{d}t + \sum_{j=1}^{m}\tau_j^2(1+\tau_j)\int_{I_j}\left\|\frac{\partial\tilde{\eta}}{\partial t}\right\|^2_{\Omega_{t_{j-1}}}\mathrm{d}t$$
$$+(T^2+1)\sum_{j=1}^{m}\|\eta_{j-1}^-\|^2_{\Omega_{j-1}}\Big)\Big)\prod_{j=0}^{m-1}(1+C_5\tau_{j+1}).$$

Now (4.53) and the inequality $1+\sigma < \exp\sigma$ valid for any $\sigma > 0$ immediately yield

$$\|\xi_m^-\|^2_{\Omega_{t_m}} + \frac{1}{4}\sum_{j=1}^m \|\{\xi\}_{j-1}\|^2_{\Omega_{t_{j-1}}} + \frac{\beta_0}{2}\sum_{j=1}^m \int_{I_j} \|\xi\|^2_{DG,t}\mathrm{d}t$$
$$\le C_{AE}\Big(\|\xi_0^-\|^2_{\Omega_{t_0}} + C_5\Big(\sum_{j=1}^m(\tau_j+1)\int_{I_j} R_t(\eta)\,\mathrm{d}t + \sum_{j=1}^m \tau_j^2(1+\tau_j)\int_{I_j}\left\|\frac{\partial\tilde\eta}{\partial t}\right\|^2_{\Omega_{t_{j-1}}}\mathrm{d}t$$
$$+(T^2+1)\sum_{j=1}^m \|\eta_{j-1}^-\|^2_{\Omega_{j-1}}\Big)\Big),$$

with constant $C_{AE} = \exp C_5\, T$.

Finally, using the relation $e = \xi + \eta$ and the standard inequalities

$$\begin{aligned}
\|e_m^-\|^2_{\Omega_{t_m}} &\le 2\left(\|\xi_m^-\|^2_{\Omega_{t_m}} + \|\eta_m^-\|^2_{\Omega_{t_m}}\right),\\
\|e\|^2_{DG,t} &\le 2\left(\|\xi\|^2_{DG,t} + \|\eta\|^2_{DG,t}\right),\\
\|\{e\}_{j-1}\|^2_{\Omega_{t_{j-1}}} &\le 2\left(\|\{\xi\}_{j-1}\|^2_{\Omega_{t_{j-1}}} + \|\{\eta\}_{j-1}\|^2_{\Omega_{t_{j-1}}}\right)
\end{aligned}$$

we obtain the abstract error estimate (4.50) for $q = 1$. It follows from the definition of ξ_0^- that $\xi_0^- = 0$. Hence, the error e is bounded only by expressions depending on η and $\tilde\eta$. □

4.2.2 Discrete characteristic function

In our further considerations, the concept of a discrete characteristic function will play an important role.

Let $m \in \{1,\dots,M\}$. For $s \in I_m$ we denote $\tilde\vartheta_s = \tilde\vartheta_s(X,t)$, $X \in \Omega_{t_{m-1}}, t \in I_m$, the discrete characteristic function to $\tilde\xi$ at a point $s \in I_m$. It is defined as $\tilde\vartheta_s \in P^q(I_m; S_h^{p,m-1})$ such that

$$\begin{aligned}
\int_{I_m} (\tilde\vartheta_s,\varphi)_{\Omega_{t_{m-1}}}\,\mathrm{d}t &= \int_{t_{m-1}}^s (\tilde\xi,\varphi)_{\Omega_{t_{m-1}}}\,\mathrm{d}t \quad \forall\varphi \in P^{q-1}(I_m; S_h^{p,m-1}), && (4.54)\\
\tilde\vartheta_s(X, t_{m-1}+) &= \tilde\xi(X, t_{m-1}+),\ X \in \Omega_{t_{m-1}}. && (4.55)
\end{aligned}$$

Further, we introduce the discrete characteristic function $\vartheta_s = \vartheta_s(x,t)$, $x \in \Omega_t, t \in I_m$ to $\xi \in S_{h,\tau}^{p,q}$ at a point $s \in I_m$:

$$\vartheta_s(x,t) = \tilde\vartheta_s(\mathcal{A}_t^{-1}(x),t),\ x \in \Omega_t,\ t \in I_m. \qquad (4.56)$$

Hence, in view of (2.20), $\vartheta_s \in S_{h,\tau}^{p,q}$ and for $X \in \Omega_{t_{m-1}}$ we have

$$\vartheta_s(X, t_{m-1}+) = \xi(X, t_{m-1}+). \qquad (4.57)$$

In what follows, we prove some important properties of the discrete characteristic function. Namely, we prove that the discrete characteristic function mapping $\xi \to \vartheta_s$ is continuous with respect of the norms $\|\cdot\|_{L^2(\Omega_t)}$ and $\|\cdot\|_{DG,t}$. In the proof we use a result from [9] for the discrete characteristic function on a reference domain: There exists a constant $\tilde c_{CH}^{(1)} > 0$ depending on q only such that

$$\int_{I_m} \|\tilde\vartheta_s\|^2_{\Omega_{t_{m-1}}}\,\mathrm{d}t \le \tilde c_{CH}^{(1)} \int_{I_m} \|\tilde\xi\|^2_{\Omega_{t_{m-1}}}\,\mathrm{d}t, \qquad (4.58)$$

for all $m = 1,\dots,M$ and $h \in (0,\overline{h})$.

Theorem 8. *There exist constants $c_{CH}^{(1)}, c_{CH}^{(2)} > 0$, such that*

$$\int_{I_m} \|\vartheta_s\|_{\Omega_t}^2 \,\mathrm{d}t \;\leq\; c_{CH}^{(1)} \int_{I_m} \|\xi\|_{\Omega_t}^2 \,\mathrm{d}t \tag{4.59}$$

$$\int_{I_m} \|\vartheta_s\|_{DG,t}^2 \,\mathrm{d}t \;\leq\; c_{CH}^{(2)} \int_{I_m} \|\xi\|_{DG,t}^2 \,\mathrm{d}t \tag{4.60}$$

for all $s \in I_m,\; m = 1, \dots, M$ and $h \in (0, \overline{h})$.

Proof. It is analogous to the proof of Theorem 1. $\square$

In what follows, because of simplicity, we use the notation $\tilde{\xi}' = \frac{\partial \tilde{\xi}}{\partial t}$ and do not write the arguments X and t in integrals.

Lemma 25. *There exists a constant $C_{L24} > 0$ such that*

$$\begin{aligned}
&\int_{I_m} (D_t\xi, \vartheta_s)_{\Omega_t} \mathrm{d}t + (\{\xi\}_{m-1}, \vartheta_s(t_{m-1}+))_{\Omega_{t_{m-1}}} \\
&\geq \frac{1}{2}\left(\|\xi(s-)\|_{\Omega_s}^2 + \|\xi(t_{m-1}+)\|_{\Omega_{t_{m-1}}}^2\right) \\
&\quad - C_{L24}\int_{I_m} \|\xi\|_{\Omega_t}^2 \,\mathrm{d}t - (\xi_{m-1}^+, \xi_{m-1}^-)_{\Omega_{t_{m-1}}}.
\end{aligned} \tag{4.61}$$

for any $s \in I_m,\; m = 1, \dots, M$ and $h \in (0, \overline{h})$.

Proof. By virtue of the definition of the ALE derivative (2.9), the definitions (4.29), (4.54), (4.55) and (4.56) of $\tilde{\xi}, \tilde{\vartheta}_s$ and ϑ_s, the fact that $\tilde{\xi}'$ is a polynomial of degree $\leq q-1$ in time and the substitution theorem, we can write

$$\begin{aligned}
&\int_{I_m} (D_t\xi, \vartheta_s)_{\Omega_t} \mathrm{d}t = \int_{I_m} \left(\tilde{\xi}', \tilde{\vartheta}_s J\right)_{\Omega_{t_{m-1}}} \mathrm{d}t \\
&= \int_{I_m} \left(\tilde{\xi}', \tilde{\vartheta}_s\right)_{\Omega_{t_{m-1}}} \mathrm{d}t + \int_{I_m} \left(\tilde{\xi}', \tilde{\vartheta}_s(J-1)\right)_{\Omega_{t_{m-1}}} \mathrm{d}t \\
&= \int_{t_{m-1}}^{s} \left(\tilde{\xi}', \tilde{\xi}\right)_{\Omega_{t_{m-1}}} \mathrm{d}t + \int_{I_m} \left(\tilde{\xi}', \tilde{\vartheta}_s(J-1)\right)_{\Omega_{t_{m-1}}} \mathrm{d}t \\
&= \int_{t_{m-1}}^{s} \left(\tilde{\xi}', \tilde{\xi}J\right)_{\Omega_{t_{m-1}}} \mathrm{d}t + \int_{t_{m-1}}^{s} \left(\tilde{\xi}', \tilde{\xi}(1-J)\right)_{\Omega_{t_{m-1}}} \mathrm{d}t \\
&\quad + \int_{I_m} \left(\tilde{\xi}', \tilde{\vartheta}_s(J-1)\right)_{\Omega_{t_{m-1}}} \mathrm{d}t \\
&= \int_{t_{m-1}}^{s} (D_t\xi, \xi)_{\Omega_t} \mathrm{d}t + \int_{t_{m-1}}^{s} \left(\tilde{\xi}', \tilde{\xi}(1-J)\right)_{\Omega_{t_{m-1}}} \mathrm{d}t \\
&\quad + \int_{I_m} \left(\tilde{\xi}', \tilde{\vartheta}_s(J-1)\right)_{\Omega_{t_{m-1}}} \mathrm{d}t.
\end{aligned} \tag{4.62}$$

Now we estimate the second and third term in the last line of (4.62). We begin with the third term. The fact that J is constant on each $\hat{K} \in \hat{\mathcal{T}}_{h,t_{m-1}}$ and the substitution theorem imply that

$$\begin{aligned}
&\left|\int_{I_m} \left(\tilde{\xi}', \tilde{\vartheta}_s(J-1)\right)_{\Omega_{t_{m-1}}} \mathrm{d}t\right| = \left|\sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \int_{I_m} (J_{\hat{K}} - 1)\left(\int_{\hat{K}} \tilde{\xi}'\tilde{\vartheta}_s \,\mathrm{d}X\right) \mathrm{d}t\right| \\
&\leq \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \max_{t\in I_m} |J_{\hat{K}} - 1| \int_{I_m} \left(\int_{\hat{K}} |\tilde{\xi}'\tilde{\vartheta}_s| \,\mathrm{d}X\right) \mathrm{d}t.
\end{aligned}$$

Using the relation $J_{\hat{K}}(t_{m-1}) = 1$ and inequality (3.9), we have

$$\max_{t\in I_m} |J_{\hat{K}} - 1| \le \int_{t_{m-1}}^{t_m} |J'_{\hat{K}}| \,\mathrm{d}t \le c_J \tau_m.$$

Then we find that

$$\begin{aligned}
&\sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \max_{t\in I_m} |J_{\hat{K}} - 1| \int_{I_m}\int_{\hat{K}} |\tilde{\xi}'\tilde{\vartheta}_s| \,\mathrm{d}X\mathrm{d}t \\
&\le c_J \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \tau_m \int_{I_m} \left(\int_{\hat{K}} |\tilde{\xi}'\tilde{\vartheta}_s| \,\mathrm{d}X\right) \mathrm{d}t \\
&= c_J\tau_m \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \int_{\hat{K}} \left(\int_{I_m} |\tilde{\xi}'\tilde{\vartheta}_s| \,\mathrm{d}t\right) \mathrm{d}X \\
&\le c_J\tau_m \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \int_{\hat{K}} \left(\left(\int_{I_m} |\tilde{\xi}'|^2 \,\mathrm{d}t\right)^{1/2} \left(\int_{I_m} |\tilde{\vartheta}_s|^2 \,\mathrm{d}t\right)^{1/2}\right) \mathrm{d}X.
\end{aligned}$$

Now we apply the inverse inequality in time: There exists a constant $\hat{c}_I$ such that

$$\left(\int_{I_m} |\tilde{\xi}'(X,t)|^2 \,\mathrm{d}t\right)^{1/2} \le \frac{\hat{c}_I}{\tau_m} \left(\int_{I_m} |\tilde{\xi}(X,t)|^2 \,\mathrm{d}t\right)^{1/2} \tag{4.63}$$

holds for every $X \in \Omega_{t_{m-1}}$, $\tau_m \in (0, \overline{\tau})$ and $m = 1, \ldots, M$.

This inequality, Young's inequality, Fubini's theorem, (4.58), substitution theorem and (3.6) imply that

$$\begin{aligned}
&\tau_m \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \int_{\hat{K}} \left(\left(\int_{I_m} |\tilde{\xi}'|^2 \,\mathrm{d}t\right)^{1/2} \left(\int_{I_m} |\tilde{\vartheta}_s|^2 \,\mathrm{d}t\right)^{1/2}\right) \mathrm{d}X \\
&\le \hat{c}_I \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \int_{\hat{K}} \left(\int_{I_m} |\tilde{\xi}|^2\mathrm{d}t\right)^{1/2} \left(\int_{I_m} |\tilde{\vartheta}_s|^2\mathrm{d}t\right)^{1/2} \mathrm{d}X \\
&\le \frac{\hat{c}_I}{2} \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \int_{\hat{K}} \left(\int_{I_m} \left(|\tilde{\xi}|^2 + |\tilde{\vartheta}_s|^2\right) \mathrm{d}t\right) \mathrm{d}X \\
&= \frac{\hat{c}_I}{2} \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \int_{I_m} \left(\int_{\hat{K}} \left(|\tilde{\xi}|^2 + |\tilde{\vartheta}_s|^2\right) \mathrm{d}X\right) \mathrm{d}t \\
&= \frac{\hat{c}_I}{2} \left(\int_{I_m} \|\tilde{\xi}\|^2_{\Omega_{t_{m-1}}} \mathrm{d}t + \int_{I_m} \|\tilde{\vartheta}_s\|^2_{\Omega_{t_{m-1}}} \mathrm{d}t\right) \\
&\le \frac{\hat{c}_I}{2}(1 + \tilde{c}^{(1)}_{CH}) \int_{I_m} \|\tilde{\xi}\|^2_{\Omega_{t_{m-1}}} \mathrm{d}t \\
&= \frac{\hat{c}_I}{2}(1 + \tilde{c}^{(1)}_{CH}) \int_{I_m} \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \int_{\hat{K}} \left(|\tilde{\xi}|^2\mathrm{d}X\right) \mathrm{d}t \\
&= \frac{\hat{c}_I}{2}(1 + \tilde{c}^{(1)}_{CH}) \int_{I_m} \left(\int_{\Omega_t} |\xi|^2 J^{-1}\mathrm{d}x\right) \mathrm{d}t \le c^* \int_{I_m} \|\xi\|^2_{\Omega_t} \,\mathrm{d}t,
\end{aligned}$$

where $c^* = (C_J^-)^{-1}\hat{c}_I(1 + \tilde{c}^{(1)}_{CH})/2$. Summarizing the obtained results, we see that we have proved the inequality

$$\left|\int_{I_m} \left(\tilde{\xi}', \tilde{\vartheta}_s(J-1)\right)_{\Omega_{t_{m-1}}} \mathrm{d}t\right| \le c^* c_J \int_{I_m} \|\xi\|^2_{\Omega_t} \,\mathrm{d}t. \tag{4.64}$$

Similarly as above we can estimate the second term on the right-hand side of (4.62):

$$\begin{aligned}
&\left|\int_{t_{m-1}}^{s}\left(\tilde{\xi}', \tilde{\xi}(1-J)\right)_{\Omega_{t_{m-1}}} \mathrm{d}t\right| \le \int_{I_m}\left|(\tilde{\xi}', \tilde{\xi}(1-J))_{\Omega_{t_{m-1}}}\right| \mathrm{d}t \\
&\le \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \max_{t\in I_m}|1-J_{\hat{K}}| \int_{I_m}\int_{\hat{K}} |\tilde{\xi}'\tilde{\xi}| \,\mathrm{d}X\mathrm{d}t \\
&\le c_J\tau_m \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \int_{I_m}\int_{\hat{K}} |\tilde{\xi}'\tilde{\xi}| \,\mathrm{d}X\mathrm{d}t \\
&= c_J\tau_m \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \int_{\hat{K}}\left(\int_{I_m} |\tilde{\xi}'\tilde{\xi}| \,\mathrm{d}t\right)\mathrm{d}X \\
&\le c_J\tau_m \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \int_{\hat{K}}\left(\left(\int_{I_m} |\tilde{\xi}'|^2 \,\mathrm{d}t\right)^{1/2}\left(\int_{I_m} |\tilde{\xi}|^2 \,\mathrm{d}t\right)^{1/2}\right)\mathrm{d}X.
\end{aligned}$$

Now the inverse inequality (4.63) in time, Young's inequality, Fubini's theorem, (4.58) and (3.6) yield the inequality

$$\left|\int_{t_{m-1}}^{s}\left(\tilde{\xi}', \tilde{\xi}(1-J)\right)_{\Omega_{t_{m-1}}} \mathrm{d}t\right| \le c_1\int_{I_m}\|\xi\|^2_{\Omega_t}\,\mathrm{d}t. \tag{4.65}$$

with $c_1 = c_J(C_J^-)^{-1}\hat{c}_I/2$.

From (4.62), (4.64) and (4.65) we find that

$$\int_{I_m}(D_t\xi, \vartheta_s)_{\Omega_t}\,\mathrm{d}t \ge \int_{t_{m-1}}^{s}(D_t\xi, \xi)_{\Omega_t}\,\mathrm{d}t - (c^*c_J + c_1)\int_{I_m}\|\xi\|^2_{\Omega_t}\,\mathrm{d}t. \tag{4.66}$$

Similarly as in the proof of Lemma 21, by (4.28), we get

$$\begin{aligned}
&\int_{t_{m-1}}^{s}(D_t\xi, \xi)_{\Omega_t}\,\mathrm{d}t \qquad (4.67)\\
&= \frac{1}{2}\int_{t_{m-1}}^{s}\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}\xi^2(x,t)\,\mathrm{d}x\right)\mathrm{d}t - \frac{1}{2}\int_{t_{m-1}}^{s}(\xi^2, \mathrm{div}\boldsymbol{z})_{\Omega_t}\,\mathrm{d}t \\
&= \frac{1}{2}\left(\|\xi(s-)\|^2_{\Omega_s} - \|\xi^+_{m-1}\|^2_{\Omega_{t_{m-1}}}\right) - \frac{1}{2}\int_{t_{m-1}}^{s}(\xi^2, \mathrm{div}\boldsymbol{z})_{\Omega_t}\,\mathrm{d}t.
\end{aligned}$$

Now, if we set $c_2 = c^*c_J + c_1$, from (4.66), (4.67), (4.57) and (3.10) we obtain the inequality

$$\begin{aligned}
&\int_{I_m}(D_t\xi, \vartheta_s)_{\Omega_t}\,\mathrm{d}t + (\{\xi\}_{m-1}, \vartheta_s(t_{m-1}+))_{\Omega_{t_{m-1}}} \qquad (4.68)\\
&\ge \frac{1}{2}\left(\|\xi(s-)\|^2_{\Omega_s} + \|\xi^+_{m-1}\|^2_{\Omega_{t_{m-1}}}\right) - (c_z/2 + c_2)\int_{I_m}\|\xi\|^2_{\Omega_t}\,\mathrm{d}t - (\xi^-_{m-1}, \xi^+_{m-1})_{\Omega_{t_{m-1}}},
\end{aligned}$$

which is (4.61) with $C_{L24} = c_z/2 + c_2$. □

4.2.3 Abstract error estimate for higher order approximation in time

In the following part of this chapter we shall consider higher order approximation in time. This means that we assume $q \ge 2$. For every $m = 1, \ldots, M$ on the interval I_m we introduce the following notation:

$$\begin{aligned} t_{m-1+l/q} &= t_{m-1} + \tau_m \frac{l}{q}, \quad \xi_{m-1} = \xi(t_{m-1}+),\ \xi_m = \xi(t_m-), \\ \xi_{m-1+l/q} &= \xi(t_{m-1+l/q}), \quad l = 0,\dots,q. \end{aligned}$$

Lemma 26. *There exist constants $L_q^*, M_q^* > 0$ such that for $m = 1,\dots,M$ we have*

$$\sum_{l=0}^{q} \|\xi_{m-1+l/q}\|^2_{\Omega_{t_{m-1+l/q}}} \geq \frac{L_q^*}{\tau_m} \int_{I_m} \|\xi\|^2_{\Omega_t} \mathrm{d}t, \tag{4.69}$$

$$\|\xi^+_{m-1}\|^2_{\Omega_{t_{m-1}}} \leq \frac{M_q^*}{\tau_m} \int_{I_m} \|\xi\|^2_{\Omega_t} \mathrm{d}t. \tag{4.70}$$

Proof. See proof of Lemma 8.

□

In the following lemmas, for simplicity we shall use the notation ϑ_l^* and $\tilde{\vartheta}_l^*$ for discrete characteristic functions to ξ and $\tilde{\xi}$, respectively, at time instant $t_{m-1+l/q}$. This means that $\vartheta_l^* = \vartheta_{t_{m-1+l/q}}$, $\tilde{\vartheta}_l^* = \tilde{\vartheta}_{t_{m-1+l/q}}$.

Now we prove an auxiliary result similar to Lemma 22.

Lemma 27. *There exists a constant $C_\eta > 0$ independent of u, U, h, τ, m, M such that*

$$\begin{aligned} &\left| \int_{I_m} (D_t\eta, \vartheta_l^*)_{\Omega_t} \mathrm{d}t + (\{\eta\}_{m-1}, (\vartheta_l^*)^+_{m-1})_{\Omega_{t_{m-1}}} \right| \\ &\leq C_\mu \left(\tau_m^2 \int_{I_m} \left\| \frac{\partial \tilde{\eta}}{\partial t} \right\|^2_{\Omega_{t_{m-1}}} \mathrm{d}t + \int_{I_m} \|\xi\|^2_{\Omega_t} \mathrm{d}t \right) \\ &\quad + \frac{1}{\delta_3} \|\eta^-_{m-1}\|^2_{\Omega_{t_{m-1}}} + \delta_3 \|\xi^+_{m-1}\|^2_{\Omega_{t_{m-1}}} \end{aligned} \tag{4.71}$$

holds for every $\delta_3 > 0$.

Proof. Similarly as in (4.33), we get

$$\begin{aligned} &\int_{I_m} (D_t\eta, \vartheta_l^*)_{\Omega_t} \mathrm{d}t \\ &= \underbrace{\int_{I_m} \left(\int_{\Omega_{t_{m-1}}} \frac{\partial \tilde{\eta}(X,t)}{\partial t} \tilde{\vartheta}_l^*(X,t)\, \mathrm{d}X \right) \mathrm{d}t}_{\gamma_1} \\ &\quad + \underbrace{\int_{I_m} \left(\int_{\Omega_{t_{m-1}}} \frac{\partial \tilde{\vartheta}_l^*(X,t)}{\partial t} \tilde{\xi}(X,t)(J(X,t) - J(X,t_{m-1}))\, \mathrm{d}X \right) \mathrm{d}t.}_{\gamma_2} \end{aligned} \tag{4.72}$$

By (4.35) and (4.59) we have

$$\begin{aligned} |\gamma_2| &\leq c_J^2 \tau_m^2 \int_{I_m} \left\| \frac{\partial \tilde{\eta}}{\partial t} \right\|^2_{\Omega_{t_{m-1}}} \mathrm{d}t + (C_J^-)^{-1} \int_{I_m} \|\vartheta_l^*\|^2_{\Omega_t} \mathrm{d}t \\ &\leq C_\mu \Big(\tau_m^2 \int_{I_m} \left\| \frac{\partial \tilde{\eta}}{\partial t} \right\|^2_{\Omega_{t_{m-1}}} \mathrm{d}t + \int_{I_m} \|\xi\|^2_{\Omega_t} \mathrm{d}t \Big), \end{aligned} \tag{4.73}$$

where $C_\mu = \max\{c_J^2, (C_J^-)^{-1} c_{CH}^{(1)}\}$.

Further, we pay attention to γ_1. Integration by parts implies

$$\gamma_1 = (\tilde{\eta}_m^-, (\tilde{\vartheta}_l^*)_m^-)_{\Omega_{t_{m-1}}} - (\tilde{\eta}_{m-1}^+, (\tilde{\vartheta}_l^*)_{m-1}^+)_{\Omega_{t_{m-1}}} - \int_{I_m} \left(\tilde{\eta}, \frac{\partial \tilde{\vartheta}_l^*}{\partial t}\right)_{\Omega_{t_{m-1}}} \mathrm{d}t. \quad (4.74)$$

Since $\frac{\partial \tilde{\vartheta}_l^*}{\partial t}$ is a polynom in t of degree $\leq q-1$, the last expression is zero. We also have $(\tilde{\eta}_m^-, (\tilde{\vartheta}_l^*)_m^-)_{\Omega_{t_{m-1}}} = 0$ (cf. [31], (6.90)), $\tilde{\eta}_{m-1}^+ = \eta_{m-1}^+$, $(\tilde{\vartheta}_l^*)_{m-1}^+ = (\vartheta_l^*)_{m-1}^+$ and thus,

$$\gamma_1 + \left(\{\eta\}_{m-1}, (\vartheta_l^*)_{m-1}^+\right)_{\Omega_{t_{m-1}}} = -\left(\eta_{m-1}^-, (\vartheta_l^*)_{m-1}^+\right)_{\Omega_{t_{m-1}}}. \quad (4.75)$$

This, the relation $(\vartheta_l^*)_{m-1}^+ = \xi_{m-1}^+$ and Young's inequality with constant $\delta_0 > 0$ imply that

$$|\gamma_1 + \left(\{\eta\}_{m-1}, (\vartheta_l^*)_{m-1}^+\right)_{\Omega_{t_{m-1}}}| \leq \frac{1}{\delta_0} \|\eta_{m-1}^-\|^2_{\Omega_{t_{m-1}}} + \delta_0 \|\xi_{m-1}^+\|^2_{\Omega_{t_{m-1}}}.$$

□

Theorem 9. *Let $q \geq 2$. There exist constants $C_{T9}, C_{T9}^* > 0$ such that*

$$\begin{aligned} \int_{I_m} \|\xi\|^2_{\Omega_t} \mathrm{d}t \quad &\leq \quad C_{T9}\, \tau_m \bigg(\int_{I_m} (R_t(\eta) + R_t^*(\eta))\, \mathrm{d}t + \tau_m^2 \int_{I_m} \left\| \frac{\partial \tilde{\eta}}{\partial t} \right\|^2_{\Omega_{t_{m-1}}} \mathrm{d}t \quad (4.76) \\ &+ \|\xi_{m-1}^-\|^2_{\Omega_{t_{m-1}}} + \delta_0 \|\{\xi\}_{m-1}\|^2_{\Omega_{t_{m-1}}} + \left(c_7 + \frac{1}{\delta_0}\right) \|\eta_{m-1}^-\|^2_{\Omega_{t_{m-1}}} \bigg), \end{aligned}$$

provided $0 < \tau_m < C_{T9}^*$.

Proof. Let $l \in \{1, \ldots, q-1\}$. Setting $\varphi := \vartheta_l^*$ in (4.6) and using identity (4.7), we get

$$\begin{aligned} &\int_{I_m} (D_t \xi, \vartheta_l^*)_{\Omega_t} \, \mathrm{d}t + \left(\{\xi\}_{m-1}, (\vartheta_l^*)_{m-1}^+\right)_{\Omega_{t_{m-1}}} \quad (4.77) \\ &= \int_{I_m} (-a_h(U, U, \vartheta_l^*, t) + a_h(U, \pi u, \vartheta_l^*, t) - \beta_0 J_h(\xi, \vartheta_l^*, t))\, \mathrm{d}t \\ &\quad + \int_{I_m} (-a_h(U, \pi u, \vartheta_l^*, t) + a_h(u, \pi u, \vartheta_l^*, t))\, \mathrm{d}t \\ &\quad + \int_{I_m} (-a_h(u, \pi u, \vartheta_l^*, t) + a_h(u, u, \vartheta_l^*, t) - \beta_0 J_h(\eta, \vartheta_l^*, t))\, \mathrm{d}t \\ &\quad + \int_{I_m} (b_h(u, \vartheta_l^*, t) - b_h(U, \vartheta_l^*, t))\, \mathrm{d}t \\ &\quad + \int_{I_m} (-d_h(\xi, \vartheta_l^*, t) - d_h(\eta, \vartheta_l^*, t))\, \mathrm{d}t \\ &\quad - \int_{I_m} (D_t \eta, \vartheta_l^*)_{\Omega_t} \, \mathrm{d}t - \left(\{\eta\}_{m-1}, (\vartheta_l^*)_{m-1}^+\right)_{\Omega_{t_{m-1}}}. \end{aligned}$$

This relation and Lemma 25 imply that

$$
\begin{aligned}
&\frac{1}{2}\left(\left\|\xi^-_{m-1+l/q}\right\|^2_{\Omega_{t_{m-1+l/q}}} + \|\xi^+_{m-1}\|^2_{\Omega_{t_{m-1}}}\right) \qquad (4.78)\\
&\le \int_{I_m} |-a_h(U,U,\vartheta_l^*,t) + a_h(U,\pi u,\vartheta_l^*,t) - \beta_0 J_h(\xi,\vartheta_l^*,t)|\,\mathrm{d}t\\
&\quad + \int_{I_m} |-a_h(U,\pi u,\vartheta_l^*,t) + a_h(u,\pi u,\vartheta_l^*,t)|\,\mathrm{d}t\\
&\quad + \int_{I_m} (|-a_h(u,\pi u,\vartheta_l^*,t) + a_h(u,u,\vartheta_l^*,t)| + |\beta_0 J_h(\eta,\vartheta_l^*,t)|)\,\mathrm{d}t\\
&\quad + \int_{I_m} |b_h(u,\vartheta_l^*,t) - b_h(U,\vartheta_l^*,t)|\,\mathrm{d}t\\
&\quad + \int_{I_m} (|d_h(\xi,\vartheta_l^*,t)| + |d_h(\eta,\vartheta_l^*,t)|)\,\mathrm{d}t\\
&\quad + \left|\int_{I_m} (D_t\eta,\vartheta_l^*)_{\Omega_t}\,\mathrm{d}t + \left(\{\eta\}_{m-1},(\vartheta_l^*)^+_{m-1}\right)_{\Omega_{t_{m-1}}}\right|\\
&\quad + \left|\left(\xi^+_{m-1},\xi^-_{m-1}\right)_{\Omega_{t_{m-1}}}\right| + C_{L24}\int_{I_m}\|\xi\|^2_{\Omega_t}\,\mathrm{d}t \equiv \mathrm{RHS}.
\end{aligned}
$$

Now we need to estimate the right-hand side of (4.78) from above. Using Lemmas 16, 17, 18, 19, 20, 27 with $k_b = k_c = k_d = 1$, $k_e = 1/2$ and Young's inequality with any $\delta_2 > 0$, we have

$$
\begin{aligned}
\mathrm{RHS} \;\le\; & \int_{I_m} \Big(C_a(\|\xi\|^2_{DG,t} + \|\vartheta_l^*\|^2_{DG,t}) + \beta_0\|\vartheta_l^*\|^2_{DG,t} + C_c\|\xi\|^2_{\Omega_t} + C_c R_t(\eta)\Big)\,\mathrm{d}t\\
& + \int_{I_m}\left(\beta_0\|\vartheta_l^*\|^2_{DG,t} + C_d\,R_t^*(\eta) + \frac{\beta_0}{2\delta}R_t(\eta) + 2\beta_0\delta\|\vartheta_l^*\|^2_{DG,t}\right)\mathrm{d}t\\
& + \int_{I_m}\Big(\beta_0\|\vartheta_l^*\|^2_{\Omega_t} + C_b\|\xi\|^2_{\Omega_t} + C_b R_t(\eta)\Big)\,\mathrm{d}t\\
& + \int_{I_m}\left(\beta_0\|\xi\|^2_{DG,t} + \frac{c_e}{2\beta_0}\|\vartheta_l^*\|^2_{\Omega_t} + \beta_0\|\eta\|^2_{DG,t} + \frac{c_e}{2\beta_0}\|\vartheta_l^*\|^2_{\Omega_t}\right)\mathrm{d}t\\
& + C_\mu\left(\tau_m^2\int_{I_m}\left\|\frac{\partial\tilde\eta}{\partial t}\right\|^2_{\Omega_{t_{m-1}}}\mathrm{d}t + \int_{I_m}\|\xi\|^2_{\Omega_t}\mathrm{d}t\right)\\
& + \frac{1}{\delta_3}\|\eta^-_{m-1}\|^2_{\Omega_{t_{m-1}}} + \delta_3\|\xi^+_{m-1}\|^2_{\Omega_{t_{m-1}}}\\
& + \frac{\|\xi^-_{m-1}\|^2_{\Omega_{t_{m-1}}}}{\delta_2} + 2\delta_2\|\xi^+_{m-1}\|^2_{\Omega_{t_{m-1}}} + C_{L24}\int_{I_m}\|\xi\|^2_{\Omega_t}\mathrm{d}t.
\end{aligned}
$$

Rearranging we get

$$
\begin{aligned}
\mathrm{RHS} \;\le\; & c_1\int_{I_m}\Big(\|\xi\|^2_{DG,t} + \|\vartheta_l^*\|^2_{DG,t} + \|\vartheta_l^*\|^2_{\Omega_t} + \|\xi\|^2_{\Omega_t} + R_t(\eta) + R_t^*(\eta)\Big)\,\mathrm{d}t\\
& + C_\mu\tau_m^2\int_{I_m}\left\|\frac{\partial\tilde\eta}{\partial t}\right\|^2_{\Omega_{t_{m-1}}}\mathrm{d}t + \frac{1}{\delta_3}\|\eta^-_{m-1}\|^2_{\Omega_{t_{m-1}}} + (\delta_3 + 2\delta_2)\|\xi^+_{m-1}\|^2_{\Omega_{t_{m-1}}}\\
& + \frac{\|\xi^-_{m-1}\|^2_{\Omega_{t_{m-1}}}}{\delta_2},
\end{aligned}
$$

where $c_1 = \max\{C_a + \beta_0, C_a + 2\beta_0 + 2\beta_0\delta, \beta_0 + c_e/\beta_0, C_c + C_b + C_\mu + C_{L24}, C_c + \beta_0/2\delta + C_b + \beta_0, C_d\}$. Now applying (4.57) and the result from Theorem 8 about the continuity of the discrete characteristic function, i.e.,

$$\begin{aligned}\int_{I_m} \|\vartheta_l^*\|^2_{\Omega_t}\mathrm{d}t &\leq c^{(1)}_{\mathrm{CH}}\int_{I_m} \|\xi\|^2_{\Omega_t}\mathrm{d}t,\\ \int_{I_m} \|\vartheta_l^*\|^2_{DG,t}\mathrm{d}t &\leq c^{(2)}_{\mathrm{CH}}\int_{I_m} \|\xi\|^2_{DG,t}\mathrm{d}t,\end{aligned}$$

we get

$$\begin{aligned}\mathrm{RHS} \leq\ & c_2\int_{I_m}\left(\|\xi\|^2_{DG,t}+\|\xi\|^2_{\Omega_t}+R_t(\eta)+R_t^*(\eta)\right)\mathrm{d}t + C_\mu\tau_m^2\int_{I_m}\left\|\frac{\partial\tilde{\eta}}{\partial t}\right\|^2_{\Omega_{t_{m-1}}}\mathrm{d}t\\ &+\frac{1}{\delta_3}\|\eta^-_{m-1}\|^2_{\Omega_{t_{m-1}}}+(\delta_3+2\delta_2)\|\xi^+_{m-1}\|^2_{\Omega_{t_{m-1}}}+\frac{\|\xi^-_{m-1}\|^2_{\Omega_{t_{m-1}}}}{\delta_2},\end{aligned}$$

with $c_2 = c_1\max\{1+c^{(1)}_{CH}, 1+c^{(2)}_{CH}\}$. Then it follows from this inequality and (4.78) that

$$\begin{aligned}&\frac{1}{2}\left(\left\|\xi^-_{m-1+l/q}\right\|^2_{\Omega_{t_{m-1+l/q}}}+\|\xi^+_{m-1}\|^2_{\Omega_{t_{m-1}}}\right)\\ &\leq c_2\int_{I_m}\left(\|\xi\|^2_{DG,t}+\|\xi\|^2_{\Omega_t}+R_t(\eta)+R_t^*(\eta)\right)\mathrm{d}t\\ &\quad+C_\mu\tau_m^2\int_{I_m}\left\|\frac{\partial\tilde{\eta}}{\partial t}\right\|^2_{\Omega_{t_{m-1}}}\mathrm{d}t+\frac{1}{\delta_3}\|\eta^-_{m-1}\|^2_{\Omega_{t_{m-1}}}\\ &\quad+(\delta_3+2\delta_2)\|\xi^+_{m-1}\|^2_{\Omega_{t_{m-1}}}+\frac{\|\xi^-_{m-1}\|^2_{\Omega_{t_{m-1}}}}{\delta_2}.\end{aligned}\tag{4.79}$$

Further, multiplying (4.79) by $\frac{\beta_0}{4c_2(q-1)}$, summing over $l = 1, \ldots, q-1$ and adding to (4.44), we find that

$$\begin{aligned}&\|\xi^-_m\|^2_{\Omega_{t_m}}+\frac{\beta_0}{8c_2(q-1)}\sum_{l=1}^{q-1}\|\xi^-_{m-1+l/q}\|^2_{\Omega_{t_{m-1+l/q}}}+\left(\frac{\beta_0}{8c_2}+1\right)\|\xi^+_{m-1}\|^2_{\Omega_{t_{m-1}}}\\ &\quad+\frac{\beta_0}{2}\int_{I_m}\|\xi\|^2_{DG,t}\mathrm{d}t\\ &\leq\frac{\beta_0}{4}\int_{I_m}\|\xi\|^2_{DG,t}\mathrm{d}t+\left(\frac{\beta_0}{4}+C_1\right)\int_{I_m}\|\xi\|^2_{\Omega_t}\mathrm{d}t\\ &\quad+\left(\frac{\beta_0}{4}+C_2\right)\int_{I_m}R_t(\eta)\mathrm{d}t+\frac{\beta_0}{4}\int_{I_m}R_t^*(\eta)\mathrm{d}t\\ &\quad+\left(\frac{\beta_0C_\mu}{4c_2}+C_3\right)\tau_m^2\int_{I_m}\left\|\frac{\partial\tilde{\eta}}{\partial t}\right\|^2_{\Omega_{t_{m-1}}}\mathrm{d}t\\ &\quad+\left(\frac{\beta_0}{4c_2\delta_2}+\frac{2}{\delta_1}\right)\|\xi^-_{m-1}\|^2_{\Omega_{t_{m-1}}}+\delta_0\|\{\xi\}_{m-1}\|^2_{\Omega_{t_{m-1}}}\\ &\quad+\left(\frac{\beta_0}{4c_2\delta_3}+\frac{1}{\delta_0}\right)\|\eta^-_{m-1}\|^2_{\Omega_{t_{m-1}}}+\left(\frac{\beta_0}{4c_2}(\delta_3+2\delta_2)+4\delta_1\right)\|\xi^+_{m-1}\|^2_{\Omega_{t_{m-1}}}.\end{aligned}$$

Setting $c_3 := \min\left\{1, \frac{\beta_0}{8c_2(q-1)}, \frac{\beta_0}{8c_2}+1\right\}$ and rearranging, we get

$$c_3\underbrace{\Big(\|\xi_m^-\|_{\Omega_{t_m}^2} + \sum_{l=1}^{q-1} \|\xi_{m-1+l/q}^-\|^2_{\Omega_{t_{m-1+l/q}}} + \|\xi_{m-1}^+\|^2_{\Omega_{t_{m-1}}} \Big)}_{=\sum_{l=0}^{q}\|\xi_{m-1+l/q}\|^2_{\Omega_{t_{m-1+l/q}}}} + \frac{\beta_0}{4}\int_{I_m}\|\xi\|^2_{DG,t}\mathrm{d}t$$
$$\le \left(\frac{\beta_0}{4}+C_1\right)\int_{I_m}\|\xi\|^2_{\Omega_t}\mathrm{d}t + \left(\frac{\beta_0}{4}+C_2\right)\int_{I_m}R_t(\eta)\mathrm{d}t$$
$$+\frac{\beta_0}{4}\int_{I_m}R_t^*(\eta)\mathrm{d}t + \left(\frac{\beta_0 C_\mu}{4c_2}+C_3\right)\tau_m^2\int_{I_m}\left\|\frac{\partial\tilde{\eta}}{\partial t}\right\|^2_{\Omega_{t_{m-1}}}\mathrm{d}t$$
$$+\left(\frac{\beta_0}{4c_2\delta_2}+\frac{2}{\delta_1}\right)\|\xi_{m-1}^-\|^2_{\Omega_{t_{m-1}}} + \delta_0\|\{\xi\}_{m-1}\|^2_{\Omega_{t_{m-1}}}$$
$$+\left(\frac{\beta_0}{4c_2\delta_3}+\frac{1}{\delta_0}\right)\|\eta_{m-1}^-\|^2_{\Omega_{t_{m-1}}} + \left(\frac{\beta_0}{4c_2}(\delta_3+2\delta_2)+4\delta_1\right)\|\xi_{m-1}^+\|^2_{\Omega_{t_{m-1}}}.$$

It follows from inequalities (4.69) and (4.70) that

$$\frac{c_3 L_q^*}{\tau_m}\int_{I_m}\|\xi\|^2_{\Omega_t}\mathrm{d}t + \frac{\beta_0}{4}\int_{I_m}\|\xi\|^2_{DG,t}\mathrm{d}t$$
$$\le \left(\frac{\beta_0(\delta_3+2\delta_2)M_q^*}{4c_2\tau_m} + \frac{4\delta_1 M_q^*}{\tau_m} + \frac{\beta_0}{4} + C_1\right)\int_{I_m}\|\xi\|^2_{\Omega_t}\mathrm{d}t$$
$$+\left(\frac{\beta_0}{4}+C_2\right)\int_{I_m}R_t(\eta)\mathrm{d}t$$
$$+\frac{\beta_0}{4}\int_{I_m}R_t^*(\eta)\mathrm{d}t + \left(\frac{\beta_0 C_\mu}{4c_2}+C_3\right)\tau_m^2\int_{I_m}\left\|\frac{\partial\tilde{\eta}}{\partial t}\right\|^2_{\Omega_{t_{m-1}}}\mathrm{d}t$$
$$+\left(\frac{\beta_0}{4c_2\delta_2}+\frac{2}{\delta_1}\right)\|\xi_{m-1}^-\|^2_{\Omega_{t_{m-1}}} + \delta_0\|\{\xi\}_{m-1}\|^2_{\Omega_{t_{m-1}}}$$
$$+\left(\frac{\beta_0}{4c_2\delta_3}+\frac{1}{\delta_0}\right)\|\eta_{m-1}^-\|^2_{\Omega_{t_{m-1}}}.$$

Setting $\delta_1 = \frac{c_3 L_q^*}{24 M_q^*}$, $\delta_2 = \frac{c_2 c_3 L_q^*}{3\beta_0 M_q^*}$, $\delta_3 = \frac{2c_2 c_3 L_q^*}{3\beta_0 M_q^*}$, $c_4 := \frac{\beta_0}{4}+C_2$, $c_5 = \frac{\beta_0 C_\mu}{4c_2}+C_3$, $c_6 := \frac{\beta_0}{4c_2\delta_2}+\frac{2}{\delta_1}$, $c_7 = \frac{\beta_0}{4c_2\delta_3}$ we get

$$\left(\frac{c_3 L_q^*}{2\tau_m} - \frac{\beta_0}{4} - C_1\right)\int_{I_m}\|\xi\|^2_{\Omega_t}\mathrm{d}t + \frac{\beta_0}{4}\int_{I_m}\|\xi\|^2_{DG,t}\mathrm{d}t \tag{4.80}$$
$$\le c_4\int_{I_m}\left(R_t(\eta)+R_t^*(\eta)\right)\mathrm{d}t + c_5\tau_m^2\int_{I_m}\left\|\frac{\partial\tilde{\eta}}{\partial t}\right\|^2_{\Omega_{t_{m-1}}}\mathrm{d}t$$
$$+c_6\|\xi_{m-1}^-\|^2_{\Omega_{t_{m-1}}} + \delta_0\|\{\xi\}_{m-1}\|^2_{\Omega_{t_{m-1}}} + \left(c_7+\frac{1}{\delta_0}\right)\|\eta_{m-1}^-\|^2_{\Omega_{t_{m-1}}}.$$

If the condition $0 < \tau_m \le C_{T9}^* := \frac{c_3 L_q^*}{\beta_0+4C_1}$ is satisfied, then $\frac{\beta_0}{4}+C_1 \ge \frac{c_3 L_q^*}{4\tau_m}$ and from (4.80) we obtain the estimate

$$
\begin{aligned}
&\frac{c_3 L_q^*}{4\tau_m}\int_{I_m}\|\xi\|^2_{\Omega_t}\mathrm{d}t+\frac{\beta_0}{4}\int_{I_m}\|\xi\|^2_{DG,t}\mathrm{d}t\\
&\le c_4\int_{I_m}\left(R_t(\eta)+R_t^*(\eta)\right)\mathrm{d}t+c_5\tau_m^2\int_{I_m}\left\|\frac{\partial\tilde{\eta}}{\partial t}\right\|^2_{\Omega_{t_{m-1}}}\mathrm{d}t\\
&\quad +c_6\|\xi^-_{m-1}\|^2_{\Omega_{t_{m-1}}}+\delta_0\|\{\xi\}_{m-1}\|^2_{\Omega_{t_{m-1}}}+\left(c_7+\frac{1}{\delta_0}\right)\|\eta^-_{m-1}\|^2_{\Omega_{t_{m-1}}},
\end{aligned}
$$

which implies (4.76) with $C_{T9} := \frac{4}{c_3 L_q^*}\max\{c_4, c_5, c_6, 1\}$. □

The error analysis will be finished by the application of the discrete Gronwall inequality. If (4.76) is substituted into (4.42), an inequality is obtained, which is a basis of the proof of our main result about the abstract error estimate:

$$
\begin{aligned}
&\|\xi^-_m\|^2_{\Omega_{t_m}}-\|\xi^-_{m-1}\|_{\Omega_{t_{m-1}}}+\frac{1}{2}\|\{\xi\}_{m-1}\|^2_{\Omega_{t_{m-1}}}+\frac{\beta_0}{2}\int_{I_m}\|\xi\|^2_{DG,t}\,\mathrm{d}t \qquad (4.81)\\
&\le C_1\,C_{T9}\,\tau_m\Big(\int_{I_m}\left(R_t(\eta)+R_t^*(\eta)\right)\mathrm{d}t+\tau_m^2\int_{I_m}\left\|\frac{\partial\tilde{\eta}}{\partial t}\right\|^2_{\Omega_{t_{m-1}}}\mathrm{d}t\\
&\quad +\|\xi^-_{m-1}\|^2_{\Omega_{t_{m-1}}}+\delta_0\|\{\xi\}_{m-1}\|^2_{\Omega_{t_{m-1}}}+\left(c_7+\frac{1}{\delta_0}\right)\|\eta^-_{m-1}\|^2_{\Omega_{t_{m-1}}}\Big)\\
&\quad +C_2\int_{I_m}R_t(\eta)\,\mathrm{d}t+C_3\,\tau_m^2\int_{I_m}\left\|\frac{\partial\tilde{\eta}}{\partial t}\right\|^2_{\Omega_{t_{m-1}}}\mathrm{d}t+4\|\eta^-_{m-1}\|^2_{\Omega_{t_{m-1}}}.\\
&\le (C_2+C_1\,C_{T9})(1+\tau_m)\int_{I_m}\left(R_t(\eta)+R_t^*(\eta)\right)\mathrm{d}t\\
&\quad +(C_1\,C_{T9}\,\tau_m+C_3)\,\tau_m^2\int_{I_m}\left\|\frac{\partial\tilde{\eta}}{\partial t}\right\|^2_{\Omega_{t_{m-1}}}\mathrm{d}t\\
&\quad +C_1\,C_{T9}\,\tau_m\,\|\xi^-_{m-1}\|^2_{\Omega_{t_{m-1}}}+C_1\,C_{T9}\,\left(c_7+\frac{1}{\delta_0}\right)\tau_m\,\|\eta^-_{m-1}\|^2_{\Omega_{t_{m-1}}}\\
&\quad +C_1\,C_{T9}\,\tau_m\,\delta_0\,\|\{\xi\}_{m-1}\|^2_{\Omega_{t_{m-1}}}\\
&\le C_1\,C_{T9}\,\tau_m\,\|\xi^-_{m-1}\|^2_{\Omega_{t_{m-1}}}+C_1\,C_{T9}\,\left(c_7+\frac{1}{\delta_0}\right)\tau_m\,\|\eta^-_{m-1}\|^2_{\Omega_{t_{m-1}}}\\
&\quad +C_1\,C_{T9}\,\tau_m\,\delta_0\,\|\{\xi\}_{m-1}\|^2_{\Omega_{t_{m-1}}}+C^*\int_{I_m}K_{t,m}(\eta)\,\mathrm{d}t,
\end{aligned}
$$

where

$$C^* = \max\{C_2+C_1\,C_{T9},\; C_1\,C_{T9}\,T+C_3\} \qquad (4.82)$$

is a constant independent on τ_m and

$$K_{t,m}(\eta) = (1+\tau_m)(R_t(\eta)+R_t^*(\eta))+\tau_m^2\left\|\frac{\partial\tilde{\eta}}{\partial t}\right\|^2_{\Omega_{t_{m-1}}}$$

for $t\in I_m$.

Theorem 10. *Let $0 < \tau_m \le C^*_{T9}$ for $m = 1, \dots, M$. Then there exists a constant $C_{T10} > 0$ such that*

$$\|\xi^-_m\|^2_{\Omega_{t_m}} + \frac{1}{4}\sum_{j=1}^{m} \|\{\xi\}_{j-1}\|^2_{\Omega_{t_{j-1}}} + \frac{\beta_0}{2}\sum_{j=1}^{m}\int_{I_j} \|\xi\|^2_{DG,t}\,\mathrm{d}t \tag{4.83}$$
$$\le C_{T10}\Big(\|\xi^-_0\|^2_{\Omega_{t_0}} + \sum_{j=1}^{m}\int_{I_j} C^* K_{t,j}(\eta)\,\mathrm{d}t$$
$$+ C_1\, C_{T9}\, (c_7 + 4C_1\, C_{T9}\, T)\sum_{j=1}^{m} \tau_j\, \|\eta^-_{j-1}\|^2_{\Omega_{t_{j-1}}}\Big), \quad m = 1, \dots, M,\ h \in (0, \overline{h}),$$

where C^* was defined in (4.82) and

$$K_{t,j}(\eta) = (1 + \tau_j)(R_t(\eta) + R^*_t(\eta)) + \tau_j^2 \left\|\frac{\partial\tilde{\eta}}{\partial t}\right\|^2_{\Omega_{t_{j-1}}} \tag{4.84}$$

for $t \in I_j$, $j = 1, \dots, m$.

Proof. Writing j instead of m in (4.81), we obtain

$$\|\xi^-_j\|^2_{\Omega_{t_j}} - \|\xi^-_{j-1}\|_{\Omega_{t_{j-1}}} + \frac{1}{2}\|\{\xi\}_{j-1}\|^2_{\Omega_{t_{j-1}}} + \frac{\beta_0}{2}\int_{I_j} \|\xi\|^2_{DG,t}\,\mathrm{d}t$$
$$\le C_1\, C_{T9}\, \tau_j\, \|\xi^-_{j-1}\|^2_{\Omega_{t_{j-1}}} + C_1\, C_{T9}\, \left(c_7 + \frac{1}{\delta_0}\right)\tau_j\, \|\eta^-_{j-1}\|^2_{\Omega_{t_{j-1}}}$$
$$+ C_1\, C_{T9}\, \tau_j\, \delta_0\, \|\{\xi\}_{j-1}\|^2_{\Omega_{t_{j-1}}} + C^*\int_{I_j} K_{t,j}(\eta)\,\mathrm{d}t$$
$$\le C_1\, C_{T9}\, \tau_j\, \|\xi^-_{j-1}\|^2_{\Omega_{t_{j-1}}} + C_1\, C_{T9}\, \left(c_7 + \frac{1}{\delta_0}\right)\tau_j\, \|\eta^-_{j-1}\|^2_{\Omega_{t_{j-1}}}$$
$$+ C_1\, C_{T9}\, T\, \delta_0\, \|\{\xi\}_{j-1}\|^2_{\Omega_{t_{j-1}}} + C^*\int_{I_j} K_{t,j}(\eta)\,\mathrm{d}t.$$

Setting $\delta_0 = \frac{1}{4C_1\, C_{T9}\, T}$ we have

$$\|\xi^-_j\|^2_{\Omega_{t_j}} - \|\xi^-_{j-1}\|_{\Omega_{t_{j-1}}} + \frac{1}{4}\|\{\xi\}_{j-1}\|^2_{\Omega_{t_{j-1}}} + \frac{\beta_0}{2}\int_{I_j} \|\xi\|^2_{DG,t}\,\mathrm{d}t$$
$$\le C_1\, C_{T9}\, \tau_j\, \|\xi^-_{j-1}\|^2_{\Omega_{t_{j-1}}} + C_1\, C_{T9}\, (c_7 + 4C_1\, C_{T9}\, T)\,\tau_j\, \|\eta^-_{j-1}\|^2_{\Omega_{t_{j-1}}}$$
$$+ C^*\int_{I_j} K_{t,j}(\eta)\,\mathrm{d}t.$$

Let $m \ge 1$. The summation over all $j = 1, \dots, m$ yields the inequality

$$\|\xi^-_m\|^2_{\Omega_{t_m}} + \frac{1}{4}\sum_{j=1}^{m} \|\{\xi\}_{j-1}\|^2_{\Omega_{t_{j-1}}} + \frac{\beta_0}{2}\sum_{j=1}^{m}\int_{I_j} \|\xi\|^2_{DG,t}\,\mathrm{d}t$$
$$\le \|\xi^-_0\|^2_{\Omega_0} + C_1\, C_{T9}\sum_{j=0}^{m-1} \tau_{j+1}\|\xi^-_j\|^2_{\Omega_{t_j}} + \sum_{j=1}^{m}\int_{I_j} C^*\, K_{t,j}(\eta)\,\mathrm{d}t$$
$$+ C_1\, C_{T9}\, (c_7 + 4C_1\, C_{T9}\, T)\sum_{j=1}^{m} \tau_j\, \|\eta^-_{j-1}\|^2_{\Omega_{t_{j-1}}}.$$

The use of the discrete Gronwall inequality from Lemma 14 with the following setting

$$
\begin{aligned}
x_0 &= a_0 = \|\xi_0^-\|^2_{\Omega_{t_0}}, \quad y_0 = 0,\\
x_m &= \|\xi_m^-\|^2_{\Omega_{t_m}},\\
y_m &= \frac{1}{4}\sum_{j=1}^m \|\{\xi\}_{j-1}\|^2_{\Omega_{t_{j-1}}} + \frac{\beta_0}{2}\sum_{j=1}^m \int_{I_j} \|\xi\|^2_{DG,t}\,\mathrm{d}t,\\
a_m &= \|\xi_0^-\|^2_{\Omega_{t_0}} + \sum_{j=1}^m \int_{I_j} C^* K_{t,j}(\eta)\,\mathrm{d}t\\
&\qquad + C_1\, C_{T9}\,(c_7 + 4C_1\, C_{T9}\, T)\sum_{j=1}^m \tau_j\,\|\eta_{j-1}^-\|^2_{\Omega_{t_{j-1}}},\\
b_j &= C_1\, C_{T9}\,\tau_{j+1}, \quad j = 0, 1, \ldots, m,
\end{aligned}
$$

yield

$$
\begin{aligned}
&\|\xi_m^-\|^2_{\Omega_{t_m}} + \frac{1}{4}\sum_{j=1}^m \|\{\xi\}_{j-1}\|^2_{\Omega_{t_{j-1}}} + \frac{\beta_0}{2}\sum_{j=1}^m \int_{I_j} \|\xi\|^2_{DG,t}\,\mathrm{d}t \qquad (4.85)\\
&\le \Bigg(\|\xi_0^-\|^2_{\Omega_{t_0}} + \sum_{j=1}^m \int_{j} C^* K_{t,j}(\eta)\,\mathrm{d}t\\
&\quad + C_1\, C_{T9}\,(c_7 + 4C_1\, C_{T9}\, T)\sum_{j=1}^m \tau_j\,\|\eta_{j-1}^-\|^2_{\Omega_{t_{j-1}}}\Bigg)\prod_{j=0}^{m-1}\Big(1 + C_1\, C_{T9}\, c_4\,\tau_{j+1}\Big).
\end{aligned}
$$

Finally (4.85) and the inequality $1+\sigma < \exp(\sigma)$ valid for any $\sigma > 0$ immediately yield (4.83) with the constant $C_{T10} := \exp(C_1\, C_{T9}\, c_4\, T)$. ☐

Theorem 11. *Let $0 < \tau_m \le C^*_{T9}$ for $m = 1, \ldots, M$. Then there exists a constant $C_{AE} > 0$ such that*

$$
\begin{aligned}
&\|e_m^-\|^2_{\Omega_{t_m}} + \frac{1}{4}\sum_{j=1}^m \|\{e\}_{j-1}\|^2_{\Omega_{t_{j-1}}} + \frac{\beta_0}{2}\sum_{j=1}^m \int_{I_j} \|e\|^2_{DG,t}\,\mathrm{d}t \qquad (4.86)\\
&\le C_{AE}\Bigg(\|\xi_0^-\|^2_{\Omega_{t_0}} + \sum_{j=1}^m \int_{I_j} C^* K_{t,j}(\eta)\,\mathrm{d}t\\
&\quad + C_1\, C_{T9}\,(c_7 + 4C_1\, C_{T9}\, T)\sum_{j=1}^m \tau_j\,\|\eta_{j-1}^-\|^2_{\Omega_{t_{j-1}}}\Bigg) + 2\|\eta_m^-\|^2_{\Omega_{t_m}}\\
&\quad + \frac{1}{2}\sum_{j=1}^m \|\{\eta\}_{j-1}\|^2_{\Omega_{t_{j-1}}} + \beta_0\sum_{j=1}^m \int_{I_j} \|\eta\|^2_{DG,t}\,\mathrm{d}t, \quad m = 1, \ldots, M,\ h \in (0, \overline{h}),
\end{aligned}
$$

where C^* and $K_{t,j}(\eta)$ was defined in (4.82) *and* (4.84)*, respectively.*

Proof. From (4.83) using the relation $e = \xi + \eta$ and the standard inequalities

$$
\begin{aligned}
\|e_m^-\|^2_{\Omega_{t_m}} &\le 2\left(\|\xi_m^-\|^2_{\Omega_{t_m}} + \|\eta^-\|^2_{\Omega_{t_m}}\right),\\
\|e\|^2_{DG,t} &\le 2\left(\|\xi\|^2_{DG,t} + \|\eta\|^2_{DG,t}\right),\\
\|\{e\}_{j-1}\|^2_{\Omega_{t_{j-1}}} &\le 2\Big(\|\{\xi\}_{j-1}\|^2_{\Omega_{t_{j-1}}} + \|\{\eta\}_{j-1}\|^2_{\Omega_{t_{j-1}}}\Big)
\end{aligned}
$$

we obtain the abstract error estimate (4.86). ☐

Taking into account the definition of ξ_0^- and (2.33), we see that $\xi_0^- = 0$. Hence, (4.86) represents the estimate of the error e in terms of expressions depending on the interpolation error η. This is the basis for the derivation of the error estimate in terms of τ and h.

4.3 Interpolation and space-time projection operator

In this section a suitable $S_h^{p,m-1}$-interpolation and a space-time projection operator will be defined with respect to the error analysis of the numerical method proposed in Chapter 2. Space $S_h^{p,m-1}$ was defined in Section 2.2.2 by (2.17).

At first we introduce the $S_h^{p,m-1}$-interpolation $\Pi_{h,m-1}$ defined for all functions $w \in L^2(\Omega_{t_{m-1}})$ as

$$\Pi_{h,m-1} w \in S_h^{p,m-1}, \quad (\Pi_{h,m-1} w - w, \varphi)_{\Omega_{t_{m-1}}} = 0 \quad \forall \varphi \in S_h^{p,m-1}. \tag{4.87}$$

Hence, $\Pi_{h,m-1}$ is the $L^2(\Omega_{t_{m-1}})$-projection on the space $S_h^{p,m-1}$.

Before defining the space-time projection operation we remind some important spaces from Section 2.2.2. By $P^q(I_m; S_h^{p,m-1})$ we denote the space of mappings of the time interval I_m into the space $S_h^{p,m-1}$ which are polynomials of degree $\leq q$ in time (p, $q \geq 1$ are integers). Then the space of piecewise polynomial functions in space and time $S_{h,\tau}^{p,q}$ was defined by 2.20.

Now we define space $\hat{S}_{h,\tau}^{p,q}$ as

$$\hat{S} = \{\varphi; \varphi|_{I_m} \in C(\bar{I}_m; L^2(\Omega_{t_{m-1}}), m = 1, \ldots, M\}. \tag{4.88}$$

By [31], Section 6.1.4, we define the space-time projection operator $\tilde{\pi} : \hat{S} \longrightarrow S_{h,\tau}^{p,q}$ in the following way:

If $\tilde{w} \in \hat{S}$, then

$$\tilde{\pi}\tilde{w} \in S_{h,\tau}^{p,q}, \tag{4.89}$$

$$(\tilde{\pi}\tilde{w})(X, t_m-) = \Pi_{h,m-1}\tilde{w}(X, t_m-), \quad X \in \Omega_{t_{m-1}}, \quad m = 1, \ldots, M, \tag{4.90}$$

$$\int_{I_m} (\tilde{\pi}\tilde{w} - \tilde{w}, \varphi)_{\Omega_{t_{m-1}}} \, \mathrm{d}t = 0, \quad \forall \varphi \in S_{h,\tau}^{p,q-1}, \quad m = 1, \ldots, M, \tag{4.91}$$

where the operator $\Pi_{h,m-1}$ is defined by (4.87). We can see, that condition (4.91) means, that the interpolation error $\tilde{\pi}\tilde{w} - \tilde{w}$ is orthogonal to polynomials of degree $q-1$ on I_m. The lower degree of test functions φ is compensated by the condition (4.90).

Hence, if $v = v(x,t)$ for $x \in \Omega_t$, $t \in \bar{I}_m$ then $\tilde{v} = \tilde{v}(X,t) = v(\mathcal{A}_{h,t}^{m-1}(X), t)$ for $X \in \Omega_{t_{m-1}}$, $t \in \bar{I}_m$ and we can define the space-time projection operator π as

$$(\pi v)(x,t) = (\tilde{\pi}\tilde{v})((\mathcal{A}_{h,t}^{m-1})^{-1}(x), t), \quad x \in \Omega_t, \, t \in I_m, \, m = 1, \ldots, M, \tag{4.92}$$

provided the expressions in (4.92) make sense.

4.4 Error estimates in terms of h and τ

Error estimates in terms of h (size of the space mesh) and τ (size of the time mesh) can be derived using the abstract error estimate (4.86), where we take into account that $\xi_0^- = 0$, as it is mentioned at the end of Section 4.2.3. Moreover, in this analysis we come to the difficult open problem how to estimate the expression $\|\{\eta\}_{j-1}\|_{\Omega_{t_{j-1}}}$. Therefore, we omit the expression $\frac{1}{4}\sum_{j=1}^m \|\{e\}_{j-1}\|^2_{\Omega_{t_{j-1}}}$ in the estimate (4.86), which will be replaced by the following relation:

$$\begin{aligned}
&\|e_m^-\|^2_{\Omega_{t_m}} + \frac{\beta_0}{2}\sum_{j=1}^m \int_{I_j} \|e\|^2_{DG,t}\, \mathrm{d}t \qquad (4.93)\\
&\le C_{AE}\Big(\sum_{j=1}^m \int_{I_j} C^* K_{t,j}(\eta)\, \mathrm{d}t + C_1\, C_{T9}\, (c_7 + 4C_1\, C_{T9}\, T) \sum_{j=1}^m \tau_j\, \|\eta_{j-1}^-\|^2_{\Omega_{t_{j-1}}}\Big)\\
&\quad +2\|\eta_m^-\|^2_{\Omega_{t_m}} + \beta_0 \sum_{j=1}^m \int_{I_j} \|\eta\|^2_{DG,t}\, \mathrm{d}t, \quad m = 1,\dots,M,\ h \in (0,\overline{h}).
\end{aligned}$$

As in the previous sections, we assume, that u and U denote the exact and the approximate solutions satisfying (4.4) and (4.2) - (4.3), respectively. According to (4.5) the error can be written in the form

$$\begin{aligned}
e(x,t) &= U(x,t) - u(x,t) \qquad (4.94)\\
&= \underbrace{(U(x,t) - \pi u(x,t))}_{\xi(x,t)} + \underbrace{(\pi u(x,t) - u(x,t))}_{\eta(x,t)}.
\end{aligned}$$

Terms $u(x,t)$, $\eta(x,t)$ and $\xi(x,t)$ can be transferred to the reference domain using the ALE-mapping $\mathcal{A}_{h,t}^{m-1}$, see (4.29):

$$\tilde{u}(X,t) = u(\mathcal{A}_{h,t}^{m-1}(X),t), \quad \tilde{\eta}(X,t) = \eta(\mathcal{A}_{h,t}^{m-1}(X),t), \quad \tilde{\xi}(X,t) = \xi(\mathcal{A}_{h,t}^{m-1}(X),t),$$

$$x = \mathcal{A}_{h,t}^{m-1}(X),\ X \in \Omega_{t_{m-1}},\ x \in \Omega_t,\ t \in [t_{m-1}, t_m].$$

Using the definition of the space-time projector operators $\tilde{\pi}$ and π (see (4.89)-(4.91) and (4.92), respectively) we can write $\tilde{\eta}(X,t)$ in the following form:

$$\begin{aligned}
\tilde{\eta}(X,t) &= \eta(\mathcal{A}_{h,t}^{m-1}(X),t) = \pi u(\mathcal{A}_{h,t}^{m-1}(X),t) - u(\mathcal{A}_{h,t}^{m-1}(X),t) \qquad (4.95)\\
&= \tilde{\pi}\tilde{u}((\mathcal{A}_{h,t}^{m-1})^{-1}\mathcal{A}_{h,t}^{m-1}(X),t) - u(\mathcal{A}_{h,t}^{m-1}(X),t)\\
&= \tilde{\pi}\tilde{u}(X,t) - \tilde{u}(X,t).
\end{aligned}$$

Now we express term $\tilde{\eta} = \tilde{\eta}(X,t)$ as

$$\begin{aligned}
&\tilde{\eta}|_{I_m} = (\tilde{\pi}\tilde{u} - \tilde{u})|_{I_m} = \tilde{\eta}^{(1)} + \tilde{\eta}^{(2)}, \quad m = 1,\dots,M, \qquad (4.96)\\
&\tilde{\eta}^{(1)} = (\Pi_{h,m-1}\tilde{u} - \tilde{u})|_{I_m}, \quad \tilde{\eta}^{(2)} = (\tilde{\pi}(\Pi_{h,m-1}\tilde{u}) - \Pi_{h,m-1}\tilde{u})|_{I_m},
\end{aligned}$$

where operators $\tilde{\pi}$ and $\Pi_{h,m-1}$ are given by (4.89)-(4.91) and (4.87), respectively. In (4.96) we used the fact, that $\tilde{\pi}(\Pi_{h,m-1}\tilde{u})|_{I_m} = \tilde{\pi}\tilde{u}|_{I_m}$, $m = 1,\dots,M$, which follows from Theorem 6.9. in [31].

Assumption on the regularity of u: We consider $p, q \ge 1$ (p and q denote polynomial degree in space and time, respectively), assume that the functions u

and $\tilde{u}$ are sufficiently regular so that $\mathcal{A}_t$ has continuous time derivative of order $q+1$ and

$$
\begin{aligned}
u &\in H^{q+1}([0,T]; H^1(\Omega_t)) \cap C^1([0,T]; H^s(\Omega_t)), && (4.97)\\
\tilde{u}|_{I_m} &\in H^{q+1}(I_m; H^1(\Omega_{t_{m-1}})) \cap C^1(\bar{I}_m; H^s(\Omega_{t_{m-1}})), \quad m = 1, \dots, M, && (4.98)
\end{aligned}
$$

where $s \geq 2$ is an integer. As usual we set $\mu = \min(p+1, s)$.

4.4.1 Estimates for $\tilde{\eta}$

Now we summarize results from [31] for the term $\tilde{\eta}$.

Lemma 28. *There exists a constant $C_{L28} > 0$ such that the following estimates hold:*

$$
\begin{aligned}
\|\tilde{\eta}_0^-\|^2_{\Omega_{t_{m-1}}} &\leq C^2_{L28}\, h^{2\mu} |\tilde{u}(0)|^2_{H^\mu(\Omega_{t_{m-1}})}, && (4.99)\\
\|\tilde{\eta}_{m-1}^-\|^2_{\Omega_{t_{m-1}}} &\leq C^2_{L28}\, h^{2\mu} |\tilde{u}(t_{m-1}-)|^2_{H^\mu(\Omega_{t_{m-1}})}, && (4.100)\\
\int_{I_m} \|\tilde{\eta}^{(1)}\|^2_{L^2(\hat{K})}\, \mathrm{d}t &\leq C^2_{L28}\, h^{2\mu}_{\hat{K}} |\tilde{u}|^2_{L^2(I_m; H^\mu(\hat{K}))}, && (4.101)\\
\int_{I_m} \|\tilde{\eta}^{(1)}\|^2_{H^1(\hat{K})}\, \mathrm{d}t &\leq C^2_{L28}\, h^{2(\mu-1)}_{\hat{K}} |\tilde{u}|^2_{L^2(I_m; H^\mu(\hat{K}))}, && (4.102)\\
h^2_{\hat{K}} \int_{I_m} \|\tilde{\eta}^{(1)}\|^2_{H^2(\hat{K})}\, \mathrm{d}t &\leq C^2_{L28}\, h^{2(\mu-1)}_{\hat{K}} |\tilde{u}|^2_{L^2(I_m; H^\mu(\hat{K}))}, && (4.103)
\end{aligned}
$$

for $\hat{K} \in \hat{\mathcal{T}}_{h,m-1}$, $m = 1, \dots, M$.

Proof. These results are consequence of Lemma 6.17 from [31]. □

Lemma 29. *There exists constant $C_{L29} > 0$ such that the following estimates hold:*

$$
\begin{aligned}
\int_{I_m} \|\tilde{\eta}^{(2)}\|^2_{L^2(\hat{K})}\, \mathrm{d}t &\leq C_{L29}\, \tau_m^{2(q+1)} |\tilde{u}|^2_{H^{q+1}(I_m; L^2(\hat{K}))}, && (4.104)\\
\int_{I_m} \|\tilde{\eta}^{(2)}\|^2_{H^1(\hat{K})}\, \mathrm{d}t &\leq C_{L29}\, \tau_m^{2(q+1)} |\tilde{u}|^2_{H^{q+1}(I_m; H^1(\hat{K}))}, && (4.105)\\
h^2_{\hat{K}} \int_{I_m} \|\tilde{\eta}^{(2)}\|^2_{H^2(\hat{K})}\, \mathrm{d}t &\leq C_{L29}\, \tau_m^{2(q+1)} |\tilde{u}|^2_{H^{q+1}(I_m; H^1(\hat{K}))}, && (4.106)
\end{aligned}
$$

for $\hat{K} \in \hat{\mathcal{T}}_{h,m-1}$, $m = 1, \dots, M$.

Proof. It follows from Lemma 6.18 from [31]. □

Finally, we shall be concerned with the estimation of $\int_{I_m} J_h(\tilde{\eta}, \tilde{\eta}, t)\, \mathrm{d}t$. We have

$$
J_h(\tilde{\eta}, \tilde{\eta}, t) \leq 2(J_h(\tilde{\eta}^{(1)}, \tilde{\eta}^{(1)}, t) + J_h(\tilde{\eta}^{(2)}, \tilde{\eta}^{(2)}, t)). \tag{4.107}
$$

From identity (6.115) of the monograph [31] we have

$$
\int_{I_m} J_h(\tilde{\eta}^{(1)}, \tilde{\eta}^{(1)}, t)\, \mathrm{d}t \leq C^2 h^{2(\mu-1)} |\tilde{u}|^2_{L^2(I_m; H^\mu(\Omega_{t_{m-1}}))}. \tag{4.108}
$$

Lemma 30. *Let the Dirichlet data $u_D = u_D(x,t)$ have the behaviour in t as a polynomial of degree $\le q$:*

$$u_D(x,t) = \sum_{j=0}^{q} \psi_j(x) t^j, \tag{4.109}$$

where $\psi_j \in H^{s-1/2}(\partial\Omega)$ for $j = 0,\dots,q$. Then there exists $C^*_{L30} > 0$ such that

$$\int_{I_m} J_h(\tilde{\eta}^{(2)}, \tilde{\eta}^{(2)}, t)\ \mathrm{d}t \le (C^*_{L30})^2 \tau_m^{2(q+1)} |\tilde{u}|^2_{H^{q+1}(I_m;H^\mu(\Omega_{t_{m-1}}))}, \quad m = 1,\dots,M. \tag{4.110}$$

For general data u_D, if there exists a constant $\overline{C} > 0$ such that

$$\tau_m \le \overline{C} h_{\hat{K}_\Gamma^{(L)}}$$

for all $\hat{\Gamma} \in \hat{\mathcal{F}}^B_{h,t_{m-1}}$, $m = 1,\dots,M$, $h \in (0,\overline{h})$, then there exists constant $C^{**}_{L30} > 0$ such that

$$\int_{I_m} J_h(\tilde{\eta}^{(2)}, \tilde{\eta}^{(2)}, t)\ \mathrm{d}t \le (C^{**}_{L30})^2\, \tau_m^{2q} (|\tilde{u}|^2_{H^{q+1}(I_m;L^2(\Omega_{t_{m-1}}))} + |\tilde{u}|^2_{H^{q+1}(I_m;H^1(\Omega_{t_{m-1}}))}), m = 1,\dots,M. \tag{4.111}$$

Proof. See Lemma 6.19 from [31]. □

From Lemma 30 we immediately have the following conclusion:

If u_D is defined by (4.109), we put $\gamma = 1$. Otherwise, if u_D has a general behaviour, we set $\gamma = 0$. Then

$$\int_{I_m} J_h(\tilde{\eta}^{(2)}, \tilde{\eta}^{(2)}, t)\ \mathrm{d}t \le C^2_{L30}\, \tau_m^{2(q+\gamma)} (|\tilde{u}|^2_{H^{q+1}(I_m;L^2(\Omega_{t_{m-1}}))} + |\tilde{u}|^2_{H^{q+1}(I_m;H^1(\Omega_{t_{m-1}}))}), \quad m = 1,\dots,M. \tag{4.112}$$

Lemma 31. *Let the exact solution u satisfy the regularity condition* (4.98). *Then there exists a constant $C_{L31} > 0$ independent of h, t_m, m, M and u such that*

$$\int_{I_m} \|\tilde{\eta}\|^2_{DG,t}\ \mathrm{d}t \le C_{L31}\Big(h^{2(\mu-1)} |\tilde{u}|^2_{L^2(I_m;H^\mu(\Omega_{t_{m-1}}))} + \tau_m^{2(q+\gamma)} (|\tilde{u}|^2_{H^{q+1}(I_m;L^2(\Omega_{t_{m-1}}))} + |\tilde{u}|^2_{H^{q+1}(I_m;H^1(\Omega_{t_{m-1}}))})\Big) \quad m = 1,\dots,M. \tag{4.113}$$

The choice of the parameter $\gamma = 0$ or $\gamma = 1$ is specified before identity (4.112).

Proof. See Lemma 6.22 from [31]. □

4.4.2 Estimates for η

Now we need to prove some auxiliary lemmas.

Lemma 32. *The following estimates hold:*

$$\|\eta(t)\|^2_{\Omega_t} \leq C_J^+\|\tilde{\eta}(t)\|^2_{\Omega_{t_{m-1}}}, \tag{4.114}$$

$$|\eta(t)|^2_{H^1(K)} \leq C_J^+(C_A^-)^2|\tilde{\eta}(t)|^2_{H^1(\hat{K})}, \tag{4.115}$$

$$|\eta(t)|^2_{H^2(K)} \leq C_J^+(C_A^-)^2|\tilde{\eta}(t)|^2_{H^2(\hat{K})}, \tag{4.116}$$

where $K = \mathcal{A}^{m-1}_{h,t}(\hat{K})$, $\hat{K} \in \Omega_{t_{m-1}}$, $t \in I_m$, $m = 1, \dots, M$.

Proof. Inequality (4.114) can be proved using the definition of the ALE-mapping (2.14), the substitution theorem and (3.6):

$$\begin{aligned}
\|\eta(t)\|^2_{\Omega_t} &= \int_{\Omega_t} |\eta(x,t)|^2 \mathrm{d}x = \int_{\Omega_{t_{m-1}}} |\eta(\mathcal{A}^{m-1}_{h,t}(X),t)|^2 J(X,t)\,\mathrm{d}X \\
&\leq C_J^+ \int_{\Omega_{t_{m-1}}} |\tilde{\eta}(X,t)|^2\,\mathrm{d}X = C_J^+\|\tilde{\eta}(t)\|^2_{\Omega_{t_{m-1}}}.
\end{aligned}$$

Using (4.29) and (3.61) we can prove inequality (4.115) in the following way:

$$\begin{aligned}
|\eta(t)|^2_{H^1(K)} &= \int_K |\nabla(\eta(x,t))|^2\,\mathrm{d}x = \int_K |\nabla(\tilde{\eta}((\mathcal{A}^{m-1}_{h,t})^{-1}(x),t))|^2\,\mathrm{d}x \\
&\leq \int_{\hat{K}} \|B_t^{-1}|_K\|^2 |\nabla\tilde{\eta}(X,t)|^2 J(X,t)\,\mathrm{d}X \\
&\leq (C_A^-)^2 C_J^+ \int_{\hat{K}} |\nabla\tilde{\eta}(X,t)|^2\,\mathrm{d}X = (C_A^-)^2 C_J^+\,|\tilde{\eta}(t)|^2_{H^1(\hat{K})}.
\end{aligned}$$

Similarly, because $\mathcal{A}^{m-1}_{h,t}$ is affine, which means that second derivatives with respect to X are zero, we get (4.116). □

Lemma 33. *The following estimates hold:*

$$\|\eta_m^-\|^2_{\Omega_t} \leq C_J^+\|\tilde{\eta}_{m-1}\|^2_{\Omega_{t_{m-1}}}, \quad m = 1, \dots, M, \tag{4.117}$$

$$\|\eta_{m-1}^-\|^2_{\Omega_{t_{m-1}}} \leq C_J^+\|\tilde{\eta}_{m-2}\|^2_{\Omega_{t_{m-2}}}, \quad m = 2, \dots, M. \tag{4.118}$$

For $m = 1$ *we set* $\eta^-_{m-1} = \eta^-_0 := \Pi_{h,0}u^0 - u^0$. *Similarly we set* $\tilde{\eta}_{-1} := \Pi_{h,0}u^0 - u^0$.

Proof. Inequality (4.117) can be proved using the definition of the ALE-mapping (2.14), the substitution theorem and (3.6):

$$\begin{aligned}
\|\eta_m^-\|^2_{\Omega_{t_m}} &= \int_{\Omega_{t_m}} |\eta(x,t_m-)|^2 \mathrm{d}x = \int_{\Omega_{t_{m-1}}} |\eta(\mathcal{A}^{m-1}_{h,t_m-}(X),t_{m-1})|^2 J(X,t_{m-1})\,\mathrm{d}X \\
&\leq C_J^+ \int_{\Omega_{t_{m-1}}} |\tilde{\eta}(X,t_{m-1})|^2\,\mathrm{d}X = C_J^+\|\tilde{\eta}_{m-1}\|^2_{\Omega_{t_{m-1}}}.
\end{aligned}$$

Inequality (4.118) can be proved similarly:

$$\begin{aligned}
\|\eta_{m-1}^-\|^2_{\Omega_{t_{m-1}}} &= \int_{\Omega_{t_{m-1}}} |\eta(x,t_{m-1}-)|^2 \mathrm{d}x \\
&= \int_{\Omega_{t_{m-2}}} |\eta(\mathcal{A}^{m-2}_{h,t_{m-1}-}(X),t_{m-2})|^2 J(X,t_{m-2})\,\mathrm{d}X \\
&\leq C_J^+ \int_{\Omega_{t_{m-2}}} |\tilde{\eta}(X,t_{m-2})|^2\,\mathrm{d}X = C_J^+\|\tilde{\eta}_{m-2}\|^2_{\Omega_{t_{m-2}}}.
\end{aligned}$$

□

Lemma 34. *The following estimate hold:*

$$J_h(\eta,\eta,t) \le C^{**}_{L7} C^{+}_{A}\, J_h(\tilde\eta,\tilde\eta,t). \tag{4.119}$$

Proof. From the definition of the form J_h (2.25) we have

$$J_h(\eta,\eta,t) = c_W \sum_{\Gamma\in\mathcal{F}^I_{h,t}} h(\Gamma)^{-1}\int_\Gamma [\eta]^2\,\mathrm{d}S + c_W \sum_{\Gamma\in\mathcal{F}^B_{h,t}} h(\Gamma)^{-1}\int_\Gamma \eta^2\,\mathrm{d}S.$$

Analogously as in the proof of Lemma 7 and (3.65) we prove inequalities

$$h(\Gamma)^{-1}\int_\Gamma [\eta]^2 \mathrm{d}S^\Gamma \le C^{**}_{L7} C^{+}_{A}\, h(\hat\Gamma)^{-1}\int_{\hat\Gamma} [\tilde\eta]^2 \mathrm{d}S^{\hat\Gamma},$$

for $\Gamma\in\mathcal{F}^I_{h,t}$, $\hat\Gamma\in\mathcal{F}^I_{h,t_{m-1}}$, $\hat\Gamma = (\mathcal{A}^{m-1}_{h,t})^{-1}(\Gamma)$, and

$$h(\Gamma)^{-1}\int_\Gamma \eta^2 \mathrm{d}S^\Gamma \le C^{**}_{L7} C^{+}_{A}\, h(\hat\Gamma)^{-1}\int_{\hat\Gamma} \tilde\eta^2 \mathrm{d}S^{\hat\Gamma},$$

where $\Gamma\in\mathcal{F}^B_{h,t}$, $\hat\Gamma\in\mathcal{F}^B_{h,t_{m-1}}$, $\hat\Gamma = (\mathcal{A}^{m-1}_{h,t})^{-1}(\Gamma)$.

Thus, we have

$$\begin{aligned} J_h(\eta,\eta,t) &\le c_W C^{**}_{L7} C^{+}_{A} \sum_{\Gamma\in\mathcal{F}^I_{h,t}} h(\hat\Gamma)^{-1}\int_{\hat\Gamma} [\tilde\eta]^2 \mathrm{d}S^{\hat\Gamma} \\ &\quad + c_W C^{**}_{L7} C^{+}_{A} \sum_{\Gamma\in\mathcal{F}^B_{h,t}} h(\hat\Gamma)^{-1}\int_{\hat\Gamma} \tilde\eta^2 \mathrm{d}S^{\hat\Gamma} \\ &\le C^{**}_{L7} C^{+}_{A}\, J_h(\tilde\eta,\tilde\eta,t). \end{aligned}$$

□

Lemma 35. *There exists a constant $C_{DG} > 0$ such that*

$$\|\eta\|^2_{DG,t} \le C_{DG}\,\|\tilde\eta\|^2_{DG,t}. \tag{4.120}$$

Proof. From (3.1), (4.115) and (4.119) we find that

$$\begin{aligned} \|\eta\|^2_{DG,t} &= \sum_{K\in\mathcal{T}_{h,t}} |\eta|^2_{H^1(K)} + J_h(\eta,\eta,t) \\ &\le C^{+}_{J}(C^{-}_{A})^2 \sum_{\hat K\in\hat{\mathcal{T}}_{h,t_{m-1}}} |\tilde\eta(t)|^2_{H^1(\hat K)} + C^{**}_{L7} C^{+}_{A}\, J_h(\tilde\eta,\tilde\eta,t) \\ &\le C_{DG}\,\|\tilde\eta\|^2_{DG,t}, \end{aligned}$$

where $C_{DG} = \max\{C^{+}_{J}(C^{-}_{A})^2, C^{**}_{L7} C^{+}_{A}\}$. □

Now we are ready to estimate all terms on the right-hand side of (4.93) in terms of h and τ.

Lemma 36. *There exists a constant $C_{L36} > 0$ such that*

$$\begin{aligned} &\sum_{j=1}^m \int_{I_j} C^* K_{t,j}(\eta)\,\mathrm{d}t \qquad (4.121)\\ &\le C_{L36}\sum_{j=1}^m \Bigg(h^{2(\mu-1)}|\tilde u|^2_{L^2(I_j;H^\mu(\Omega_{t_{j-1}}))} + h^{2(\mu-1)}\,\tau_j^2 \left|\frac{\partial\tilde u}{\partial t}\right|^2_{L^2(I_j;H^\mu(\Omega_{t_{j-1}}))} \\ &\quad + \tau_j^{2(q+\gamma)}\big(|\tilde u|^2_{H^{q+1}(I_j;L^2(\Omega_{t_{j-1}}))} + |\tilde u|^2_{H^{q+1}(I_j;H^1(\Omega_{t_{j-1}}))}\big) \\ &\quad + \tau_j^{2(q+1)} \left|\frac{\partial\tilde u}{\partial t}\right|^2_{H^{q}(I_j;H^1(\Omega_{t_{j-1}}))} \Bigg), \end{aligned}$$

where choice of the parameter $\gamma = 0$ *or* $\gamma = 1$ *is specified before identity* (4.112).

Proof. From (4.84) we have

$$K_{t,j}(\eta) = (1+\tau_j)(R_t(\eta) + R_t^*(\eta)) + \tau_j^2 \left\| \frac{\partial \tilde{\eta}}{\partial t} \right\|^2_{\Omega_{t_{j-1}}}, \quad t \in I_j,\ j = 1,\dots,M,$$

where $R_t(\eta)$ and $R_t^*(\eta)$ was defined in (4.9) and (4.10), respectively. Using estimates (4.120), (4.114), (4.115) and (4.116) we find that

$$\begin{aligned} R_t(\eta) &= \|\eta\|^2_{DG,t} + \|\eta\|^2_{\Omega_t} + \sum_{K\in\mathcal{T}_{h,t}} \left(|\eta|^2_{H^1(K)} + h_K^2 |\eta|^2_{H^2(K)} \right) \\ &\le C_{DG} \|\tilde{\eta}\|^2_{DG,t} + C_J^+ \|\tilde{\eta}\|^2_{\Omega_{t_{j-1}}} \\ &\quad + C_J^+ (C_A^-)^2 \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{j-1}}} \left(|\tilde{\eta}|^2_{H^1(\hat{K})} + h_{\hat{K}}^2 |\tilde{\eta}|^2_{H^2(\hat{K})} \right) \end{aligned}$$

and

$$\begin{aligned} R_t^*(\eta) &= \|\eta\|^2_{DG,t} + \sum_{K\in\mathcal{T}_{h,t}} \left(h_K^2 |\eta|^2_{H^2(K)} \right) \\ &\le C_{DG} \|\tilde{\eta}\|^2_{DG,t} + C_J^+ (C_A^-)^2 \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{j-1}}} h_{\hat{K}}^2 |\tilde{\eta}|^2_{H^2(\hat{K})}, \end{aligned}$$

which in total gives

$$\begin{aligned} \sum_{j=1}^m \int_{I_j} C^* K_{t,j}(\eta)\,\mathrm{d}t \ \le\ & C^* \sum_{j=1}^m \int_{I_j} (1+\tau_j) \Big(2C_{DG} \|\tilde{\eta}\|^2_{DG,t} + C_J^+ \|\tilde{\eta}\|^2_{\Omega_{t_{j-1}}} \\ & + C_J^+ (C_A^-)^2 \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{j-1}}} (|\tilde{\eta}|^2_{H^1(\hat{K})} + h_{\hat{K}}^2 |\tilde{\eta}|^2_{H^2(\hat{K})}) \Big)\,\mathrm{d}t \\ & + C^* \sum_{j=1}^m \int_{I_j} \tau_j^2 \left\| \frac{\partial \tilde{\eta}}{\partial t} \right\|^2_{\Omega_{t_{j-1}}} \mathrm{d}t. \end{aligned}$$

Concerning term $\left\| \frac{\partial \tilde{\eta}}{\partial t} \right\|^2_{\Omega_{t_{j-1}}}$, from (4.98) we have

$$\frac{\partial \tilde{\eta}}{\partial t} \in H^q(I_m; H^1(\Omega_{t_{m-1}})) \cap C(\bar{I}_m; H^s(\Omega_{t_{m-1}})),$$

and from (4.102) and (4.105) we obtain

$$\begin{aligned} \tau_m^2 \int_{I_m} \left\| \frac{\partial \tilde{\eta}^{(1)}}{\partial t} \right\|^2_{L^2(\hat{K})} \mathrm{d}t \ &\le\ C_{L28}^2\, h_{\hat{K}}^{2(\mu-1)}\, \tau_m^2 \left| \frac{\partial \tilde{u}}{\partial t} \right|^2_{L^2(I_m; H^\mu(\hat{K}))}, \\ \tau_m^2 \int_{I_m} \left\| \frac{\partial \tilde{\eta}^{(2)}}{\partial t} \right\|^2_{L^2(\hat{K})} \mathrm{d}t \ &\le\ C_{L28}\, \tau_m^{2(q+1)} \left| \frac{\partial \tilde{u}}{\partial t} \right|^2_{H^q(I_m; H^1(\hat{K}))}, \end{aligned}$$

for $\hat{K} \in \hat{\mathcal{T}}_{h,m-1}$, $m = 1,\dots,M$.

Now we can apply results from (4.113), (4.101), (4.104), (4.102), (4.105), (4.103), (4.106) and obtain the following estimate in terms of h and τ:

$$
\begin{aligned}
&\sum_{j=1}^{m} \int_{I_j} C^* K_{t,j}(\eta)\,\mathrm{d}t \\
&\le 2C^* C_{DG}\, C_{L31} \sum_{j=1}^{m} (1+\tau_j)\Big(h^{2(\mu-1)} |\tilde{u}|^2_{L^2(I_j;H^\mu(\Omega_{t_{j-1}}))} \\
&\qquad\qquad + \tau_j^{2(q+\gamma)} (|\tilde{u}|^2_{H^{q+1}(I_j;L^2(\Omega_{t_{j-1}}))} + |\tilde{u}|^2_{H^{q+1}(I_j;H^1(\Omega_{t_{j-1}}))})\Big) \\
&\quad + C^* C_J^+ \sum_{j=1}^{m} (1+\tau_j) \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{j-1}}} \Big(C_{L28}^2\, h_{\hat{K}}^{2\mu} |\tilde{u}|^2_{L^2(I_j;H^\mu(\hat{K}))} \\
&\qquad\qquad + C_{L29}\, \tau_j^{2(q+1)} |\tilde{u}|^2_{H^{q+1}(I_j;L^2(\hat{K}))}\Big) \\
&\quad + C^* C_J^+ (C_A^-)^2 \sum_{j=1}^{m} (1+\tau_j) \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{j-1}}} \Big(C_{L28}^2\, h_{\hat{K}}^{2(\mu-1)} |\tilde{u}|^2_{L^2(I_j;H^\mu(\hat{K}))} \\
&\qquad\qquad + C_{L29}\, \tau_j^{2(q+1)} |\tilde{u}|^2_{H^{q+1}(I_j;H^1(\hat{K}))}\Big) \\
&\quad + 2C^* C_J^+ (C_A^-)^2 \sum_{j=1}^{m} (1+\tau_j) \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{j-1}}} \Big(C_{L28}^2\, h_{\hat{K}}^{2(\mu-1)} |\tilde{u}|^2_{L^2(I_j;H^\mu(\hat{K}))} \\
&\qquad\qquad + C_{L29}\, \tau_j^{2(q+1)} |\tilde{u}|^2_{H^{q+1}(I_j;H^1(\hat{K}))}\Big) \\
&\quad + C^* \sum_{j=1}^{m} \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{j-1}}} \Big(C_{L28}^2\, h_{\hat{K}}^{2(\mu-1)}\, \tau_j^2 \left|\frac{\partial\tilde{u}}{\partial t}\right|^2_{L^2(I_j;H^\mu(\hat{K}))} \\
&\qquad\qquad + C_{L28}\, \tau_j^{2(q+1)} \left|\frac{\partial\tilde{u}}{\partial t}\right|^2_{H^q(I_j;H^1(\hat{K}))}\Big) \\
&\le C_{L36} \sum_{j=1}^{m} \Big(h^{2(\mu-1)} |\tilde{u}|^2_{L^2(I_j;H^\mu(\Omega_{t_{j-1}}))} + h^{2(\mu-1)}\, \tau_j^2 \left|\frac{\partial\tilde{u}}{\partial t}\right|^2_{L^2(I_j;H^\mu(\Omega_{t_{j-1}}))} \\
&\quad + \tau_j^{2(q+\gamma)} (|\tilde{u}|^2_{H^{q+1}(I_j;L^2(\Omega_{t_{j-1}}))} + |\tilde{u}|^2_{H^{q+1}(I_j;H^1(\Omega_{t_{j-1}}))}) \\
&\quad + \tau_j^{2(q+1)} \left|\frac{\partial\tilde{u}}{\partial t}\right|^2_{H^q(I_j;H^1(\Omega_{t_{j-1}}))} \Big),
\end{aligned}
$$

where we used that $h_K \le h$ for all $\hat{K} \in \hat{\mathcal{T}}_{h,t_{j-1}}$, $j = 1, \ldots, M$ and

$$
\begin{aligned}
C_{L36} = \max\ \{ & 2C^* C_{DG}\, C_{L31}\, (1+T), \\
& C^* C_J^+\, (1+T) \max\{C_{L28}^2,\, C_{L29}\}, \\
& 2C^* C_J^+\, (C_A^-)^2\, (1+T) \max\{C_{L28}^2,\, C_{L29}\}, \\
& C^* \max\{C_{L28}^2,\, C_{L29}\}\},
\end{aligned}
$$

for $\gamma = 1$ and

$$
\begin{aligned}
C_{L36} = \quad \max \quad & \{2C^* \, C_{DG} \, C_{L31} \, (1+T), \\
& C^* \, C_J^+ \, (1+T) \, \max\{C_{L28}^2, \, T \, C_{L29}\}, \\
& 2C^* \, C_J^+ \, (C_A^-)^2 \, (1+T) \max\{C_{L28}^2, \, T \, C_{L29}\}, \\
& C^* \max\{C_{L28}^2, \, C_{L29}\}\},
\end{aligned}
$$

for $\gamma = 0$. □

Lemma 37. *The following estimates hold:*

$$
\sum_{j=2}^{m} \tau_j \|\eta_{j-1}\|^2_{\Omega_{t_{j-1}}} \leq C_J^+ \, C_{L28}^2 \, h^{2\mu} \sum_{j=2}^{m} \tau_j |\tilde{u}(t_{j-2}-)|^2_{H^\mu(\Omega_{t_{j-2}})}, \tag{4.122}
$$

$$
\|\eta_m^-\|^2_{\Omega_{t_m}} \leq C_J^+ \, C_{L28}^2 \, h^{2\mu} |\tilde{u}(t_{m-1}-)|^2_{H^\mu(\Omega_{t_{m-1}})}. \tag{4.123}
$$

Proof. From (4.118) and (4.100) we have

$$
\begin{aligned}
\sum_{j=2}^{m} \tau_j \|\eta_{j-1}\|^2_{\Omega_{t_{j-1}}} &\leq C_J^+ \sum_{j=2}^{m} \tau_j \|\tilde{\eta}_{j-2}\|^2_{\Omega_{t_{j-2}}} \\
&\leq C_J^+ \, C_{L28}^2 \, h^{2\mu} \sum_{j=2}^{m} \tau_j |\tilde{u}(t_{j-2}-)|^2_{H^\mu(\Omega_{t_{j-2}})}.
\end{aligned}
$$

Moreover from (4.117) and (4.100) we get

$$
\begin{aligned}
\|\eta_m^-\|^2_{\Omega_{t_m}} &\leq C_J^+ \|\tilde{\eta}_{m-1}\|^2_{\Omega_{t_{m-1}}} \\
&\leq C_J^+ \, C_{L28}^2 \, h^{2\mu} |\tilde{u}(t_{m-1}-)|^2_{H^\mu(\Omega_{t_{m-1}})}.
\end{aligned}
$$

□

Lemma 38. *The following estimate holds:*

$$
\begin{aligned}
\sum_{j=1}^{m} \int_{I_j} \|\eta\|^2_{DG,t} \, \mathrm{d}t \leq \; & C_{DG} \, C_{L31} \sum_{j=1}^{m} \Big(h^{2(\mu-1)} |\tilde{u}|^2_{L^2(I_j; H^\mu(\Omega_{t_{j-1}}))} \\
& + \tau_j^{2(q+\gamma)} \big(|\tilde{u}|^2_{H^{q+1}(I_j; L^2(\Omega_{t_{j-1}}))} + |\tilde{u}|^2_{H^{q+1}(I_j; H^1(\Omega_{t_{j-1}}))} \big) \Big).
\end{aligned} \tag{4.124}
$$

The choice of the parameter $\gamma = 0$ or $\gamma = 1$ is specified before identity (4.112).

Proof. From (4.120) and (4.113) we get

$$
\begin{aligned}
\sum_{j=1}^{m} \int_{I_j} \|\eta\|^2_{DG,t} \, \mathrm{d}t &\leq C_{DG} \sum_{j=1}^{m} \int_{I_j} \|\tilde{\eta}\|^2_{DG,t} \, \mathrm{d}t \\
&\leq C_{DG} \, C_{L31} \sum_{j=1}^{m} \Big(h^{2(\mu-1)} |\tilde{u}|^2_{L^2(I_j; H^\mu(\Omega_{t_{j-1}}))} \\
&\qquad + \tau_j^{2(q+\gamma)} \big(|\tilde{u}|^2_{H^{q+1}(I_j; L^2(\Omega_{t_{j-1}}))} + |\tilde{u}|^2_{H^{q+1}(I_j; H^1(\Omega_{t_{j-1}}))} \big) \Big).
\end{aligned}
$$

□

4.4.3 Main result

Now we can formulate our main theorem about the error estimate in terms of h and τ:

Theorem 12. *There exists a constant $C_{T12} > 0$ such that*

$$
\begin{aligned}
&\|e_m^-\|^2_{\Omega_{t_m}} + \frac{\beta_0}{2}\sum_{j=1}^m \int_{I_j} \|e\|^2_{DG,t}\,\mathrm{d}t \qquad (4.125)\\
&\leq C_{T12}\Bigg(\sum_{j=1}^m \Bigg(h^{2(\mu-1)}|\tilde{u}|^2_{L^2(I_j;H^\mu(\Omega_{t_{j-1}}))} + h^{2(\mu-1)}\,\tau^2\left|\frac{\partial\tilde{u}}{\partial t}\right|^2_{L^2(I_j;H^\mu(\Omega_{t_{j-1}}))}\\
&\quad +\tau^{2(q+\gamma)}(|\tilde{u}|^2_{H^{q+1}(I_j;L^2(\Omega_{t_{j-1}}))} + |\tilde{u}|^2_{H^{q+1}(I_j;H^1(\Omega_{t_{j-1}}))}) + \tau^{2(q+1)}\left|\frac{\partial\tilde{u}}{\partial t}\right|^2_{H^q(I_j;H^1(\Omega_{t_{j-1}}))}\Bigg)\\
&\quad +h^{2\mu}\sum_{j=2}^m \tau|\tilde{u}(t_{j-2}-)|^2_{H^\mu(\Omega_{t_{j-2}})} + h^{2\mu}|\tilde{u}(t_{m-1}-)|^2_{H^\mu(\Omega_{t_{m-1}})}\Bigg).
\end{aligned}
$$

The choice of the parameter $\gamma = 0$ or $\gamma = 1$ is specified before identity (4.112)*.*

Proof. From (4.93), (4.121), (4.123), (4.122) and (4.124) we get

$$
\begin{aligned}
&\|e_m^-\|^2_{\Omega_{t_m}} + \frac{\beta_0}{2}\sum_{j=1}^m \int_{I_j} \|e\|^2_{DG,t}\,\mathrm{d}t\\
&\leq C_{AE}C_{L36}\sum_{j=1}^m \Bigg(h^{2(\mu-1)}|\tilde{u}|^2_{L^2(I_j;H^\mu(\Omega_{t_{j-1}}))} + h^{2(\mu-1)}\,\tau_j^2\left|\frac{\partial\tilde{u}}{\partial t}\right|^2_{L^2(I_j;H^\mu(\Omega_{t_{j-1}}))}\\
&\qquad +\tau_j^{2(q+\gamma)}(|\tilde{u}|^2_{H^{q+1}(I_j;L^2(\Omega_{t_{j-1}}))} + |\tilde{u}|^2_{H^{q+1}(I_j;H^1(\Omega_{t_{j-1}}))})\\
&\qquad +\tau_j^{2(q+1)}\left|\frac{\partial\tilde{u}}{\partial t}\right|^2_{H^q(I_j;H^1(\Omega_{t_{j-1}}))}\Bigg)\\
&\quad +C_{AE}C_1\,C_{T9}\,(c_7 + 4C_1\,C_{T9}\,T)\,C_J^+\,C_{L28}^2\,h^{2\mu}\sum_{j=2}^m \tau_j|\tilde{u}(t_{j-2}-)|^2_{H^\mu(\Omega_{t_{j-2}})}\\
&\quad +2C_J^+\,C_{L28}^2\,h^{2\mu}|\tilde{u}(t_{m-1}-)|^2_{H^\mu(\Omega_{t_{m-1}})}\\
&\quad +\beta_0 C_{DG}\,C_{L31}\sum_{j=1}^m \Bigg(h^{2(\mu-1)}|\tilde{u}|^2_{L^2(I_j;H^\mu(\Omega_{t_{j-1}}))}\\
&\qquad +\tau_j^{2(q+\gamma)}(|\tilde{u}|^2_{H^{q+1}(I_j;L^2(\Omega_{t_{j-1}}))} + |\tilde{u}|^2_{H^{q+1}(I_j;H^1(\Omega_{t_{j-1}}))})\Bigg).
\end{aligned}
$$

Setting $C_{T12} = \max\{C_{AE}C_{L36},\, C_{AE}C_1\,C_{T9}\,(c_7 + 4C_1\,C_{T9}\,T)\,C_J^+\,C_{L28}^2,\, 2C_J^+\,C_{L28}^2,\, \beta_0 C_{DG}\,C_{L31}\}$ and using $\tau = \max_{j=1,\dots,M}\tau_j$ we obtain (4.125). ☐

5. Applications of STDGM

Let us now focus our attention on applications of STDGM to the solution of the compressible Navier-Stokes equations (written in the conservative ALE form) in a time-dependent domain coupled with linear or nonlinear elasticity. The developed method is applied to the numerical simulation of air flow in a simplified model of human vocal tract and flow induced vocal folds vibrations.

5.1 Formulation of the continuous problem

In Chapter 1 we introduced the formulation of a nonstationary viscous compressible flow in a time-dependent domain Ω_t. Now we reformulate this problem with the aid of the ALE method.

5.1.1 Compressible Navier-Stokes equations in the ALE form

The system describing compressible flow, consisting of the continuity equation, the Navier-Stokes equations and the energy equation, can be written in the form

$$\frac{\partial \boldsymbol{w}}{\partial t} + \sum_{s=1}^{2} \frac{\partial \boldsymbol{f}_s(\boldsymbol{w})}{\partial x_s} = \sum_{s=1}^{2} \frac{\partial \boldsymbol{R}_s(\boldsymbol{w}, \nabla \boldsymbol{w})}{\partial x_s}, \tag{5.1}$$

where $\boldsymbol{w}$ is the state vector, $\boldsymbol{f}_s(\boldsymbol{w})$ represents the inviscid fluxes and $\boldsymbol{R}_s(\boldsymbol{w}, \nabla \boldsymbol{w})$ denotes the viscous terms, see (1.7).

Then, introducing a regular one-to-one ALE mapping (2.7) we can define the domain velocity $\boldsymbol{z}$ in (2.8) and the ALE derivative of a function $w_i = w_i(x,t)$, $x \in \Omega_t$, $t \in [0,T]$, $i = 1, \dots, 4$ as

$$\frac{D}{Dt} w_i(x,t) = \frac{\partial \widetilde{w_i}}{\partial t}(X,t), \tag{5.2}$$

where $\widetilde{w_i}(X,t) = w_i(\mathcal{A}_t(X),t)$, $X \in \Omega_0$, and $x = \mathcal{A}_t(X) \in \Omega_t$.

Using the chain rule we can rewrite this formula to the following form

$$\frac{Dw_i}{Dt} = \frac{\partial w_i}{\partial t} + \boldsymbol{z} \cdot \nabla w_i = \frac{\partial w_i}{\partial t} + \operatorname{div}(\boldsymbol{z} w_i) - w_i \operatorname{div} \boldsymbol{z}, \quad i = 1, \dots, 4. \tag{5.3}$$

Finally, introducing $\boldsymbol{g}_s(\boldsymbol{w}) = \boldsymbol{f}_s(\boldsymbol{w}) - \boldsymbol{z}\boldsymbol{w}$, $s = 1,2$ and using (5.3) we can write system (5.1) in the ALE form

$$\frac{D\boldsymbol{w}}{Dt} + \sum_{s=1}^{2} \frac{\partial \boldsymbol{g}_s(\boldsymbol{w})}{\partial x_s} + \boldsymbol{w} \operatorname{div} \boldsymbol{z} = \sum_{s=1}^{2} \frac{\partial \boldsymbol{R}_s(\boldsymbol{w}, \nabla \boldsymbol{w})}{\partial x_s}. \tag{5.4}$$

5.1.2 Dynamic elasticity system

We assume that the elastic body is represented by a bounded polygonal domain $\Omega^b \subset \mathbb{R}^2$ with boundary $\partial\Omega^b = \Gamma_D^b \cup \Gamma_N^b$, where $\Gamma_D^b \cap \Gamma_N^b = \emptyset$. On Γ_D^b and Γ_N^b we prescribe the Dirichlet and the Neumann boundary condition, respectively. The deformation of the body is described by the displacement $\boldsymbol{u} : \overline{\Omega^b} \times [0,T] \to \mathbb{R}^2$ and the deformation mapping

$$\boldsymbol{\psi}(\boldsymbol{X},t) = \boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X},t),\ \boldsymbol{X} \in \overline{\Omega^b},\ t \in [0,T]. \tag{5.5}$$

Further, we introduce the deformation gradient, the Jacobian and the cofactor $\mathrm{Cof}\boldsymbol{F}$ of the matrix $\boldsymbol{F}$:

$$\boldsymbol{F} = \nabla\boldsymbol{\psi}, \quad J = \det\boldsymbol{F} > 0, \quad \mathrm{Cof}\boldsymbol{F} = J(\boldsymbol{F}^{-T}). \tag{5.6}$$

Here $\boldsymbol{F}^{-T} = (\boldsymbol{F}^{-1})^T$. If we set $\boldsymbol{F} = (F_{ij})_{i,j=1}^2$, then $F_{ij} = \frac{\partial\psi_i}{\partial x_j}$ and

$$\boldsymbol{F}^{-T} = \frac{1}{\det\boldsymbol{F}}\begin{pmatrix} F_{22} & -F_{12} \\ -F_{21} & F_{11}\end{pmatrix}^T = \frac{1}{\det\boldsymbol{F}}\begin{pmatrix} F_{22} & -F_{21} \\ -F_{12} & F_{11}\end{pmatrix}.$$

Now, we introduce the first Piola-Kirchhoff stress tensor $\boldsymbol{P} = \boldsymbol{P}(\boldsymbol{F})$,

$$\boldsymbol{P}(\boldsymbol{F}) = \begin{pmatrix} P_{11}(\boldsymbol{F}) & P_{12}(\boldsymbol{F}) \\ P_{21}(\boldsymbol{F}) & P_{22}(\boldsymbol{F})\end{pmatrix}.$$

Its form depends on the chosen elasticity model (cf. [25]). Moreover we denote the divergence of the first Piola-Kirchhoff stress tensor $\boldsymbol{P}$ as

$$\mathrm{div}\boldsymbol{P}(\boldsymbol{F}) = \begin{pmatrix} \frac{\partial P_{11}(\boldsymbol{F})}{\partial x_1} + \frac{\partial P_{21}(\boldsymbol{F})}{\partial x_2} \\ \frac{\partial P_{12}(\boldsymbol{F})}{\partial x_1} + \frac{\partial P_{22}(\boldsymbol{F})}{\partial x_2}\end{pmatrix}.$$

The general dynamic elasticity problem is formulated in the following way: Find a displacement function $\boldsymbol{u} : \overline{\Omega^b} \times [0,T] \to \mathbb{R}^2$ such that

$$\rho^b \frac{\partial^2\boldsymbol{u}}{\partial t^2} + C_M^b \rho^b \frac{\partial\boldsymbol{u}}{\partial t} - \mathrm{div}\boldsymbol{P}(\boldsymbol{F}) = \boldsymbol{f} \qquad \text{in } \Omega^b \times [0,T], \tag{5.7}$$

$$\boldsymbol{u} = \boldsymbol{u}_D \quad \text{in } \Gamma_D^b \times [0,T], \tag{5.8}$$

$$\boldsymbol{P}(\boldsymbol{F})\,\boldsymbol{n} = \boldsymbol{g}_N \quad \text{in } \Gamma_N^b \times [0,T], \tag{5.9}$$

$$\boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0, \quad \frac{\partial\boldsymbol{u}}{\partial t}(\cdot,0) = \boldsymbol{z}_0 \qquad \text{in } \Omega^b, \tag{5.10}$$

where $\boldsymbol{f} : \Omega^b \times [0,T] \to \mathbb{R}^2$ is the density of the acting volume force, $\boldsymbol{g}_N : \Gamma_N^b \times [0,T] \to \mathbb{R}^2$ is the surface traction, $\boldsymbol{u}_D : \Gamma_D^b \times [0,T] \to \mathbb{R}^2$ is the prescribed displacement, $\boldsymbol{u}_0 : \Omega^b \to \mathbb{R}^2$ is the initial displacement, $\boldsymbol{z}_0 : \Omega^b \to \mathbb{R}^2$ is the initial deformation velocity, $\rho^b > 0$ is the material density and $C_M^b \geq 0$ is the damping coefficient.

In the stationary case (static problem) we seek $\boldsymbol{u} : \overline{\Omega^b} \to \mathbb{R}^2$ such that

$$-\mathrm{div}\boldsymbol{P}(\boldsymbol{F}) = \boldsymbol{f} \quad \text{in } \Omega^b, \tag{5.11}$$

$$\boldsymbol{u} = \boldsymbol{u}_D \quad \text{on } \Gamma_D^b, \quad \boldsymbol{P}(\boldsymbol{F})\boldsymbol{n} = \boldsymbol{g}_N \quad \text{on } \Gamma_N^b. \tag{5.12}$$

Linear elasticity

In case of linear elasticity the stress tensor $\boldsymbol{P}(\boldsymbol{F})$ is denoted as $\boldsymbol{\sigma}(\boldsymbol{u})$, which depends linearly on the strain tensor $\boldsymbol{e}(\boldsymbol{u}) = (\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)/2$ according to the relation

$$\boldsymbol{P}(\boldsymbol{F}) := \boldsymbol{\sigma}(\boldsymbol{u}) = \lambda^b \,\mathrm{tr}(\boldsymbol{e}(\boldsymbol{u}))\mathbb{I} + 2\mu^b\boldsymbol{e}(\boldsymbol{u}). \tag{5.13}$$

Here λ^b and μ^b are the Lamé parameters that can be expressed with the aid of the Young modulus E^b and the Poisson ratio ν^b:

$$\lambda^b = \frac{E^b \nu^b}{(1+\nu^b)(1-2\nu^b)}, \quad \mu^b = \frac{E^b}{2(1+\nu^b)}. \tag{5.14}$$

If we set $\boldsymbol{e}(\boldsymbol{u}) = (e_{ij}(\boldsymbol{u}))_{i,j=1}^2$, then $e_{ij}(\boldsymbol{u}) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$ and

$$\mathrm{tr}(\boldsymbol{e}(\boldsymbol{u})) = \sum_{i=1}^{2} e_{ii}(\boldsymbol{u}) = \sum_{i=1}^{2} \frac{\partial u_i}{\partial x_i} = \mathrm{div}\boldsymbol{u}.$$

Nonlinear elasticity

In the nonlinear case we consider two elasticity models. First, the neo-Hookean model with Piola-Kirchhoff stress tensor

$$\boldsymbol{P}(\boldsymbol{F}) = \mu^b(\boldsymbol{F} - \boldsymbol{F}^{-T}) + \lambda^b \log(\det \boldsymbol{F})\boldsymbol{F}^{-T}. \tag{5.15}$$

Moreover we consider the nonlinear St. Venant-Kirchhoff model, which is related to the linear elasticity model by using the nonlinear Green strain tensor $\boldsymbol{E}$ instead of the linearized strain tensor $\boldsymbol{e}$. The first Piola-Kirchhoff stress tensor is then defined as

$$\boldsymbol{P}(\boldsymbol{F}) = \boldsymbol{F}\boldsymbol{\Sigma}, \tag{5.16}$$

where

$$\boldsymbol{\Sigma} = \lambda^b \mathrm{tr}(\boldsymbol{E})\mathbb{I} + 2\mu^b \boldsymbol{E} \tag{5.17}$$

and

$$\boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}\right), \quad \boldsymbol{E} = (E_{ij})_{i,j=1}^2 \tag{5.18}$$

is the second Piola-Kirchhoff stress tensor with components

$$E_{ij} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{e_{ij}-\text{linear part}} + \underbrace{\frac{1}{2}\sum_{k=1}^{2}\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}}_{E^*_{ij}-\text{nonlinear part}}. \tag{5.19}$$

Writing $\boldsymbol{\Sigma}(\boldsymbol{u}) = (\Sigma_{ij})_{i,j=1}^2$, we get

$$\begin{aligned}\Sigma_{ij} = {} & \lambda^b\left(\sum_{l=1}^{2}\frac{\partial u_l}{\partial x_l} + \frac{1}{2}\sum_{l=1}^{2}\sum_{k=1}^{2}\left(\frac{\partial u_k}{\partial x_l}\right)^2\right)\delta_{ij} \\ & + \mu^b\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum_{k=1}^{2}\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right).\end{aligned} \tag{5.20}$$

For a detailed description we can refer the reader to monographs [25] and [13].

5.1.3 Fluid-structure coupling

In the FSI problem the coupling of the discrete flow problem and the structural problem is realized via the transmission conditions representing the continuity of the velocity and normal stress on the common boundary $\tilde{\Gamma}_{W_t}$ between fluid and structure. We assume that

$$\tilde{\Gamma}_{W_t} = \left\{ \boldsymbol{x} \in \mathbb{R}^2;\ \boldsymbol{x} = \boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X}, t),\ \boldsymbol{X} \in \Gamma_N^b \right\} \subset \Gamma_{W_t}. \tag{5.21}$$

Then we use the following transmission conditions:

a) For linear elasticity we assume that

$$\boldsymbol{\sigma}(\boldsymbol{u}(\boldsymbol{X}, t))\boldsymbol{n}(\boldsymbol{X}) = \boldsymbol{\tau}^f(\boldsymbol{x}, t)\boldsymbol{n}(\boldsymbol{X}), \quad \boldsymbol{v}(\boldsymbol{x}, t) = \frac{\partial \boldsymbol{u}(\boldsymbol{X}, t)}{\partial t}. \tag{5.22}$$

b) For nonlinear elasticity we use the following conditions:

$$\boldsymbol{P}(\boldsymbol{F}(\boldsymbol{X}, t))\boldsymbol{n}(\boldsymbol{X}) = \boldsymbol{\tau}^f(\boldsymbol{x}, t)\mathrm{Cof}(\boldsymbol{F}(\boldsymbol{X}, t))\boldsymbol{n}(\boldsymbol{X}), \ \ \boldsymbol{v}(\boldsymbol{x}, t) = \frac{\partial \boldsymbol{u}(\boldsymbol{X}, t)}{\partial t}. \tag{5.23}$$

In the above relations, $\boldsymbol{x} = \boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X}, t),\ \boldsymbol{X} \in \Gamma_N^b,\ \boldsymbol{x} \in \tilde{\Gamma}_{Wt}$, $\boldsymbol{v}$ is the flow velocity, expression $\boldsymbol{\tau}^f = \{\tau_{ij}^f\}_{i,j=1}^2 = \{-p\delta_{ij} + \tau_{ij}^V\}_{i,j=1}^2$ represents the aerodynamical stress tensor and $\boldsymbol{n}(\boldsymbol{X})$ is the unit outward normal to $\partial\Omega^b$ on Γ_N^b (by δ_{ij} we denote the Kronecker symbol).

5.1.4 Determination of the ALE mapping

The ALE mapping $\mathcal{A}_t$ is determined with the aid of an artificial stationary linear elasticity problem proposed in [80]. We seek $\boldsymbol{d} = (d_1, d_2)$ defined in Ω_{ref} as a solution of the elastic static system

$$\sum_{j=1}^{2} \frac{\partial \tau_{ij}^a(\boldsymbol{d})}{\partial X_j} = 0 \text{ in } \Omega_{ref}, \quad i = 1, 2, \tag{5.24}$$

where τ_{ij}^a are the components of the artificial stress tensor $\tau_{ij}^a = \delta_{ij}\lambda^a \mathrm{div}\boldsymbol{d} + 2\mu^a e_{ij}^a(\boldsymbol{d}),\ e_{ij}^a(\boldsymbol{d}) = \frac{1}{2}\left(\frac{\partial d_i}{\partial X_j} + \frac{\partial d_j}{\partial X_i}\right),\ i = 1, 2$. The Lamé coefficients λ^a and μ^a are related to the artificial Young modulus E^a and the artificial Poisson number ν^a similarly as in Section 5.1.2. The boundary conditions for $\boldsymbol{d}$ are prescribed by

$$\boldsymbol{d}|_{\Gamma_I \cup \Gamma_O} = 0,\ \boldsymbol{d}|_{\Gamma_{W_0} \setminus \Gamma_N^b} = 0,\ \boldsymbol{d}(\boldsymbol{X}, t) = \boldsymbol{u}(\boldsymbol{X}, t),\ \boldsymbol{X} \in \Gamma_N^b \tag{5.25}$$

(for the definition of the boundary parts see Chapter 1 and 5.1.2). The solution of the problem (5.24)–(5.25) gives us the ALE mapping of $\overline{\Omega}_{ref}$ onto $\overline{\Omega}_t$ in the form

$$\mathcal{A}_t(\boldsymbol{X}) = \boldsymbol{X} + \boldsymbol{d}(\boldsymbol{X}, t), \quad \boldsymbol{X} \in \overline{\Omega}_{ref}, \tag{5.26}$$

for each time instant t.

For simplicity we set $\Omega_{ref} = \Omega_0$ for all time steps, but it is possible to proceed in such a way as in the previous sections, i.e. define the reference domain separately for each time interval $[t_{m-1}, t_m]$.

5.2 Discrete problem

The following section is devoted to the description of the STDGM discretization of the flow and structural problems.

5.2.1 Discretization of the flow problem

We describe the discretization as it is carried out in the program system used in our practical computations. We assume that Ω_t is a polygonal domain for every $t \in [0, T]$. We denote by $\mathcal{T}_{ht}$ a partition of the closure $\overline{\Omega}_t$ into a finite number of closed triangles with disjoint interiors satisfying standard properties (3.3). We suppose that $\mathcal{T}_{ht}$ is an image of $\mathcal{T}_{h0}$ under the regular mapping "$t \to \mathcal{A}_t$". Moreover, we assume that the ALE mapping $\mathcal{A}_t$ is continuous and piecewise affine in $\overline{\Omega_0}$.

By $\mathcal{F}$ we denote the system of all faces of all elements $K \in \mathcal{T}_{ht}$. Further, we introduce the set of boundary faces $\mathcal{F}^B = \{\Gamma \in \mathcal{F};\ \Gamma \subset \partial\Omega_t\}$, the set of "Dirichlet" boundary faces $\mathcal{F}^D = \{\Gamma \in \mathcal{F}^B$; a Dirichlet condition is prescribed on $\Gamma\}$ and the set of inner faces $\mathcal{F}^I = \mathcal{F} \setminus \mathcal{F}^B$. Each $\Gamma \in \mathcal{F}$ is associated with a unit normal vector $\boldsymbol{n}_\Gamma$ to Γ. For $\Gamma \in \mathcal{F}^B$ the normal $\boldsymbol{n}_\Gamma$ has the same orientation as the outer normal to $\partial\Omega_t$.

For each $\Gamma \in \mathcal{F}^I$ there exist two neighbouring elements $K_\Gamma^{(L)}, K_\Gamma^{(R)} \in \mathcal{T}_{ht}$ such that $\Gamma \subset \partial K_\Gamma^{(R)} \cap \partial K_\Gamma^{(L)}$. We use the convention that $K_\Gamma^{(R)}$ lies in the direction of $\boldsymbol{n}_\Gamma$ and $K_\Gamma^{(L)}$ lies in the opposite direction to $\boldsymbol{n}_\Gamma$. If $\Gamma \in \mathcal{F}^B$, then the element adjacent to Γ will be denoted by $K_\Gamma^{(L)}$.

Now we introduce the space of piecewise polynomial functions

$$\boldsymbol{S}_{ht}^r = [S_{ht}^r]^4, \quad \text{with} \ \ S_{ht}^r = \{v; v|_K \in P^r(K)\ \forall\, K \in \mathcal{T}_{ht}\}, \tag{5.27}$$

where $r > 0$ is an integer and $P^r(K)$ denotes the space of all polynomials on K of degree $\le r$. It is possible to see that $S_{ht}^r = \{v; v = \mathcal{A}_t(\hat{v}),\ \hat{v} \in S_{h0}^r\}$. A function $\boldsymbol{\varphi} \in \boldsymbol{S}_{ht}^r$ is, in general, discontinuous on interfaces $\Gamma \in \mathcal{F}^I$. If $\boldsymbol{\varphi}$ is a function defined on $K_\Gamma^{(L)} \cup K_\Gamma^{(R)}$, then by $\boldsymbol{\varphi}_\Gamma^{(L)}$ and $\boldsymbol{\varphi}_\Gamma^{(R)}$ we denote the values of $\boldsymbol{\varphi}$ on Γ considered from the interior of $K_\Gamma^{(L)}$ and $K_\Gamma^{(R)}$, respectively, and set

$$\begin{aligned}
\langle \boldsymbol{\varphi} \rangle_\Gamma &= (\boldsymbol{\varphi}_\Gamma^{(L)} + \boldsymbol{\varphi}_\Gamma^{(R)})/2, \\
[\boldsymbol{\varphi}]_\Gamma &= \boldsymbol{\varphi}_\Gamma^{(L)} - \boldsymbol{\varphi}_\Gamma^{(R)}.
\end{aligned}$$

The discrete problem is derived in the following way: We multiply system (5.4) by a test function $\boldsymbol{\varphi}_h \in \boldsymbol{S}_{ht}^r$, integrate over $K \in \mathcal{T}_{ht}$, apply Green's theorem, sum over all elements $K \in \mathcal{T}_{ht}$, use the concept of the numerical flux and introduce suitable terms mutually vanishing for a regular exact solution and linearize the resulting forms on the basis of properties (1.11), (1.12) of the functions $\boldsymbol{f}_s$ and $\boldsymbol{R}_s$. It is a generalization of approaches from [29], [22] and [31]. In this way we get the following forms (followed by the explanation of symbols appearing in their definitions):

$$
\begin{aligned}
\hat{a}_h(\overline{\boldsymbol{w}}_h, \boldsymbol{w}_h, \boldsymbol{\varphi}_h, t) = & \sum_{K\in\mathcal{T}_{ht}} \int_K \sum_{s=1}^{2}\sum_{k=1}^{2} \mathbb{K}_{s,k}(\overline{\boldsymbol{w}}_h)\ \frac{\partial \boldsymbol{w}_h}{\partial x_k}\cdot\frac{\partial \boldsymbol{\varphi}_h}{\partial x_s}\ \mathrm{d}x \\
& - \sum_{\Gamma\in\mathcal{F}_{ht}^I} \int_\Gamma \sum_{s=1}^{2} \left\langle \sum_{k=1}^{2} \mathbb{K}_{s,k}(\overline{\boldsymbol{w}}_h)\frac{\partial \boldsymbol{w}_h}{\partial x_k} \right\rangle (\boldsymbol{n}_\Gamma)_s\cdot[\boldsymbol{\varphi}_h]\ \mathrm{d}S \\
& - \sum_{\Gamma\in\mathcal{F}_{ht}^D} \int_\Gamma \sum_{s=1}^{2}\sum_{k=1}^{2} \mathbb{K}_{s,k}(\overline{\boldsymbol{w}}_h)\frac{\partial \boldsymbol{w}_h}{\partial x_k}(\boldsymbol{n}_\Gamma)_s\cdot\boldsymbol{\varphi}_h\ \mathrm{d}S \\
& - \Theta \sum_{\Gamma\in\mathcal{F}_{ht}^I} \int_\Gamma \sum_{s=1}^{2} \left\langle \sum_{k=1}^{2} \mathbb{K}^T_{k,s}(\overline{\boldsymbol{w}}_h)\frac{\partial \boldsymbol{\varphi}_h}{\partial x_k} \right\rangle (\boldsymbol{n}_\Gamma)_s\cdot[\boldsymbol{w}_h]\ \mathrm{d}S \\
& - \Theta \sum_{\Gamma\in\mathcal{F}_{ht}^D} \int_\Gamma \sum_{s=1}^{2}\sum_{k=1}^{2} \mathbb{K}^T_{k,s}(\overline{\boldsymbol{w}}_h)\frac{\partial \boldsymbol{\varphi}_h}{\partial x_k}(\boldsymbol{n}_\Gamma)_s\cdot\boldsymbol{w}_h\ \mathrm{d}S,
\end{aligned} \tag{5.28}
$$

$$
d_h(\boldsymbol{w}_h, \boldsymbol{\varphi}_h, t) = \sum_{K\in\mathcal{T}_{ht}} \int_K (\boldsymbol{w}_h\cdot\boldsymbol{\varphi}_h)\,\mathrm{div}\boldsymbol{z}\ \mathrm{d}x, \tag{5.29}
$$

$$
J_h(\boldsymbol{w}_h, \boldsymbol{\varphi}_h, t) = \sum_{\Gamma\in\mathcal{F}_{ht}^I} \int_\Gamma \frac{\mu C_W}{h_\Gamma}[\boldsymbol{w}_h]\cdot[\boldsymbol{\varphi}_h]\ \mathrm{d}S + \sum_{\Gamma\in\mathcal{F}_{ht}^D} \int_\Gamma \frac{\mu C_W}{h_\Gamma}\boldsymbol{w}_h\cdot\boldsymbol{\varphi}_h\ \mathrm{d}S, \tag{5.30}
$$

$$
\begin{aligned}
\ell_h(\overline{\boldsymbol{w}}_h, \boldsymbol{w}_B, \boldsymbol{\varphi}_h, t) = & \sum_{\Gamma\in\mathcal{F}_{ht}^D} \int_\Gamma \frac{\mu C_W}{h_\Gamma}\boldsymbol{w}_B\cdot\boldsymbol{\varphi}_h\ \mathrm{d}S \\
& - \Theta \sum_{\Gamma\in\mathcal{F}_{ht}^D} \int_\Gamma \sum_{s=1}^{2}\sum_{k=1}^{2} \mathbb{K}^T_{k,s}(\overline{\boldsymbol{w}}_h)\frac{\partial \boldsymbol{\varphi}_h}{\partial x_k}(\boldsymbol{n}_\Gamma)_s\cdot\boldsymbol{w}_B\ \mathrm{d}S,
\end{aligned} \tag{5.31}
$$

$$
\begin{aligned}
\hat{b}_h(\overline{\boldsymbol{w}}_h, \boldsymbol{w}_h, \boldsymbol{\varphi}_h, t) = & \\
& - \sum_{K\in\mathcal{T}_{ht_{k+1}}} \int_K \sum_{s=1}^{2} (\mathbb{A}_s(\overline{\boldsymbol{w}}_h(x)) - z_s(x))\mathbb{I})\boldsymbol{w}_h(x))\cdot\frac{\partial \boldsymbol{\varphi}_h(x)}{\partial x_s}\ \mathrm{d}x \\
& + \sum_{\Gamma\in\mathcal{F}_{ht}^I} \int_\Gamma \Big(\mathbb{P}_g^+\big(\langle\overline{\boldsymbol{w}}_h\rangle_\Gamma, \boldsymbol{n}_\Gamma\big)\boldsymbol{w}_h^{(L)} + \mathbb{P}_g^-\big(\langle\overline{\boldsymbol{w}}_h\rangle_\Gamma, \boldsymbol{n}_\Gamma\big)\boldsymbol{w}_h^{(R)}\Big)\cdot[\boldsymbol{\varphi}_h]\ \mathrm{d}S \\
& + \sum_{\Gamma\in\mathcal{F}_{ht}^B} \int_\Gamma \Big(\mathbb{P}_g^+\big(\langle\overline{\boldsymbol{w}}_h\rangle_\Gamma, \boldsymbol{n}_\Gamma\big)\boldsymbol{w}_h^{(L)} + \mathbb{P}_g^-\big(\langle\overline{\boldsymbol{w}}_h\rangle_\Gamma, \boldsymbol{n}_\Gamma\big)\overline{\boldsymbol{w}}_h^{(R)}\Big)\cdot\boldsymbol{\varphi}_h\ \mathrm{d}S.
\end{aligned} \tag{5.32}
$$

We set $\Theta = 1$, $\Theta = 0$ or $\Theta = -1$ and get the so-called symmetric (SIPG), incomplete (IIPG) or nonsymmetric (NIPG) version, respectively, of the discretization of viscous terms. In (5.30) and (5.31), C_W denotes a positive sufficiently large constant.

In the form (5.32) we follow the ideas from [76]. The symbols $\mathbb{P}_g^+(\boldsymbol{w}, \boldsymbol{n})$ and $\mathbb{P}_g^-(\boldsymbol{w}, \boldsymbol{n})$ denote the "positive" and "negative" parts of the matrix $\mathbb{P}_g(\boldsymbol{w}, \boldsymbol{n}) = \sum_{s=1}^{2}(\mathbb{A}_s(\boldsymbol{w}) - z_s\mathbb{I})n_s$ defined in the following way. By [41], this matrix is diagonalizable. It means that there exists a nonsingular matrix $\mathbb{T} = \mathbb{T}(\boldsymbol{w}, \boldsymbol{n})$ such that

$$
\mathbb{P}_g = \mathbb{T}\Lambda\mathbb{T}^{-1}, \quad \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_4), \tag{5.33}
$$

where $\lambda_i = \lambda_i(\boldsymbol{w}, \boldsymbol{n})$ are eigenvalues of the matrix $\mathbb{P}_g$. Now we define the "positive" and "negative" parts of the matrix $\mathbb{P}_g$ by

$$
\mathbb{P}_g^\pm = \mathbb{T}\Lambda^\pm\mathbb{T}^{-1}, \quad \Lambda^\pm = \mathrm{diag}(\lambda_1^\pm, \ldots, \lambda_4^\pm), \tag{5.34}
$$

where $\lambda^+ = \max(\lambda, 0)$, $\lambda^- = \min(\lambda, 0)$.

The boundary state $\boldsymbol{w}_B$ is defined on the basis of the Dirichlet boundary conditions (1.8), (1.9), (1.10)

$$\boldsymbol{w}_B = (\rho_D, \rho_D v_{D1}, \rho_D v_{D2}, c_v \rho_D \theta_\Gamma^{(L)} + \frac{1}{2}\rho_D |\boldsymbol{v}_D|^2) \quad \text{on } \Gamma_I, \tag{5.35}$$

$$\boldsymbol{w}_B = \boldsymbol{w}_\Gamma^{(L)} \quad \text{on } \Gamma_O, \tag{5.36}$$

$$\boldsymbol{w}_B = (\rho_\Gamma^{(L)}, \rho_\Gamma^{(L)} z_{D1}, \rho_\Gamma^{(L)} z_{D2}, c_v \rho_\Gamma^{(L)} \theta_\Gamma^{(L)} + \frac{1}{2}\rho_\Gamma^{(L)} |\boldsymbol{z}_D|^2) \quad \text{on } \Gamma_{Wt}. \tag{5.37}$$

Here quantities $\theta_\Gamma^{(L)}$, $\boldsymbol{w}_\Gamma^{(L)}$ and $\rho_\Gamma^{(L)}$ are obtained by extrapolation. For $\Gamma \in \mathcal{F}^B$ we set $\langle \overline{\boldsymbol{w}}_h \rangle_\Gamma = (\overline{\boldsymbol{w}}_\Gamma^{(L)} + \overline{\boldsymbol{w}}_\Gamma^{(R)})/2$ and the boundary state $\overline{\boldsymbol{w}}_\Gamma^{(R)}$ is defined with the aid of the solution of the 1D linearized initial-boundary Riemann problem as in [39].

In order to avoid spurious oscillations in the approximate solution in the vicinity of discontinuities or steep gradients, we apply artificial viscosity forms. They are based on the discontinuity indicator

$$g_t(K) = \frac{1}{h_K |K|^{3/4}} \int_{\partial K} [\overline{\rho}_h]^2 \, \mathrm{d}S, \quad K \in \mathcal{T}_{ht}, \tag{5.38}$$

introduced in [33]. By $[\overline{\rho}_h]$ we denote the jump of the function $\overline{\rho}_h$ on the boundary ∂K and $|K|$ denotes the area of the element K. Then we define the discrete discontinuity indicator $G_t(K) = 0$ if $g_t(K) < 1$, $G_t(K) = 1$ if $g_t(K) \geq 1$, and the artificial viscosity forms (see [45])

$$\hat{\beta}_h(\overline{\boldsymbol{w}}_h, \boldsymbol{w}_h, \boldsymbol{\varphi}_h, t) = \nu_1 \sum_{K \in \mathcal{T}_{ht}} h_K G_t(K) \int_K \nabla \boldsymbol{w}_h \cdot \nabla \boldsymbol{\varphi}_h \, \mathrm{d}x, \tag{5.39}$$

$$\hat{J}_h(\overline{\boldsymbol{w}}_h, \boldsymbol{w}_h, \boldsymbol{\varphi}_h, t) = \nu_2 \sum_{\Gamma \in \mathcal{F}_h^I} \frac{1}{2}\Big(G_t(K_\Gamma^{(L)}) + G_t(K_\Gamma^{(R)})\Big) \int_\Gamma [\boldsymbol{w}_h] \cdot [\boldsymbol{\varphi}_h] \, \mathrm{d}S,$$

with parameters ν_1, $\nu_2 = O(1)$.

Because of the time discretization we consider a partition

$$0 = t_0 < t_1 < \ldots < t_M = T$$

of the time interval $[0, T]$ and denote $I_m = (t_{m-1}, t_m)$, $\overline{I}_m = [t_{m-1}, t_m]$, $\tau_m = t_m - t_{m-1}$, for $m = 1, \ldots, M$.

We define the space $\mathbf{S}_{h\tau}^{rq} = (S_{h\tau}^{rq})^4$, where $S_{h\tau}^{rq}$ is defined in a similar way as in (2.20), (2.21),

$$S_{h\tau}^{rq} = \Big\{\phi;\ \phi(x,t) = \sum_{i=0}^{q} t^i \phi_i(x),\ \phi_i \in S_{ht}^r,\ t \in I_m,\ x \in \Omega_t,\ m = 1, \ldots, M\Big\}, \tag{5.40}$$

with integers $r, q \geq 1$. For $\boldsymbol{\varphi} \in \mathbf{S}_{h\tau}^{rq}$ we introduce one-sided limits and jump at time instant t_m as

$$\boldsymbol{\varphi}_m^\pm = \boldsymbol{\varphi}(t_m^\pm) = \lim_{t \to t_m \pm} \boldsymbol{\varphi}(t), \quad \{\boldsymbol{\varphi}\}_m = \boldsymbol{\varphi}_m^+ - \boldsymbol{\varphi}_m^-. \tag{5.41}$$

In order to bind the solution on intervals I_{m-1} and I_m, we augment the resulting identity by the penalty expression $(\{\boldsymbol{w}_{h\tau}\}_{m-1}, \boldsymbol{\varphi}_{h\tau}(t_{m-1}+))_{t_{m-1}}$. The initial state $\boldsymbol{w}_{h\tau}(0-) \in \mathbf{S}_{h0}^r$ is defined as the $L^2(\Omega_{h0})$-projection of $\boldsymbol{w}^0$ on $\mathbf{S}_{h0}^r$, i.e.

$$(\boldsymbol{w}_{h\tau}(0-), \boldsymbol{\varphi}_h)_{\Omega_{t_0}} = \left(\boldsymbol{w}^0, \boldsymbol{\varphi}_h\right)_{\Omega_{t_0}} \quad \forall \boldsymbol{\varphi}_h \in \mathbf{S}_{h0}^r. \tag{5.42}$$

Moreover, we introduce the prolongation $\overline{\boldsymbol{w}}_{h\tau}(t)$ of $\boldsymbol{w}_{h\tau}|_{I_{m-1}}$ on time interval I_m (the space-time DG technique with prolongation was analyzed theoretically in [78] on a scalar model problem with a domain Ω independent of time t).

In what follows we denote

$$(a, b)_\omega = \int_\omega ab \ \mathrm{d}x, \tag{5.43}$$

for functions $a,\ b$ defined in a set $\omega \subset \mathbb{R}^2$.

Now the *space-time DG approximate solution* of the flow problem is defined as a function $\boldsymbol{w}_{h\tau} \in \mathbf{S}_{h\tau}^{rq}$ satisfying (5.42) and the following relation for $m = 1, \dots, M$:

$$\begin{aligned}
&\int_{I_m} \left(\left(\frac{D^{\mathcal{A}} \boldsymbol{w}_{h\tau}}{Dt}, \boldsymbol{\varphi}_{h\tau} \right)_{\Omega_t} + \hat{a}_h(\overline{\boldsymbol{w}}_{h\tau}, \boldsymbol{w}_{h\tau}, \boldsymbol{\varphi}_{h\tau}, t) \right) \ \mathrm{d}t \qquad (5.44)\\
&+ \int_{I_m} \left(\hat{b}_h(\overline{\boldsymbol{w}}_{h\tau}, \boldsymbol{w}_{h\tau}, \boldsymbol{\varphi}_{h\tau}, t) + J_h(\boldsymbol{w}_{h\tau}, \boldsymbol{\varphi}_{h\tau}, t) + d_h(\boldsymbol{w}_{h\tau}, \boldsymbol{\varphi}_{h\tau}, t) \right) \ \mathrm{d}t \\
&+ \int_{I_m} \left(\hat{\beta}_h(\overline{\boldsymbol{w}}_{h\tau}, \boldsymbol{w}_{h\tau}, \boldsymbol{\varphi}_{h\tau}, t) + \hat{J}_h(\overline{\boldsymbol{w}}_{h\tau}, \boldsymbol{w}_{h\tau}, \boldsymbol{\varphi}_{h\tau}, t) \right) \ \mathrm{d}t \\
&+ (\{\boldsymbol{w}_{h\tau}\}_{m-1}, \boldsymbol{\varphi}_{h\tau}(t_{m-1}+))_{\Omega_{t_{m-1}}} = \int_{I_m} \ell_h(\overline{\boldsymbol{w}}_{h\tau}, \boldsymbol{w}_B, \boldsymbol{\varphi}_{h\tau}, t) \ \mathrm{d}t, \quad \forall \boldsymbol{\varphi}_{h\tau} \in \mathbf{S}_{h\tau}^{rq}.
\end{aligned}$$

Remark 1. *In the derivation of the discrete problem, the approximate solution and the test functions are considered as elements of the space $\boldsymbol{S}_{h\tau}^{rq}$. In practical computations, integrals appearing in the definitions of the forms $\hat{a}_h$, $\hat{b}_h$, d_h, J_h, $\hat{J}_h$ and $\hat{\beta}_h$ and also the time integrals over I_m are evaluated with the aid of quadrature formulas using values of the approximate solution at discrete points of intervals I_m. Therefore, the space $\mathbf{S}_{h\tau}^{rq}$ is finite dimensional and the discrete problem is equivalent with a finite algebraic system for every $m = 1, \dots, M$.*

5.2.2 Discretization of the elasticity problem

In the discretization of the structural problem we consider the displacement $\boldsymbol{u}$ and the deformation velocity $\boldsymbol{y}$ and split the basic system into two systems of first-order in time

$$\rho^b \frac{\partial \boldsymbol{y}}{\partial t} + c_M^b \rho^b \boldsymbol{y} - \mathrm{div} \boldsymbol{P}(\boldsymbol{F}) = \boldsymbol{f}, \quad \frac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{y} = 0 \qquad \text{in } \Omega^b \times [0, T], \tag{5.45}$$

$$\boldsymbol{u} = \boldsymbol{u}_D \quad \text{in } \Gamma_D^b \times [0, T], \tag{5.46}$$

$$\boldsymbol{P}(\boldsymbol{F})\boldsymbol{n} = \boldsymbol{g}_N \quad \text{in } \Gamma_N^b \times [0, T], \tag{5.47}$$

$$\boldsymbol{u}(\cdot, 0) = \boldsymbol{u}_0, \quad \boldsymbol{y}(\cdot, 0) = \boldsymbol{y}_0 \qquad \text{in } \Omega^b. \tag{5.48}$$

We construct a partition $\mathcal{T}_h^b$ of $\overline{\Omega}^b$ into a finite number of closed triangles K with mutually disjoint interiors satisfying standard properties formulated in previous sections. The approximate solution at every time instant $t \in [0, T]$ will be sought in the finite-dimensional space

$$\boldsymbol{S}_h^{b,s} = \left\{ v \in L^2(\Omega); v|_K \in P^s(K), K \in \mathcal{T}_h^b \right\}^2, \tag{5.49}$$

where $s > 0$ is an integer and $P^s(K)$ denotes the space of polynomials of degree not greater than s on K. By $\mathcal{F}_h^b$ we denote the system of all faces of all elements $K \in \mathcal{T}_h^b$ and distinguish there sets of boundary, "Dirichlet", "Neumann" and inner faces: $\mathcal{F}_h^{b,B} = \left\{ \Gamma \in \mathcal{F}_h^b; \Gamma \subset \partial\Omega^b \right\}$, $\mathcal{F}_h^{b,D} = \left\{ \Gamma \in \mathcal{F}_h^b; \Gamma \subset \Gamma_D^b \right\}$, $\mathcal{F}_h^{b,N} = \left\{ \Gamma \in \mathcal{F}_h^b; \Gamma \subset \Gamma_N^b \right\}$ and $\mathcal{F}_h^{b,I} = \mathcal{F}_h^b \backslash \mathcal{F}_h^{b,B}$. Triangulation $\mathcal{T}_h^b$ is of course constructed in such a way that the set $\bar{\Gamma}_D^b \cap \bar{\Gamma}_N^b$ is formed by vertices of elements laying on $\partial\Omega^b$. For each $\Gamma \in \mathcal{F}_h$ we define a unit normal vector $\boldsymbol{n}_\Gamma$. We assume that for $\Gamma \in \mathcal{F}_h^{b,B}$ the normal $\boldsymbol{n}_\Gamma$ has the same orientation as the outer normal to $\partial\Omega^b$. By $h(\Gamma)$ we denote the length of Γ. For $\boldsymbol{\varphi} \in \boldsymbol{S}_h^{b,s}$ symbols $\boldsymbol{\varphi}_\Gamma^{(L)}$ and $\boldsymbol{\varphi}_\Gamma^{(R)}$ denote the traces of $\boldsymbol{\varphi}$ on Γ from the sides of elements $K_\Gamma^{(L)}$ and $K_\Gamma^{(R)}$ adjacent to Γ. We assume that $\boldsymbol{n}_\Gamma$ is the outer normal to $\partial K_\Gamma^{(L)}$. In integrals over Γ, instead of $\boldsymbol{n}_\Gamma$ we write only $\boldsymbol{n}$. Further, $\langle \boldsymbol{\varphi} \rangle_\Gamma$ denotes the average of the traces on Γ and $[\boldsymbol{\varphi}]_\Gamma = \boldsymbol{\varphi}_\Gamma^{(L)} - \boldsymbol{\varphi}_\Gamma^{(R)}$ is the jump of $\boldsymbol{\varphi}$ on Γ.

If $\boldsymbol{a} = (a_{ij})_{i,j=1}^2, \boldsymbol{b} = (b_{ij})_{i,j=1}^2$ are tensors, then we set $\boldsymbol{a} : \boldsymbol{b} = \sum_{i,j=1}^2 a_{ij} b_{ij}$.

The DG discretization in space is formulated with the use of the following forms.

Linear elasticity form:

$$\begin{aligned} a_h^b(\boldsymbol{u}, \boldsymbol{\varphi}) = & \sum_{K \in \mathcal{T}_h^b} \int_K \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{e}(\boldsymbol{\varphi}) \ \mathrm{d}x - \sum_{\Gamma \in \mathcal{F}_h^{b,I}} \int_\Gamma (\langle \boldsymbol{\sigma}(\boldsymbol{u}) \rangle \boldsymbol{n}) \cdot [\boldsymbol{\varphi}] \ \mathrm{d}S \\ & - \sum_{\Gamma \in \mathcal{F}_h^{b,D}} \int_\Gamma (\boldsymbol{\sigma}(\boldsymbol{u}) \boldsymbol{n}) \cdot \boldsymbol{\varphi} \ \mathrm{d}S - \Theta \sum_{\Gamma \in \mathcal{F}_h^{b,I}} \int_\Gamma (\langle \boldsymbol{\sigma}(\boldsymbol{\varphi}) \rangle \boldsymbol{n}) \cdot [\boldsymbol{u}] \ \mathrm{d}S \\ & - \Theta \sum_{\Gamma \in \mathcal{F}_h^{b,D}} \int_\Gamma (\boldsymbol{\sigma}(\boldsymbol{\varphi}) \boldsymbol{n}) \cdot \boldsymbol{u} \ \mathrm{d}S, \end{aligned} \tag{5.50}$$

where $\boldsymbol{\sigma}(\boldsymbol{u})$ is defined by (5.13). Here the parameter Θ is chosen as $1, 0, -1$ for SIPG, IIPG, NIPG, respectively, version of the linear elasticity form.

Nonlinear IIPG elasticity form ($\Theta = 0$):

$$\begin{aligned} a_h^b(\boldsymbol{u}, \boldsymbol{\varphi}) = & \sum_{K \in \mathcal{T}_h^b} \int_K \boldsymbol{P}(\boldsymbol{F}) : \nabla \boldsymbol{\varphi} \ \mathrm{d}x - \sum_{\Gamma \in \mathcal{F}_h^{b,I}} \int_\Gamma (\langle \boldsymbol{P}(\boldsymbol{F}) \rangle \boldsymbol{n}) \cdot [\boldsymbol{\varphi}] \ \mathrm{d}S \\ & - \sum_{\Gamma \in \mathcal{F}_h^{b,D}} \int_\Gamma (\boldsymbol{P}(\boldsymbol{F}) \boldsymbol{n}) \cdot \boldsymbol{\varphi} \ \mathrm{d}S. \end{aligned} \tag{5.51}$$

Penalty form:

$$J_h^b(\boldsymbol{u}, \boldsymbol{\varphi}) = \sum_{\Gamma \in \mathcal{F}_h^I} \int_\Gamma \frac{C_W^b}{h(\Gamma)} [\boldsymbol{u}] \cdot [\boldsymbol{\varphi}] \ \mathrm{d}S + \sum_{\Gamma \in \mathcal{F}_h^D} \int_\Gamma \frac{C_W^b}{h(\Gamma)} \boldsymbol{u} \cdot \boldsymbol{\varphi} \ \mathrm{d}S. \tag{5.52}$$

Here $C_W^b > 0$ is a sufficiently large constant.

Right-hand side form:

$$\ell_h^b(\boldsymbol{\varphi})(t) = \sum_{K\in\mathcal{T}_h^b} \int_K \boldsymbol{f}(t)\cdot\boldsymbol{\varphi}\ \mathrm{d}x + \sum_{\Gamma\in\mathcal{F}_h^{b,N}} \int_\Gamma \boldsymbol{g}_N(t)\cdot\boldsymbol{\varphi}\ \mathrm{d}S \tag{5.53}$$
$$- \Theta \sum_{\Gamma\in\mathcal{F}_h^{b,D}} \int_\Gamma (\boldsymbol{\sigma}(\boldsymbol{\varphi})\cdot\boldsymbol{n})\cdot\boldsymbol{u}_D(t)\ \mathrm{d}S + \sum_{\Gamma\in\mathcal{F}_h^{b,D}} \int_\Gamma \frac{C_W^b}{h(\Gamma)}\boldsymbol{u}_D(t)\cdot\boldsymbol{\varphi}\ \mathrm{d}S.$$

In the nonlinear case, it is not clear how to define the SIPG and NIPG versions of the elasticity forms so that the form a_h^b is linear with respect to the test function $\boldsymbol{\varphi}$. For this reason we will consider only the IIPG version (5.51) of a_h^b.

STDGM for the structural problem

An approximate solution of problem (5.45)–(5.48), i. e., the approximations of the functions $\boldsymbol{u}$, $\boldsymbol{y}$ will be sought in the space of piecewise polynomial vector functions $\boldsymbol{S}_{h\tau}^{b,sq^*} = [S_{h\tau}^{b,sq^*}]^2$, where

$$\begin{aligned}\mathcal{V} &= S_{h\tau}^{b,sq^*} \qquad (5.54)\\ &= \left\{ v\in L^2(\Omega^b\times(0,T)); v|_{I_m} = \sum_{i=0}^{q^*} t^i\varphi_i \text{ with } \varphi_i\in\boldsymbol{S}_h^{b,s}, m=1,\dots,M\right\}.\end{aligned}$$

By s and q^* we denote positive integers representing the degrees of polynomial approximations in space and time in the discretization of the structural problem. We introduce the one-sided limits and jump of a function $\boldsymbol{\varphi}\in[S_{h\tau}^{b,sq^*}]^2$ at time t_m similarly as in (5.41). Now, the approximate STDG solution of problem (5.45)–(5.48) is defined as a couple $\boldsymbol{u}_{h\tau}, \boldsymbol{y}_{h\tau}\in\boldsymbol{S}_{h\tau}^{b,sq^*}$ such that

$$\begin{aligned}&\int_{I_m}\left(\rho^b\Big(\frac{\partial\boldsymbol{y}_{h\tau}}{\partial t},\boldsymbol{\varphi}_{h\tau}\Big)_{\Omega^b} + c_M^b\left(\rho^b\boldsymbol{y}_{h\tau},\boldsymbol{\varphi}_{h\tau}\right)_{\Omega^b}\right) + a_h^b(\boldsymbol{u}_{h\tau},\boldsymbol{\varphi}_{h\tau}) \qquad (5.55)\\ &\quad +J_h^b(\boldsymbol{u}_{h\tau},\boldsymbol{\varphi}_{h\tau})\Big)\ \mathrm{d}t + (\{\boldsymbol{y}_{h\tau}\}_{m-1},\boldsymbol{\varphi}_{h\tau}(t_{m-1}+))_{\Omega^b}\\ &= \int_{I_m}\ell_h^b(\boldsymbol{\varphi}_{h\tau})\ \mathrm{d}t \quad \forall\boldsymbol{\varphi}_{h\tau}\in\boldsymbol{S}_{h\tau}^{b,sq^*},\\ &\int_{I_m}\left(\left(\frac{\partial\boldsymbol{u}_{h\tau}}{\partial t},\boldsymbol{\varphi}_{h\tau}\right)_{\Omega^b} - (\boldsymbol{y}_{h\tau},\boldsymbol{\varphi}_{h\tau})_{\Omega^b}\right)\ \mathrm{d}t \qquad (5.56)\\ &\quad +(\{\boldsymbol{u}_{h\tau}\}_{m-1},\boldsymbol{\varphi}_{h\tau}(t_{m-1}+))_{\Omega^b} = 0, \quad \forall\boldsymbol{\varphi}_{h\tau}\in\boldsymbol{S}_{h\tau}^{b,sq^*}, \quad m=1,\dots,M.\end{aligned}$$

Similarly as in (5.42) we define the initial states $\boldsymbol{u}_h(0-), \boldsymbol{y}_h(0-)\in\boldsymbol{S}_h^{b,s}$ by

$$\begin{aligned}(\boldsymbol{u}_h(0-),\boldsymbol{\varphi}_h)_{\Omega^b} &= (\boldsymbol{u}^0,\boldsymbol{\varphi}_h)_{\Omega^b} \quad \forall\boldsymbol{\varphi}_h\in\boldsymbol{S}_h^{b,s}, \qquad (5.57)\\ (\boldsymbol{y}_h(0-),\boldsymbol{\varphi}_h)_{\Omega^b} &= (\boldsymbol{y}^0,\boldsymbol{\varphi}_h)_{\Omega^b} \quad \forall\boldsymbol{\varphi}_h\in\boldsymbol{S}_h^{b,s}.\end{aligned}$$

5.2.3 Coupling procedure

In the solution of the complete coupled FSI problem it is necessary to apply a suitable coupling procedure. See, e.g. [5] for a general framework. Here we apply the following algorithm, in which we proceed successively from one time interval $[t_{m-1}, t_m]$ to the next interval $[t_m, t_{m+1}]$ using an iterative process with few subiterations. The approximate solutions obtained during the subiterations are denoted with the index l, i.e. $\boldsymbol{w}_{h\tau,l}^m$, $\boldsymbol{u}_{h\tau,l}^m$, $\Omega_{t_m,l}$, $\mathcal{A}_{t_m,l}$, $\boldsymbol{z}_l^m$.

1. Assume that the approximate solution $\boldsymbol{w}_{h\tau}^m$ of the flow problem in time interval $\overline{I}_m = [t_{m-1}, t_m]$ and the displacement of the structure $\boldsymbol{u}_{h\tau}^m$ at time instant t_m are known.

2. Set $\boldsymbol{u}_{h\tau,0}^{m+1} := \boldsymbol{u}_{h\tau}^m$, $l := 1$ and apply the iterative process:

   (a) Interpolate $\boldsymbol{u}_{h,l-1}^{m+1}$ on the common boundary between the fluid and the structure domain - in order to get a continuous function on this interface.

   (b) The approximation $\Omega_{t_{m+1},l}$ of the fluid domain $\Omega_{t_{m+1}}$ is determined by the interpolated displacement of the moving part of the fluid domain boundary.

   (c) Determine the ALE mapping $\mathcal{A}_{t_{m+1},l}$ and approximate the domain velocity $\boldsymbol{z}_l^{m+1}$.

   (d) Solve the flow problem in the domain $\Omega_{t_{m+1},l}$ to obtain the approximate solution $\boldsymbol{w}_{h\tau,l}^{m+1}$ in time interval $\overline{I}_{m+1} = [t_m, t_{m+1}]$.

   (e) Compute the stress tensor and the aerodynamical force acting on the structure and transform it to the interface Γ_N^b.

   (f) Solve the elasticity problem, compute the displacement $\boldsymbol{u}_{h\tau,l}^{m+1}$ at time t_{m+1}.

   (g) If the variation of the displacement $\left|\boldsymbol{u}_{h\tau,l}^{m+1} - \boldsymbol{u}_{h\tau,l-1}^{m+1}\right|$ is larger than a prescribed tolerance, set $l := l + 1$ and go to (a), else, continue with (h).

   (h) Set $\boldsymbol{u}_h^{m+1} := \boldsymbol{u}_{h,l}^{m+1}$, $\boldsymbol{w}_{h\tau}^{m+1} := \boldsymbol{w}_{h\tau,l}^{m+1}$, $m := m + 1$ and go to step 2.

This algorithm represents the so-called strong coupling. If in the step (g) we set $m := m + 1$ and go to (2) already in the case when $l = 1$, then we get the weak (loose) coupling.

5.3 Algorithmization and numerical realization of the coupled problem

The linear algebraic systems equivalent to (5.42) and (5.44) are solved either by the direct solver UMFPACK ([27]) or by the GMRES ([68]) method with block diagonal preconditioning. These methods are also used for the solution of the structure problem (5.55)–(5.56). In the case of nonlinear elasticity on each time level the nonlinear system is solved by the Newton method.

5.3.1 Newton method

In case of nonlinear elasticity model, the form $a_h^b(\boldsymbol{u}, \boldsymbol{\varphi})$ is linear with respect to $\boldsymbol{\varphi}$, but nonlinear in $\boldsymbol{u}$. As a consequence, the STDGM discrete scheme results in systems of nonlinear algebraic equations. For their solution we apply the Newton method (see [28]), which was applied in, e.g., [48] and [67], where incompressible flow model and conforming finite element discretization were employed.

Let $\boldsymbol{f} : \mathbb{R}^N \to \mathbb{R}^N$. We seek a solution $\boldsymbol{\alpha} \in \mathbb{R}^N$ such that $\boldsymbol{f}(\boldsymbol{\alpha}) = 0$. The Newton method to obtain a solution can be described in the following way: let $\boldsymbol{\alpha}^{(0)}$ be an initial guess of the sought solution and let $\varepsilon > 0$ be a given tolerance. For $i \geq 0$ iterate:

1. Evaluate the residual $\boldsymbol{r}^{(i)} = \boldsymbol{f}\left(\boldsymbol{\alpha}^{(i)}\right)$.

2. If the residual is smaller then the given tolerance: $\left\|\boldsymbol{r}^{(i)}\right\| \leq \varepsilon$, stop iterations and set $\boldsymbol{\alpha} := \boldsymbol{\alpha}^{(i)}$.

3. If the residual is greater then the given tolerance, compute $\delta\boldsymbol{\alpha}$ from a system of linear algebraic equations

$$\nabla_{\boldsymbol{\alpha}} \boldsymbol{f}\left(\boldsymbol{\alpha}^{(i)}\right) \delta\boldsymbol{\alpha} = \boldsymbol{r}^{(i)}. \tag{5.58}$$

4. Update $\boldsymbol{\alpha}^{(i+1)} := \boldsymbol{\alpha}^{(i)} - \delta\boldsymbol{\alpha}$, set $i := i + 1$ and go to step 1.

5.3.2 Application of the Newton method

Now we shall explain the application of the Newton method from Section 5.3.1 to the discretization of the nonlinear elasticity problem.

Let $\psi_i,\ i = 1, \ldots, N = \dim \mathcal{V}$, be a basis of $\mathcal{V}$. The approximate solution $\boldsymbol{u}_{h\tau}$ of our problem can be expressed as a linear combination of basis functions of the space $[\mathcal{V}]^2$:

$$\boldsymbol{u}_{h\tau} = \boldsymbol{u}_{h\tau}(\boldsymbol{\alpha}) = \sum_{i=1}^{2N} \alpha_i \boldsymbol{\phi}_i, \tag{5.59}$$

where $\boldsymbol{\alpha} = (\alpha_i)_{i=1}^{2N}$ are the coefficients and $\boldsymbol{\phi}_i = (\psi_i, 0)$ for $1 \leq i \leq N$ and $\boldsymbol{\phi}_i = (0, \psi_{i-N})$ for $N < i \leq 2N$ form the basis of $[\mathcal{V}]^2$.

That one may apply the Newton method to the discretization of the nonlinear elasticity problem, it is necessary to differentiate the form $a_h^b(\boldsymbol{u}_{h\tau}(\boldsymbol{\alpha}), \boldsymbol{\varphi})$, defined by (5.51), and thereafter the tensor $\boldsymbol{P}(\boldsymbol{F})$ with respect to the coefficients $\boldsymbol{\alpha}$. In what follows, $\nabla_{\boldsymbol{X}}$ and $\nabla_{\boldsymbol{\alpha}}$ will denote the gradient with respect to $\boldsymbol{X}$ and $\boldsymbol{\alpha}$, respectively. We have

$$\begin{aligned}
\frac{\partial}{\partial \alpha_k} \boldsymbol{u}_{h\tau} &= (\psi_i, 0), \quad 1 \leq k \leq N,\ i = k, \\
\frac{\partial}{\partial \alpha_k} \boldsymbol{u}_{h\tau} &= (0, \psi_i), \quad N < k \leq 2N,\ i = k - N,
\end{aligned}$$

and

$$\begin{aligned}
\nabla_{\boldsymbol{X}} \boldsymbol{u}_{h\tau} &= \sum_{i=1}^{N} \alpha_i \nabla_{\boldsymbol{X}} (\psi_i, 0) + \sum_{i=1}^{N} \alpha_{i+N} \nabla_{\boldsymbol{X}} (0, \psi_i) = \\
&= \begin{pmatrix} \sum_{i=1}^{N} \alpha_i \frac{\partial \psi_i}{\partial x_1}, & \sum_{i=1}^{N} \alpha_i \frac{\partial \psi_i}{\partial x_2} \\ \sum_{i=1}^{N} \alpha_{i+N} \frac{\partial \psi_i}{\partial x_1}, & \sum_{i=1}^{N} \alpha_{i+N} \frac{\partial \psi_i}{\partial x_2} \end{pmatrix}.
\end{aligned}$$

By (5.5) and (5.6),

$$\boldsymbol{P}(\boldsymbol{F}) = \boldsymbol{P}(\nabla_{\boldsymbol{X}} \boldsymbol{X} + \nabla_{\boldsymbol{X}} \boldsymbol{u}). \tag{5.60}$$

Taking into account that $\nabla_{\boldsymbol{X}}(\boldsymbol{X})$ is the constant unit matrix $\mathbb{I}$, we introduce the notation

$$\tilde{\boldsymbol{P}}(\nabla_{\boldsymbol{X}}\mathbf{u}) = \mathbf{P}(\mathbb{I} + \nabla_{\mathbf{X}}\mathbf{u}). \tag{5.61}$$

Now the gradient of the form a_h^b can be expressed as

$$\begin{aligned} \nabla_{\boldsymbol{\alpha}} a_h^b(\boldsymbol{u}_{h\tau}(\boldsymbol{\alpha}), \boldsymbol{\varphi}) = &\sum_{K\in\mathcal{T}_h^b} \int_K \nabla_{\boldsymbol{\alpha}} \left(\tilde{\boldsymbol{P}}\left(\nabla_{\boldsymbol{X}}\boldsymbol{u}_{h\tau}(\boldsymbol{\alpha})\right) : \nabla_{\boldsymbol{X}}\boldsymbol{\varphi}\right) \, \mathrm{d}x \\ &- \sum_{\Gamma\in\mathcal{F}_h^{b,I}} \int_\Gamma \nabla_{\boldsymbol{\alpha}} \left(\left\langle \tilde{\boldsymbol{P}}\left(\nabla_{\boldsymbol{X}}\boldsymbol{u}_{h\tau}(\boldsymbol{\alpha})\right)\right\rangle \boldsymbol{n} \cdot [\boldsymbol{\varphi}]\right) \, \mathrm{d}S \\ &+ \sum_{\Gamma\in\mathcal{F}_h^{b,D}} \int_\Gamma \nabla_{\boldsymbol{\alpha}} \left(\tilde{\boldsymbol{P}}\left(\nabla_{\boldsymbol{X}}\boldsymbol{u}_{h\tau}(\boldsymbol{\alpha})\right) \boldsymbol{n} \cdot \boldsymbol{\varphi}\right) \, \mathrm{d}S. \end{aligned} \tag{5.62}$$

Let $\tilde{\boldsymbol{P}}\left(\nabla_{\boldsymbol{X}}\boldsymbol{u}_{h\tau}(\boldsymbol{\alpha})\right) = (P_{ij})_{i,j=1}^2$, where for simplicity we shall not write the dependence of P_{ij} on $\nabla_{\boldsymbol{X}}\boldsymbol{u}_{h\tau}(\boldsymbol{\alpha})$, and let $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)$. From

$$\tilde{\boldsymbol{P}}\left(\nabla_{\boldsymbol{X}}\boldsymbol{u}_{h\tau}(\boldsymbol{\alpha})\right) : \nabla_{\boldsymbol{X}}\boldsymbol{\varphi} = P_{11}\frac{\partial\varphi_1}{\partial x_1} + P_{12}\frac{\partial\varphi_1}{\partial x_2} + P_{21}\frac{\partial\varphi_2}{\partial x_1} + P_{22}\frac{\partial\varphi_2}{\partial x_2}, \tag{5.63}$$

we find that

$$\begin{aligned} \frac{\partial}{\partial\alpha_k}\left(\tilde{\boldsymbol{P}}\left(\nabla_{\boldsymbol{X}}\boldsymbol{u}_{h\tau}(\boldsymbol{\alpha})\right) : \nabla_{\boldsymbol{X}}\boldsymbol{\varphi}\right) &= \frac{\partial}{\partial\alpha_k}P_{11}\frac{\partial\varphi_1}{\partial x_1} + \frac{\partial}{\partial\alpha_k}P_{12}\frac{\partial\varphi_1}{\partial x_2} \\ &+ \frac{\partial}{\partial\alpha_k}P_{21}\frac{\partial\varphi_2}{\partial x_1} + \frac{\partial}{\partial\alpha_k}P_{22}\frac{\partial\varphi_2}{\partial x_2}, \\ \frac{\partial}{\partial\alpha_k}\left(\left\langle\tilde{\boldsymbol{P}}\left(\nabla_{\boldsymbol{X}}\boldsymbol{u}_{h\tau}(\boldsymbol{\alpha})\right)\right\rangle \boldsymbol{n}\cdot[\boldsymbol{\varphi}]\right) &= \left(\frac{\partial}{\partial\alpha_k}\left\langle P_{11}\right\rangle n_1 + \frac{\partial}{\partial\alpha_k}\left\langle P_{12}\right\rangle n_2\right)[\varphi_1] \\ &+ \left(\frac{\partial}{\partial\alpha_k}\left\langle P_{21}\right\rangle n_1 + \frac{\partial}{\partial\alpha_k}\left\langle P_{22}\right\rangle n_2\right)[\varphi_2], \\ \frac{\partial}{\partial\alpha_k}\left(\tilde{\boldsymbol{P}}\left(\nabla_{\boldsymbol{X}}\boldsymbol{u}_{h\tau}(\boldsymbol{\alpha})\right)\boldsymbol{n}\cdot\boldsymbol{\varphi}\right) &= \left(\frac{\partial}{\partial\alpha_k}P_{11}\, n_1 + \frac{\partial}{\partial\alpha_k}P_{12}\, n_2\right)\varphi_1 \\ &+ \left(\frac{\partial}{\partial\alpha_k}P_{21}\, n_1 + \frac{\partial}{\partial\alpha_k}P_{22}\, n_2\right)\varphi_2. \end{aligned}$$

Now for $\boldsymbol{\varphi} = (\psi_i, 0)$ we have

$$\tilde{\boldsymbol{P}}\left(\nabla_{\boldsymbol{X}}\boldsymbol{u}_{h\tau}(\boldsymbol{\alpha})\right) : \nabla_{\boldsymbol{X}}\boldsymbol{\varphi} = P_{11}\frac{\partial\psi_i}{\partial x_1} + P_{12}\frac{\partial\psi_i}{\partial x_2}, \tag{5.64}$$

$$\frac{\partial}{\partial\alpha_k}\left(\tilde{\boldsymbol{P}}\left(\nabla_{\boldsymbol{X}}\boldsymbol{u}_{h\tau}(\boldsymbol{\alpha})\right) : \nabla_{\boldsymbol{X}}\boldsymbol{\varphi}\right) = \frac{\partial}{\partial\alpha_k}P_{11}\frac{\partial\psi_i}{\partial x_1} + \frac{\partial}{\partial\alpha_k}P_{12}\frac{\partial\psi_i}{\partial x_2}, \tag{5.65}$$

$$\frac{\partial}{\partial\alpha_k}\left(\left\langle\tilde{\boldsymbol{P}}\left(\nabla_{\boldsymbol{X}}\boldsymbol{u}_{h\tau}(\boldsymbol{\alpha})\right)\right\rangle \boldsymbol{n}\cdot[\boldsymbol{\varphi}]\right) = \left(\frac{\partial}{\partial\alpha_k}\left\langle P_{11}\right\rangle n_1 + \frac{\partial}{\partial\alpha_k}\left\langle P_{12}\right\rangle n_2\right)[\psi_i], \tag{5.66}$$

$$\frac{\partial}{\partial\alpha_k}\left(\tilde{\boldsymbol{P}}\left(\nabla_{\boldsymbol{X}}\boldsymbol{u}_{h\tau}(\boldsymbol{\alpha})\right)\boldsymbol{n}\cdot\boldsymbol{\varphi}\right) = \left(\frac{\partial}{\partial\alpha_k}P_{11}\, n_1 + \frac{\partial}{\partial\alpha_k}P_{12}\, n_2\right)\psi_i, \tag{5.67}$$

while for $\boldsymbol{\varphi} = (0, \psi_i)$ we get

$$\tilde{\boldsymbol{P}}\left(\nabla_{\boldsymbol{X}} \boldsymbol{u}_{h\tau}(\boldsymbol{\alpha})\right) : \nabla_{\boldsymbol{X}} \boldsymbol{\varphi} = P_{21} \frac{\partial \psi_i}{\partial x_1} + P_{22} \frac{\partial \psi_i}{\partial x_2}, \tag{5.68}$$

$$\frac{\partial}{\partial \alpha_k} \left(\tilde{\boldsymbol{P}}\left(\nabla_{\boldsymbol{X}} \boldsymbol{u}_{h\tau}(\boldsymbol{\alpha})\right) : \nabla_{\boldsymbol{X}} \boldsymbol{\varphi} \right) = \frac{\partial}{\partial \alpha_k} P_{21} \frac{\partial \psi_i}{\partial x_1} + \frac{\partial}{\partial \alpha_k} P_{22} \frac{\partial \psi_i}{\partial x_2}, \tag{5.69}$$

$$\frac{\partial}{\partial \alpha_k} \left(\left\langle \tilde{\boldsymbol{P}}\left(\nabla_{\boldsymbol{X}} \boldsymbol{u}_{h\tau}(\boldsymbol{\alpha})\right) \right\rangle \boldsymbol{n} \cdot [\boldsymbol{\varphi}] \right) = \left(\frac{\partial}{\partial \alpha_k} \langle P_{21} \rangle \, n_1 + \frac{\partial}{\partial \alpha_k} \langle P_{22} \rangle \, n_2 \right) [\psi_i] \,, \tag{5.70}$$

$$\frac{\partial}{\partial \alpha_k} \left(\tilde{\boldsymbol{P}}\left(\nabla_{\boldsymbol{X}} \boldsymbol{u}_{h\tau}(\boldsymbol{\alpha})\right) \boldsymbol{n} \cdot \boldsymbol{\varphi} \right) = \left(\frac{\partial}{\partial \alpha_k} P_{21} \, n_1 + \frac{\partial}{\partial \alpha_k} P_{22} \, n_2 \right) \psi_i. \tag{5.71}$$

It remains to express the derivatives of the tensor $\tilde{\boldsymbol{P}}$ for the neo-Hookean and for the St. Venant-Kirchhoff material.

The Newton method is applied at each time step for the solution of the non-linear discrete problem. Each iteration of the Newton method represents a linear algebraic system and is solved by the direct solver UMFPACK (cf. [27]).

The next two sections 5.3.3 and 5.3.4 follows the work [60].

5.3.3 Neo-Hookean material - derivatives

Let $\tilde{\boldsymbol{P}} = \tilde{\boldsymbol{P}}(\nabla_{\boldsymbol{X}} \boldsymbol{u}_{h\tau}(\boldsymbol{\alpha})) = (P_{ij})_{i,j=1}^2$ be the first Piola-Kirchhoff tensor of the neo-Hookean material, defined in (5.15). Let $\boldsymbol{u}_{h\tau}(\boldsymbol{\alpha}) = (u_1, u_2)$. From (5.6) and (5.15) we get

$$P_{11} = \mu^b \left(1 + \frac{\partial u_1}{\partial x_1}\right) + c_1 \left(1 + \frac{\partial u_2}{\partial x_2}\right), \tag{5.72}$$

$$P_{12} = \mu^b \frac{\partial u_1}{\partial x_2} - c_1 \frac{\partial u_2}{\partial x_1}, \tag{5.73}$$

$$P_{21} = \mu^b \frac{\partial u_2}{\partial x_1} - c_1 \frac{\partial u_1}{\partial x_2}, \tag{5.74}$$

$$P_{22} = \mu^b \left(1 + \frac{\partial u_2}{\partial x_2}\right) + c_1 \left(1 + \frac{\partial u_1}{\partial x_1}\right), \tag{5.75}$$

where

$$c_1 = \frac{\lambda^b \log(\det \boldsymbol{F}) - \mu^b}{\det \boldsymbol{F}}. \tag{5.76}$$

Now let $\boldsymbol{u}_{h\tau}(\boldsymbol{\alpha}) = (u_1, u_2) = \sum_{k=1}^{2N} \alpha_k \boldsymbol{\xi}_k$, where $\boldsymbol{\xi}_k = (\xi_k, 0)$ for $1 \le k \le N$ and $\boldsymbol{\xi}_k = (0, \xi_{k-N})$ for $N < k \le 2N$.

At first we express the derivative of the determinant of $\boldsymbol{F}$ with respect to the coefficient α_k. If $1 \le k \le N$ and $i := k$, then

$$\frac{\partial}{\partial \alpha_k} (\det \boldsymbol{F}) = \frac{\partial \xi_i}{\partial x_1} \left(\frac{\partial u_2}{\partial x_2} + 1 \right) - \frac{\partial \xi_i}{\partial x_2} \frac{\partial u_2}{\partial x_1}, \tag{5.77}$$

and for $N < k \le 2N$, $i := k - N$:

$$\frac{\partial}{\partial \alpha_k} (\det \boldsymbol{F}) = \frac{\partial \xi_i}{\partial x_2} \left(\frac{\partial u_1}{\partial x_1} + 1 \right) - \frac{\partial \xi_i}{\partial x_1} \frac{\partial u_1}{\partial x_2}. \tag{5.78}$$

The derivatives of $\tilde{\boldsymbol{P}}(\nabla_{\boldsymbol{X}} \boldsymbol{u}_{h\tau}(\boldsymbol{\alpha}))$ with respect to the coefficient α_k are given as follows: If $1 \leq k \leq N$ and $i := k$, then

$$\frac{\partial}{\partial \alpha_k} P_{11} = \mu^b \frac{\partial \xi_i}{\partial x_1} + c_2 \frac{\partial}{\partial \alpha_k} (\det \boldsymbol{F}) \left(1 + \frac{\partial u_2}{\partial x_2}\right), \tag{5.79}$$

$$\frac{\partial}{\partial \alpha_k} P_{12} = \mu^b \frac{\partial \xi_i}{\partial x_2} - c_2 \frac{\partial}{\partial \alpha_k} (\det \boldsymbol{F}) \frac{\partial u_2}{\partial x_1}, \tag{5.80}$$

$$\frac{\partial}{\partial \alpha_k} P_{21} = -c_1 \frac{\partial \xi_i}{\partial x_2} - c_2 \frac{\partial}{\partial \alpha_k} (\det \boldsymbol{F}) \frac{\partial u_1}{\partial x_2}, \tag{5.81}$$

$$\frac{\partial}{\partial \alpha_k} P_{22} = c_1 \frac{\partial \xi_i}{\partial x_1} + c_2 \frac{\partial}{\partial \alpha_k} (\det \boldsymbol{F}) \left(1 + \frac{\partial u_1}{\partial x_1}\right), \tag{5.82}$$

where c_1 is as in (5.76),

$$c_2 = \frac{\lambda^b - \lambda^b \log(\det \boldsymbol{F}) + \mu^b}{(\det \boldsymbol{F})^2}, \tag{5.83}$$

and $\frac{\partial}{\partial \alpha_k} (\det \boldsymbol{F})$ is expressed in (5.77).

Finally for $N < k \leq 2N$ we set $i = k - N$ and get

$$\frac{\partial}{\partial \alpha_k} P_{11} = c_1 \frac{\partial \xi_i}{\partial x_2} + c_2 \frac{\partial}{\partial \alpha_k} (\det \boldsymbol{F}) \left(1 + \frac{\partial u_2}{\partial x_2}\right), \tag{5.84}$$

$$\frac{\partial}{\partial \alpha_k} P_{12} = -c_1 \frac{\partial \xi_i}{\partial x_1} - c_2 \frac{\partial}{\partial \alpha_k} (\det \boldsymbol{F}) \frac{\partial u_2}{\partial x_1}, \tag{5.85}$$

$$\frac{\partial}{\partial \alpha_k} P_{21} = \mu^b \frac{\partial \xi_i}{\partial x_1} - c_2 \frac{\partial}{\partial \alpha_k} (\det \boldsymbol{F}) \frac{\partial u_1}{\partial x_2}, \tag{5.86}$$

$$\frac{\partial}{\partial \alpha_k} P_{22} = \mu^b \frac{\partial \xi_i}{\partial x_2} + c_2 \frac{\partial}{\partial \alpha_k} (\det \boldsymbol{F}) \left(1 + \frac{\partial u_1}{\partial x_1}\right), \tag{5.87}$$

where c_1 is as in (5.76), c_2 as in (5.83) and $\frac{\partial}{\partial \alpha_k} (\det \boldsymbol{F})$ is expressed in (5.78).

5.3.4 St. Venant-Kirchhoff material - derivatives

Let $\tilde{\boldsymbol{P}} = \tilde{\boldsymbol{P}}(\nabla_{\boldsymbol{X}} \boldsymbol{u}_{h\tau}(\boldsymbol{\alpha})) = (P_{ij})_{i,j=1}^2$ be the first Piola-Kirchhoff tensor of the St. Venant-Kirchoff material as defined in (5.16) - (5.20). Let $\boldsymbol{u}_{h\tau}(\boldsymbol{\alpha}) = (u_1, u_2)$. Then we get

$$
\begin{aligned}
P_{11} &= \mu^b \frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1}\left(\frac{\partial u_2}{\partial x_2}+1\right) + \frac{\lambda^b}{2}\left(\frac{\partial u_1}{\partial x_1}+1\right)\left(\left(\frac{\partial u_2}{\partial x_2}+1\right)^2-1\right) && (5.88)\\
&\quad + \left(\mu^b + \frac{\lambda^b}{2}\right)\left(\frac{\partial u_1}{\partial x_1}+1\right)\left(\frac{\partial u_1^2}{\partial x_2}+\frac{\partial u_2^2}{\partial x_1}+\left(\frac{\partial u_1}{\partial x_1}+1\right)^2-1\right),\\
P_{12} &= \mu^b\left(\frac{\partial u_1}{\partial x_1}+1\right)\frac{\partial u_2}{\partial x_1}\left(\frac{\partial u_2}{\partial x_2}+1\right)+\frac{\lambda^b}{2}\frac{\partial u_1}{\partial x_2}\left(\frac{\partial u_2^2}{\partial x_1}-1\right) && (5.89)\\
&\quad + \left(\mu^b + \frac{\lambda^b}{2}\right)\frac{\partial u_1}{\partial x_2}\left(\frac{\partial u_1^2}{\partial x_2}+\left(\frac{\partial u_1}{\partial x_1}+1\right)^2+\left(\frac{\partial u_2}{\partial x_2}+1\right)^2-1\right),\\
P_{21} &= \mu^b\left(\frac{\partial u_2}{\partial x_2}+1\right)\frac{\partial u_1}{\partial x_2}\left(\frac{\partial u_1}{\partial x_1}+1\right)+\frac{\lambda^b}{2}\frac{\partial u_2}{\partial x_1}\left(\frac{\partial u_1^2}{\partial x_2}-1\right) && (5.90)\\
&\quad + \left(\mu^b + \frac{\lambda^b}{2}\right)\frac{\partial u_2}{\partial x_1}\left(\frac{\partial u_2^2}{\partial x_1}+\left(\frac{\partial u_1}{\partial x_1}+1\right)^2+\left(\frac{\partial u_2}{\partial x_2}+1\right)^2-1\right),\\
P_{22} &= \mu^b \frac{\partial u_2}{\partial x_1}\frac{\partial u_1}{\partial x_2}\left(\frac{\partial u_1}{\partial x_1}+1\right) + \frac{\lambda^b}{2}\left(\frac{\partial u_2}{\partial x_2}+1\right)\left(\left(\frac{\partial u_1}{\partial x_1}+1\right)^2-1\right) && (5.91)\\
&\quad + \left(\mu^b + \frac{\lambda^b}{2}\right)\left(\frac{\partial u_2}{\partial x_2}+1\right)\left(\frac{\partial u_1^2}{\partial x_2}+\frac{\partial u_2^2}{\partial x_1}+\left(\frac{\partial u_2}{\partial x_2}+1\right)^2-1\right),
\end{aligned}
$$

Now let $\boldsymbol{u}_{h\tau}(\boldsymbol{\alpha}) = (u_1, u_2) = \sum_{k=1}^{2N} \alpha_k \boldsymbol{\xi}_k$, where $\boldsymbol{\xi}_k = (\xi_k, 0)$ for $1 \leq k \leq N$ and $\boldsymbol{\xi}_k = (0, \xi_{k-N})$ for $N < k \leq 2N$.

The derivatives of $\tilde{\boldsymbol{P}}(\nabla_{\boldsymbol{X}} \boldsymbol{u}_{h\tau}(\boldsymbol{\alpha}))$ with respect to the coefficient α_k are given as follows: If $1 \leq k \leq N$ and $i := k$, then

$$
\begin{aligned}
\frac{\partial}{\partial \alpha_k} P_{11} &= \mu^b \frac{\partial \xi_i}{\partial x_2}\frac{\partial u_2}{\partial x_1}\left(\frac{\partial u_2}{\partial x_2}+1\right) + \frac{\lambda^b}{2}\frac{\partial \xi_i}{\partial x_1}\left(\left(\frac{\partial u_2}{\partial x_2}+1\right)^2-1\right) && (5.92)\\
&\quad + \left(\mu^b + \frac{\lambda^b}{2}\right)\frac{\partial \xi_i}{\partial x_1}\left(\frac{\partial u_1^2}{\partial x_2}+\frac{\partial u_2^2}{\partial x_1}+\left(\frac{\partial u_1}{\partial x_1}+1\right)^2-1\right)\\
&\quad + 2\left(\mu^b + \frac{\lambda^b}{2}\right)\left(\frac{\partial u_1}{\partial x_1}+1\right)\left(\frac{\partial u_1}{\partial x_2}\frac{\partial \xi_i}{\partial x_2}+\left(\frac{\partial u_1}{\partial x_1}+1\right)\frac{\partial \xi_i}{\partial x_1}\right),\\
\frac{\partial}{\partial \alpha_k} P_{12} &= \mu^b \frac{\partial \xi_i}{\partial x_1}\frac{\partial u_2}{\partial x_1}\left(\frac{\partial u_2}{\partial x_2}+1\right) + \frac{\lambda^b}{2}\frac{\partial \xi_i}{\partial x_2}\left(\frac{\partial u_2^2}{\partial x_1}-1\right) && (5.93)\\
&\quad + \left(\mu^b + \frac{\lambda^b}{2}\right)\frac{\partial \xi_i}{\partial x_2}\left(\frac{\partial u_1^2}{\partial x_2}+\left(\frac{\partial u_1}{\partial x_1}+1\right)^2+\left(\frac{\partial u_2}{\partial x_2}+1\right)^2-1\right)\\
&\quad + 2\left(\mu^b + \frac{\lambda^b}{2}\right)\frac{\partial u_1}{\partial x_2}\left(\frac{\partial u_1}{\partial x_2}\frac{\partial \xi_i}{\partial x_2}+\left(\frac{\partial u_1}{\partial x_1}+1\right)\frac{\partial \xi_i}{\partial x_1}\right),
\end{aligned}
$$

$$\frac{\partial}{\partial \alpha_k} P_{21} = \mu^b \left(\frac{\partial u_2}{\partial x_2} + 1\right) \frac{\partial \xi_i}{\partial x_2} \left(\frac{\partial u_1}{\partial x_1} + 1\right) + \mu^b \left(\frac{\partial u_2}{\partial x_2} + 1\right) \frac{\partial u_1}{\partial x_2} \frac{\partial \xi_i}{\partial x_1} \tag{5.94}$$
$$+ \lambda^b \frac{\partial u_2}{\partial x_1} \frac{\partial u_1}{\partial x_2} \frac{\partial \xi_i}{\partial x_2} + 2\left(\mu^b + \frac{\lambda^b}{2}\right) \frac{\partial u_2}{\partial x_1} \left(\frac{\partial u_1}{\partial x_1} + 1\right) \frac{\partial \xi_i}{\partial x_1},$$

$$\frac{\partial}{\partial \alpha_k} P_{22} = \mu^b \frac{\partial u_2}{\partial x_1} \frac{\partial \xi_i}{\partial x_2} \left(\frac{\partial u_1}{\partial x_1} + 1\right) + \mu^b \frac{\partial u_2}{\partial x_1} \frac{\partial u_1}{\partial x_2} \frac{\partial \xi_i}{\partial x_1} \tag{5.95}$$
$$+ \lambda^b \left(\frac{\partial u_2}{\partial x_2} + 1\right) \left(\frac{\partial u_1}{\partial x_1} + 1\right) \frac{\partial \xi_i}{\partial x_1} + 2\left(\mu^b + \frac{\lambda^b}{2}\right) \left(\frac{\partial u_2}{\partial x_2} + 1\right) \frac{\partial u_1}{\partial x_2} \frac{\partial \xi_i}{\partial x_2}.$$

Finally for $N < k \leq 2N$ we set $i = k - N$ and get

$$\frac{\partial}{\partial \alpha_k} P_{11} = \mu^b \frac{\partial u_1}{\partial x_2} \frac{\partial \xi_i}{\partial x_1} \left(\frac{\partial u_2}{\partial x_2} + 1\right) + \mu^b \frac{\partial u_1}{\partial x_2} \frac{\partial u_2}{\partial x_1} \frac{\partial \xi_i}{\partial x_2} \tag{5.96}$$
$$+ \lambda^b \left(\frac{\partial u_1}{\partial x_1} + 1\right) \left(\frac{\partial u_2}{\partial x_2} + 1\right) \frac{\partial \xi_i}{\partial x_2} + 2\left(\mu^b + \frac{\lambda^b}{2}\right) \left(\frac{\partial u_1}{\partial x_1} + 1\right) \frac{\partial u_2}{\partial x_1} \frac{\partial \xi_i}{\partial x_1},$$

$$\frac{\partial}{\partial \alpha_k} P_{12} = \mu^b \left(\frac{\partial u_1}{\partial x_1} + 1\right) \frac{\partial \xi_i}{\partial x_1} \left(\frac{\partial u_2}{\partial x_2} + 1\right) + \mu^b \left(\frac{\partial u_1}{\partial x_1} + 1\right) \frac{\partial u_2}{\partial x_1} \frac{\partial \xi_i}{\partial x_2} \tag{5.97}$$
$$+ \lambda^b \frac{\partial u_1}{\partial x_2} \frac{\partial u_2}{\partial x_1} \frac{\partial \xi_i}{\partial x_1} + 2\left(\mu^b + \frac{\lambda^b}{2}\right) \frac{\partial u_1}{\partial x_2} \left(\frac{\partial u_2}{\partial x_2} + 1\right) \frac{\partial \xi_i}{\partial x_2},$$

$$\frac{\partial}{\partial \alpha_k} P_{21} = \mu^b \frac{\partial \xi_i}{\partial x_2} \frac{\partial u_1}{\partial x_2} \left(\frac{\partial u_1}{\partial x_1} + 1\right) + \frac{\lambda^b}{2} \frac{\partial \xi_i}{\partial x_1} \left(\frac{\partial u_1^2}{\partial x_2} - 1\right) \tag{5.98}$$
$$+ \left(\mu^b + \frac{\lambda^b}{2}\right) \frac{\partial \xi_i}{\partial x_1} \left(\frac{\partial u_2^2}{\partial x_1} + \left(\frac{\partial u_1}{\partial x_1} + 1\right)^2 + \left(\frac{\partial u_2}{\partial x_2} + 1\right)^2 - 1\right)$$
$$+ 2\left(\mu^b + \frac{\lambda^b}{2}\right) \frac{\partial u_2}{\partial x_1} \left(\frac{\partial u_2}{\partial x_1} \frac{\partial \xi_i}{\partial x_1} + \left(\frac{\partial u_2}{\partial x_2} + 1\right) \frac{\partial \xi_i}{\partial x_2}\right),$$

$$\frac{\partial}{\partial \alpha_k} P_{22} = \mu^b \frac{\partial \xi_i}{\partial x_1} \frac{\partial u_1}{\partial x_2} \left(\frac{\partial u_1}{\partial x_1} + 1\right) + \frac{\lambda^b}{2} \frac{\partial \xi_i}{\partial x_2} \left(\left(\frac{\partial u_1}{\partial x_1} + 1\right)^2 - 1\right) \tag{5.99}$$
$$+ \left(\mu^b + \frac{\lambda^b}{2}\right) \frac{\partial \xi_i}{\partial x_2} \left(\frac{\partial u_1^2}{\partial x_2} + \frac{\partial u_2^2}{\partial x_1} + \left(\frac{\partial u_2}{\partial x_2} + 1\right)^2 - 1\right)$$
$$+ 2\left(\mu^b + \frac{\lambda^b}{2}\right) \left(\frac{\partial u_2}{\partial x_2} + 1\right) \left(\frac{\partial u_2}{\partial x_1} \frac{\partial \xi_i}{\partial x_1} + \left(\frac{\partial u_2}{\partial x_2} + 1\right) \frac{\partial \xi_i}{\partial x_2}\right).$$

5.4 Numerical experiments

Now we present numerical results to demonstrate the performance of the proposed ALE-STDGM. Section 5.4.1 is devoted to the investigation of the Turek-Hron nonlinear elasticity benchmark problem [75]. Here the STDGM is applied to solve the motion of an elastic beam. Finally, in Section 5.4.2 the main attention is paid to fluid-structure interaction, i.e. the modeling of flow induced vocal fold vibrations in a simplified human vocal tract.

5.4.1 Nonlinear elasticity benchmark problem

We consider a 2D domain of an elastic beam attached to a rigid cylinder with a radius $r = 0.05$ m. The beam is $l = 0.35$ m long and $h = 0.02$ m high. We will evaluate the displacement of the point $A = A(t)$, which is defined in the middle of the right-hand side end of the beam, see Figure 5.1. In the Turek-Hron benchmark problem [75] the elastic beam is modelled with the aid of the St. Venant-Kirchhoff material. In our computations we consider the neo-Hookean material as well.

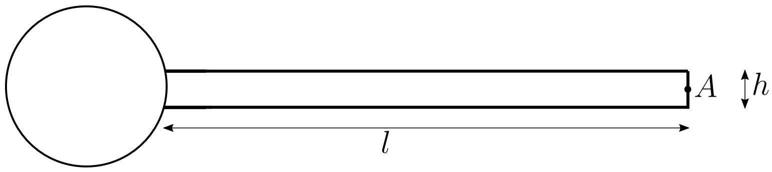

Figure 5.1: Setup of the benchmark problem: elastic beam attached to a rigid cylinder.

The domain Ω^b defined in the previous sections represents the elastic beam. Homogenous Dirichlet boundary condition is prescribed on the part of the boundary, where the beam is attached to the rigid cylinder

$$\boldsymbol{u}_D = \boldsymbol{0} \quad \text{on} \quad \Gamma_D^b \times [0, T],$$

and on the rest of the boundary we prescribe Neumann boundary condition with no surface traction

$$\boldsymbol{g}_N = \boldsymbol{0} \quad \text{on} \quad \Gamma_N^b \times [0, T],$$

The initial condition for the time-dependent problem is given by

$$\boldsymbol{u}_0 = \boldsymbol{0}, \quad \boldsymbol{z}_0 = \boldsymbol{0}, \quad \text{in} \quad \Omega^b.$$

We also prescribe the acting body force density $\boldsymbol{f}$ by

$$\boldsymbol{f} = \rho^b \boldsymbol{b}, \quad \text{where} \quad \boldsymbol{b} = (0, -2)^T \; [\mathrm{m\,s^{-2}}], \quad \rho^b = 1000 \; [\mathrm{kg\,m^{-3}}].$$

We set the damping coefficient $C_M^b = 0$, Young's modulus $E^b = 1.4 \cdot 10^6$ and Poisson ratio $\nu^b = 0.4$. The Lamé parameters are determined by relations (5.14).

In our computations we used three different computational meshes, see Figure 5.2, generated by the finite element grid generator Gmsh. Characteristics of these meshes are summarized in Table 5.1.

The nonlinear benchmark problem was solved by the proposed STDGM (5.55)-(5.57) in the space of piecewise polynomial vector functions $\boldsymbol{S}_{h\tau}^{b,sq^*}$ defined by (5.54). We used piecewise linear approximation in space ($s = 1$) and piecewise constant ($q^* = 0$), linear ($q^* = 1$) and quadratic ($q^* = 2$) approximation in time with a constant time step τ. For all computations we set $C_W^b = 6 \cdot 10^6$.

In our computations we compare the time-dependent values of the displacement of the point A, which are represented by the mean value, amplitude and

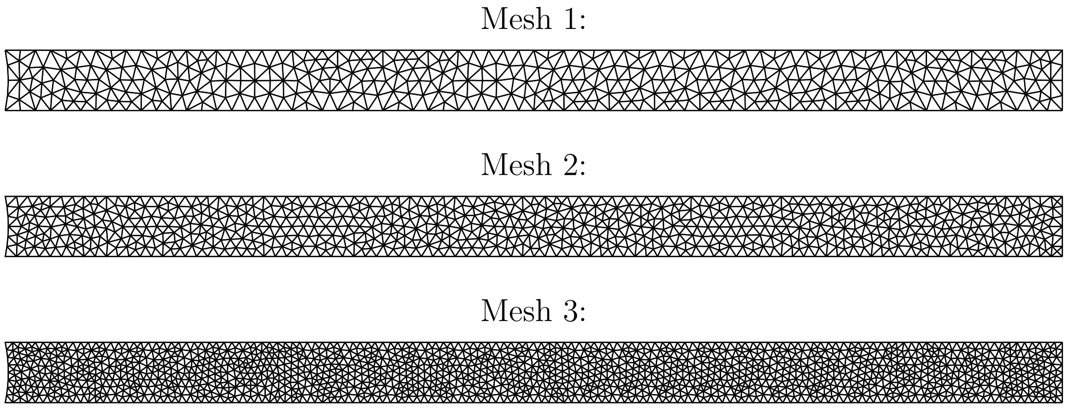

Figure 5.2: Triangular computational meshes for the benchmark problem.

	number of elements	mesh size [$\times 10^{-3}$]
Mesh 1	722	7.31
Mesh 2	1348	5.17
Mesh 3	2822	3.40

Table 5.1: Mesh statistics for the nonlinear elasticity benchmark problem.

frequency. The mean values and amplitudes are computed from the last period of the oscillations by taking the maximum (max) and minimum (min) values. Then mean value $= 1/2(\max + \min)$, and amplitude $= 1/2(\max - \min)$. The frequency of the oscillations is computed by the fast Fourier transform (FFT) taking the lowest significant frequency present in the spectrum. The data denoted by "ref" represent results from [75] for the St. Venant-Kirchhoff material. For the neo-Hookean material there are not available any reference results.

In Tables 5.2, 5.3 and 5.4 we summarize results obtained for the St. Venant-Kirchhoff material for different time steps τ on Mesh 1. It can be seen, that in case of piecewise constant time approximation ($q^* = 0$) we need even smaller time step to obtain satisfactory results. On the other hand results obtained by the piecewise linear ($q^* = 1$) and quadratic ($q^* = 2$) approximation in time show a very good agreement with computations from [75] for all time steps.

The evolution of the displacement of the point A for the St. Venant-Kirchhoff material for different time steps is shown in Figures 5.3, 5.4 and 5.5, for piecewise constant ($q^* = 0$), linear ($q^* = 1$) and quadratic ($q^* = 2$) approximation in time, respectively. It can be seen that in case of $q^* = 1$ and especially in case of $q^* = 2$ the results for different time steps τ are almost identical.

Finally in Table 5.5 we compare results for different computational meshes obtained for the St. Venant-Kirchhoff material with piecewise linear approximation in space and time for the time step $\tau = 0.02$.

method	τ	$u_1\,[\times 10^{-3}]$		$u_2\,[\times 10^{-3}]$	
ref		-14.305 ± 14.305	[1.0995]	-63.607 ± 65.160	[1.0995]
STDGM	0.04	-7.203 ± 0.002	[1.0712]	-66.214 ± 0.011	[1.0725]
STDGM	0.02	-7.186 ± 0.175	[1.0800]	-66.130 ± 0.789	[1.0775]
STDGM	0.01	-7.200 ± 1.564	[1.0887]	-65.705 ± 7.079	[1.0862]
STDGM	0.005	-7.840 ± 4.708	[1.0920]	-65.409 ± 21.393	[1.0900]

Table 5.2: Comparison of the displacement of the point A for STDGM with $s = 1$, $q^* = 0$, St. Venant-Kirchhoff material and different time steps τ. The values are written in the format "*mean value* ± *amplitude [frequency]*".

method	τ	$u_1\,[\times 10^{-3}]$		$u_2\,[\times 10^{-3}]$	
ref		-14.305 ± 14.305	[1.0995]	-63.607 ± 65.160	[1.0995]
STDGM	0.04	-14.072 ± 14.043	[1.0925]	-66.374 ± 61.499	[1.0925]
STDGM	0.02	-14.337 ± 14.316	[1.0925]	-66.456 ± 62.556	[1.0925]
STDGM	0.01	-14.546 ± 14.526	[1.0950]	-66.580 ± 62.994	[1.0950]
STDGM	0.005	-14.628 ± 14.608	[1.0930]	-66.623 ± 63.153	[1.0930]

Table 5.3: Comparison of the displacement of the point A for STDGM with $s = 1$, $q^* = 1$, St. Venant-Kirchhoff material and different time steps τ. The values are written in the format "*mean value* ± *amplitude [frequency]*".

method	τ	$u_1\,[\times 10^{-3}]$		$u_2\,[\times 10^{-3}]$	
ref		-14.305 ± 14.305	[1.0995]	-63.607 ± 65.160	[1.0995]
STDGM	0.04	-14.497 ± 14.497	[1.0925]	-64.743 ± 64.748	[1.0925]
STDGM	0.02	-14.627 ± 14.627	[1.0925]	-65.088 ± 64.711	[1.0925]
STDGM	0.01	-14.672 ± 14.672	[1.0950]	-64.879 ± 65.025	[1.0900]

Table 5.4: Comparison of the displacement of the point A for STDGM with $s = 1$, $q^* = 2$, St. Venant-Kirchhoff material and different time steps τ. The values are written in the format "*mean value* ± *amplitude [frequency]*".

# elements	τ	$u_1\,[\times 10^{-3}]$		$u_2\,[\times 10^{-3}]$	
ref		-14.305 ± 14.305	[1.0995]	-63.607 ± 65.160	[1.0995]
722	0.02	-14.337 ± 14.316	[1.0925]	-66.456 ± 62.556	[1.0925]
1348	0.02	-14.117 ± 14.112	[1.0962]	-64.508 ± 63.514	[1.0962]
2822	0.02	-14.113 ± 14.110	[1.0962]	-64.523 ± 63.518	[1.0962]

Table 5.5: Comparison of the displacement of the point A for STDGM with $s = 1$, $q^* = 1$ for St. Venant-Kirchhoff material and different meshes. The values are written in the format "*mean value* ± *amplitude [frequency]*".

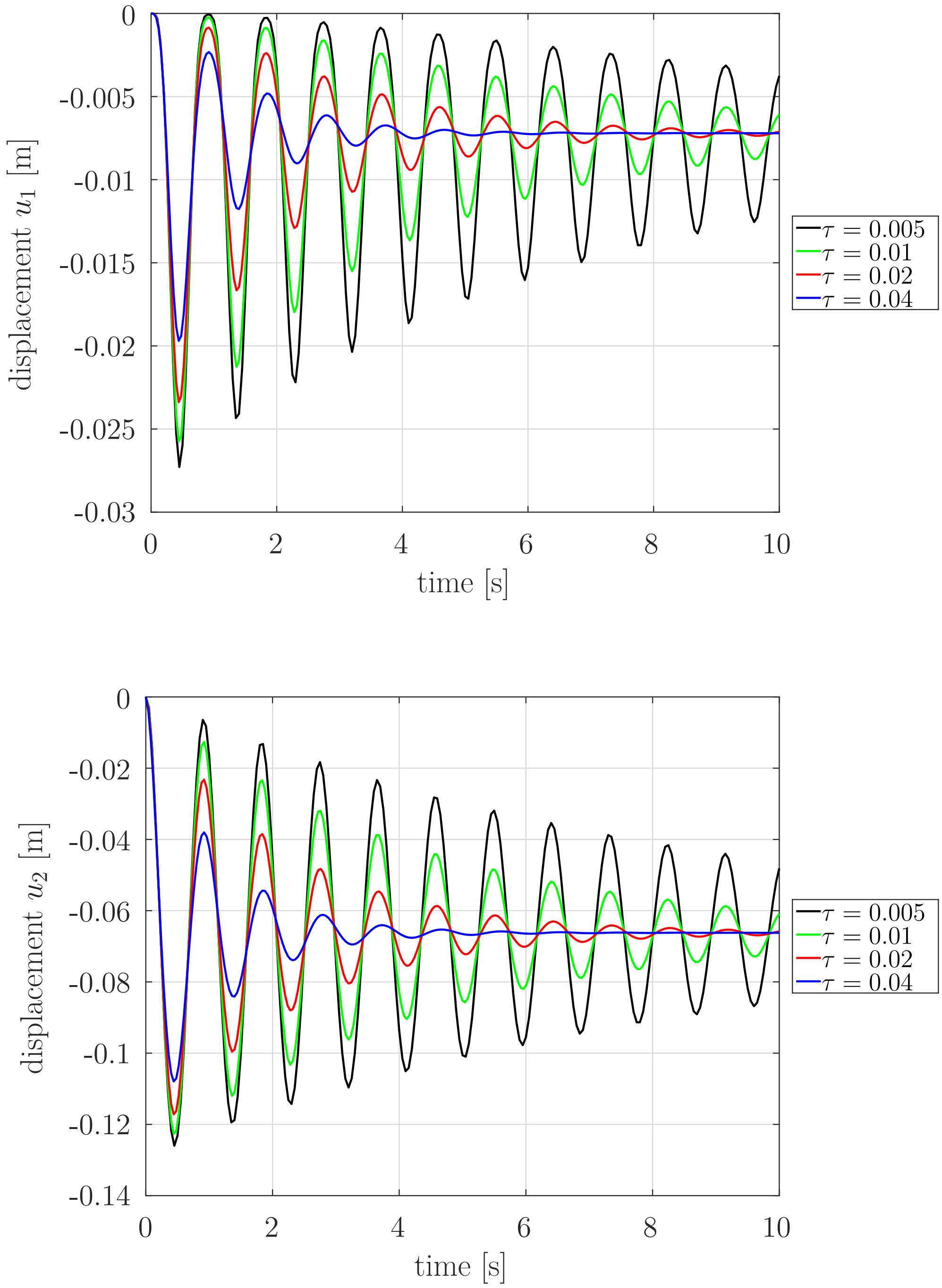

Figure 5.3: St. Venant-Kirchhoff material - displacement of the point A for STDGM with $s = 1$, $q^* = 0$ for different time steps τ.

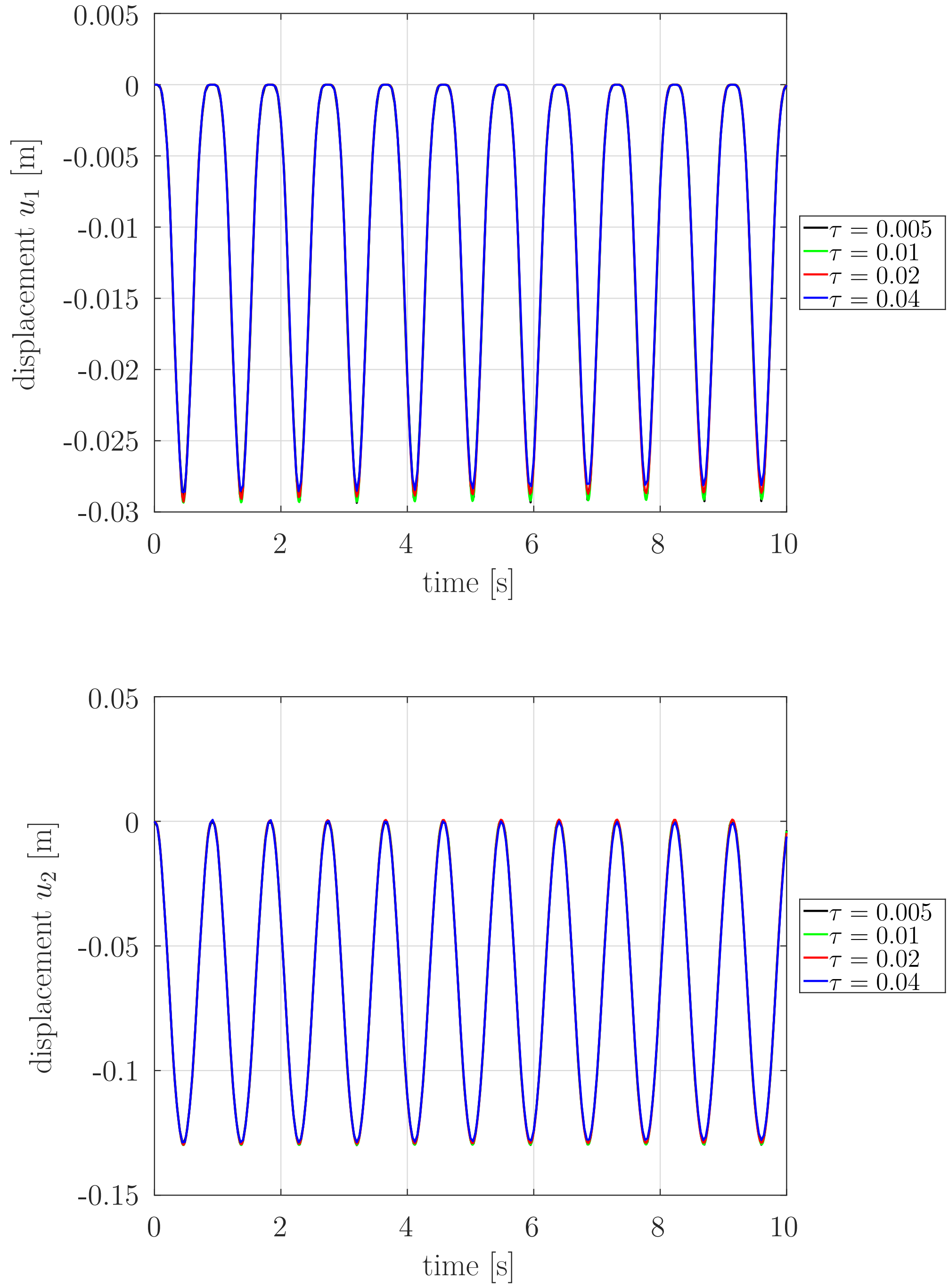

Figure 5.4: St. Venant-Kirchhoff material - displacement of the point A for STDGM with $s = 1$, $q^* = 1$ for different time steps τ.

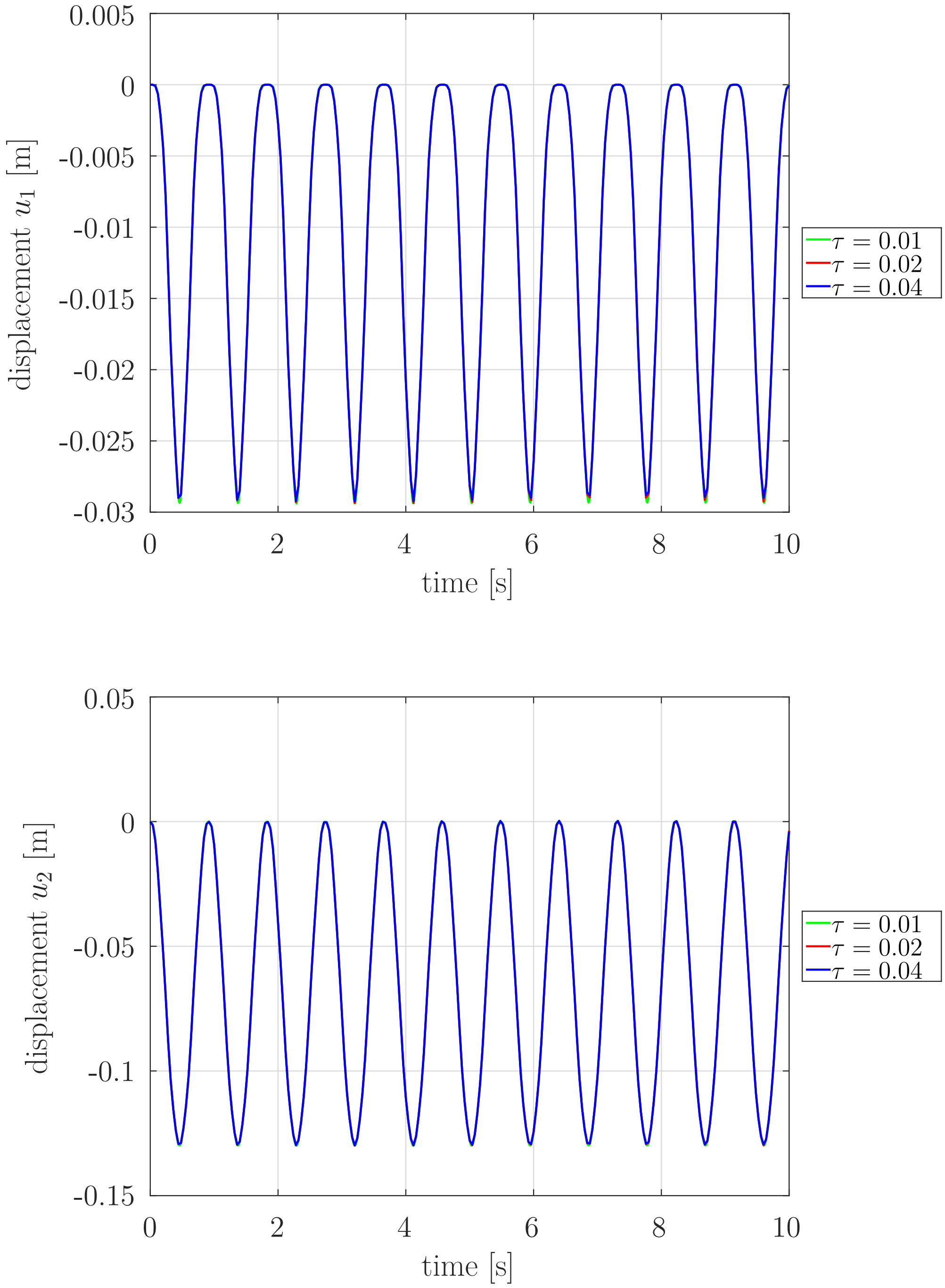

Figure 5.5: St. Venant-Kirchhoff material - displacement of the point A for STDGM with $s = 1$, $q^* = 2$ for different time steps τ.

In Tables 5.6, 5.7 and 5.8 we summarize results obtained for the neo-Hookean material for different time steps τ on Mesh 1. In this case we do not have reference data, so we can compare these results only by the results for the St. Venant-Kirchhoff material.

The evolution of the displacement of the point A for the neo-Hookean material for different time steps is shown in Figure 5.6, 5.7 and 5.8, for piecewise constant ($q^* = 0$), linear ($q^* = 1$) and quadratic ($q^* = 2$) approximation in time, respectively. It can be seen that in case of $q^* = 1$ and especially in case of $q^* = 2$ the results for different time steps τ are almost identical. In all cases we get very similar computational results as for the St. Venant-Kirchhoff material.

method	τ	$u_1\,[\times 10^{-3}]$		$u_2\,[\times 10^{-3}]$	
STDGM	0.04	-7.176 ± 0.002	[1.0712]	-66.209 ± 0.011	[1.0725]
STDGM	0.02	-7.164 ± 0.174	[1.0800]	-66.149 ± 0.788	[1.0775]
STDGM	0.01	-7.174 ± 1.558	[1.0887]	-65.798 ± 7.078	[1.0862]
STDGM	0.005	-7.813 ± 4.690	[1.0920]	-65.414 ± 21.387	[1.0900]

Table 5.6: Comparison of the displacement of the point A for STDGM with $s = 1,\ q^* = 0$, neo-Hookean material and different time steps τ. The values are written in the format "*mean value* ± *amplitude [frequency]*".

method	τ	$u_1\,[\times 10^{-3}]$		$u_2\,[\times 10^{-3}]$	
STDGM	0.04	-14.027 ± 13.992	[1.0937]	-66.625 ± 61.263	[1.0925]
STDGM	0.02	-14.290 ± 14.264	[1.0937]	-66.710 ± 62.311	[1.0925]
STDGM	0.01	-14.505 ± 14.480	[1.0937]	-66.824 ± 62.277	[1.0925]
STDGM	0.005	-14.590 ± 14.566	[1.0930]	-66.863 ± 62.944	[1.0930]

Table 5.7: Comparison of the displacement of the point A for STDGM with $s = 1,\ q^* = 1$, neo-Hookean material and different time steps τ. The values are written in the format "*mean value* ± *amplitude [frequency]*".

method	τ	$u_1\,[\times 10^{-3}]$		$u_2\,[\times 10^{-3}]$	
STDGM	0.04	-14.454 ± 14.454	[1.0937]	-64.829 ± 64.686	[1.0925]
STDGM	0.02	-14.587 ± 14.587	[1.0937]	-65.172 ± 64.657	[1.0925]
STDGM	0.01	-14.596 ± 14.595	[1.0937]	-65.227 ± 64.634	[1.0925]

Table 5.8: Comparison of the displacement of the point A for STDGM with $s = 1,\ q^* = 2$, neo-Hookean material and different time steps τ. The values are written in the format "*mean value* ± *amplitude [frequency]*".

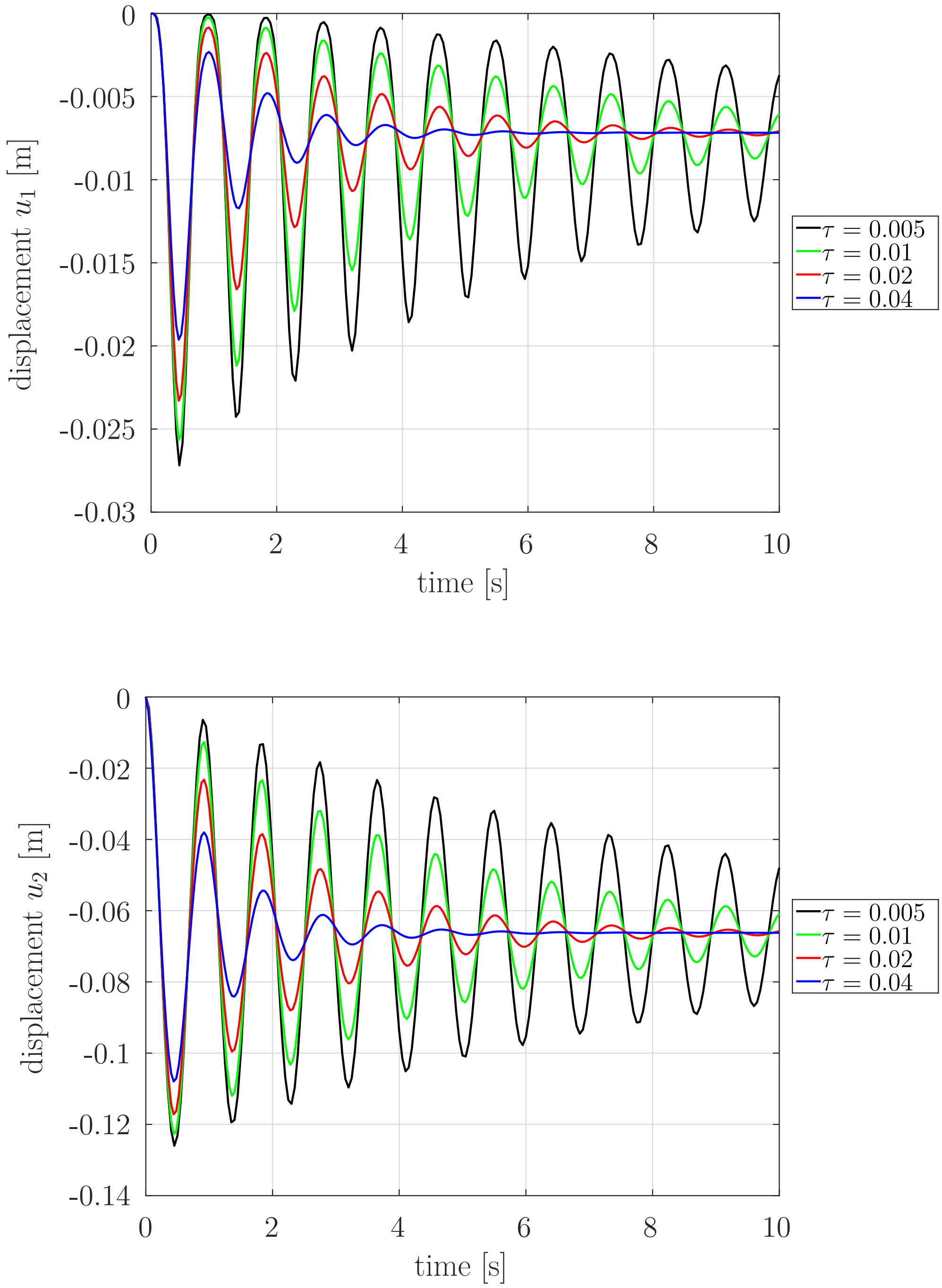

Figure 5.6: Neo-Hookean material - displacement of the point A for STDGM with $s = 1$, $q^* = 0$ for different time steps τ.

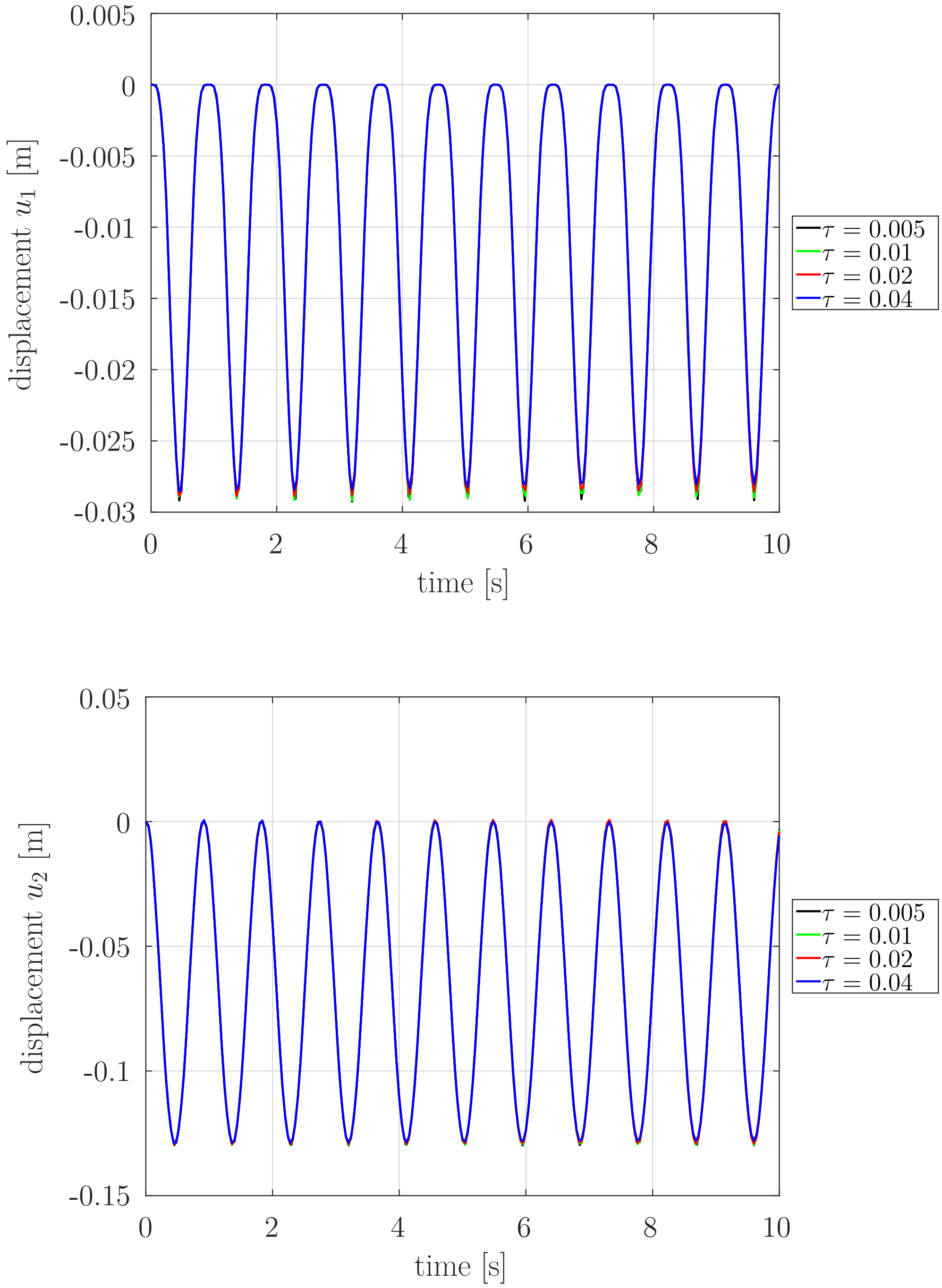

Figure 5.7: Neo-Hookean material - displacement of the point A for STDGM with $s = 1$, $q^* = 1$ for different time steps τ.

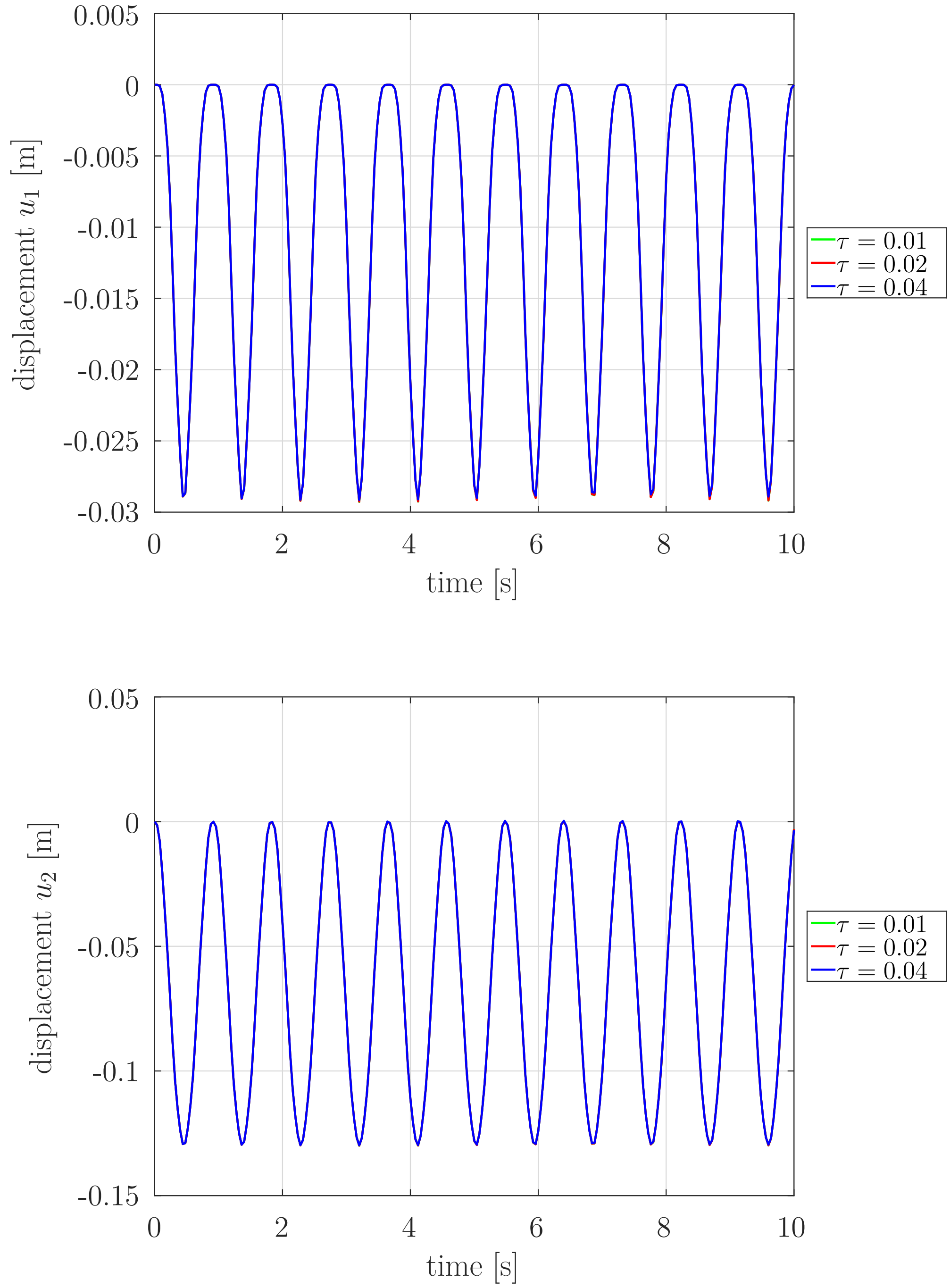

Figure 5.8: Neo-Hookean material - displacement of the point A for STDGM with $s = 1$, $q^* = 2$ for different time steps τ.

5.4.2 Flow induced vocal folds vibrations

Now we present our numerical results for a model of vocal folds in a simplified human vocal tract. The geometry of the domain occupied by the fluid and its size are given in Figure 5.9. On the right-hand side of the geometry a semicircle subdomain with a radius 3.0 cm is added - it represents the outlet Γ_O.

We prescribe the inlet boundary conditions on Γ_I (left part of the bound-

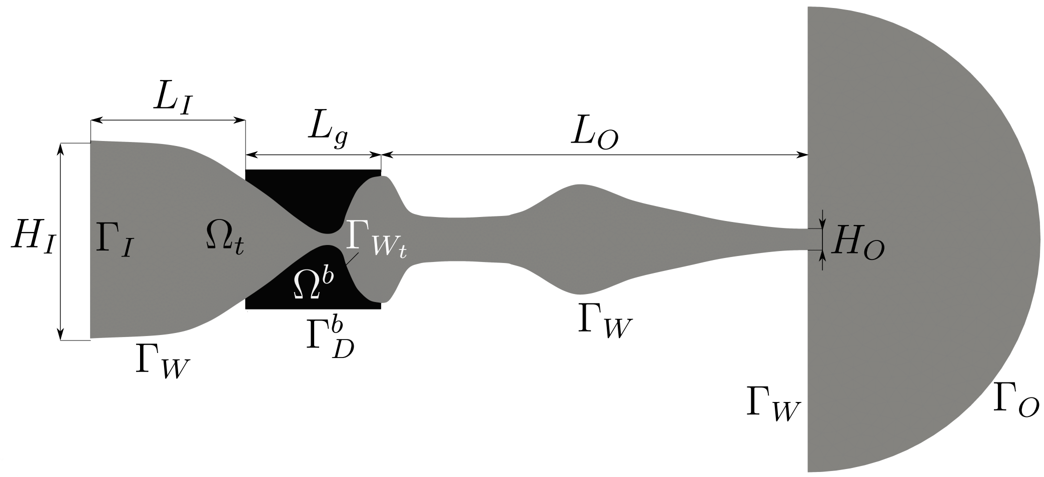

Figure 5.9: Geometry of the computational domain at time $t = 0$ and the description of its size: $L_I = 20.0$ mm, $L_g = 17.5$ mm, $L_O = 55.0$ mm, $H_I = 25.5$ mm, $H_O = 2.76$ mm. The radius of the semicircle subdomain is 3.0 cm.

ary), the outlet boundary conditions on Γ_O (right part of the boundary, which is a semicircle), and we prescribe boundary conditions on the impermeable fixed walls Γ_W (including the vertical segments of the semicircle) and on the moving impermeable walls denoted in Figure 5.9 by Γ_{W_t}. The fluid flow problem is computed on the triangulation with 17652 elements. Further, for the definition of the fluid flow problem the following data are used:

magnitude of the inlet velocity	$v_{in} = 4 \text{ m s}^{-1}$,
dynamic viscosity	$\mu = 1.80 \cdot 10^{-5} \text{ kg m}^{-1} \text{ s}^{-1}$,
inlet density	$\rho_{in} = 1.225 \text{ kg m}^{-3}$,
outlet pressure	$p_{out} = 97611$ Pa,
Reynolds number	$Re = \rho_{in} v_{in} H_I / \mu = 6941.7$,
heat conduction coefficient	$\kappa = 2.428 \cdot 10^{-2} \text{ kg m s}^{-3} \text{ K}^{-1}$,
specific heat	$c_v = 721.428 \text{ m}^2 \text{ s}^{-2} \text{ K}^{-1}$,
Poisson adiabatic constant	$\gamma = 1.4$.

For the fluid solver we use the STDGM with polynomial approximation of degree 2 in space and degree 1 in time. We employ the IIPG version of the DGM with the penalization constant $C_W = 500$ for inner faces and $C_W = 5000$ for boundary edges. The stabilization parameters ν_1 and ν_2 from (5.39) are set to 0.1. The time step τ is set to $1.0 \cdot 10^{-6}$ s. For the first 1000 time steps the fluid flow is computed with the fixed boundary. Then the part Γ_{W_t} of the boundary is released and we solve the FSI problem.

We assume that the elastic bodies motivated by a cut of vocal folds are isotropic with constant material density $\rho^b = 1040 \text{ kg m}^{-3}$. The triangulation used for the solution of the structure problem has 5118 elements, see Figure 5.10. The division of the domain into 4 regions with different material characteristics is

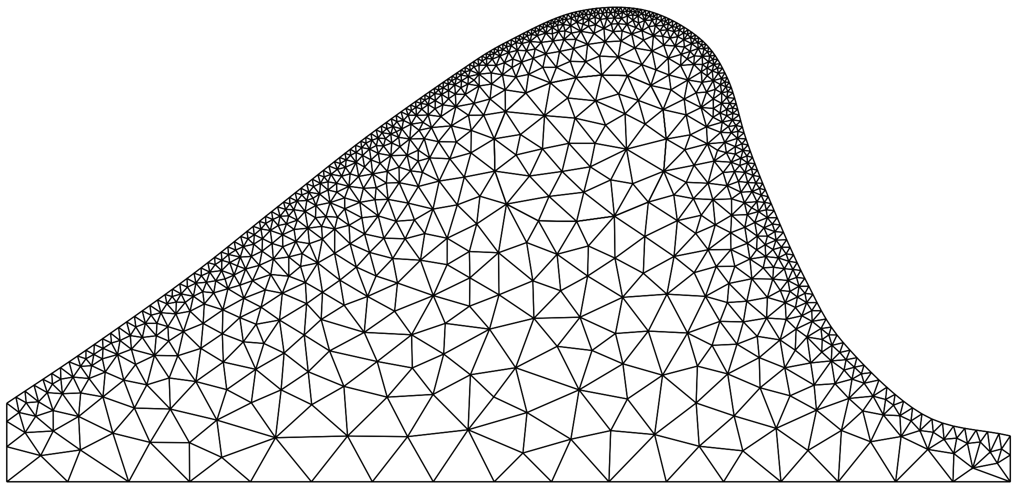

Figure 5.10: Computational mesh of vocal folds

illustrated in Figure 5.11 by the Lamé parameters and the setting of the material characteristics is described in Table 5.9.

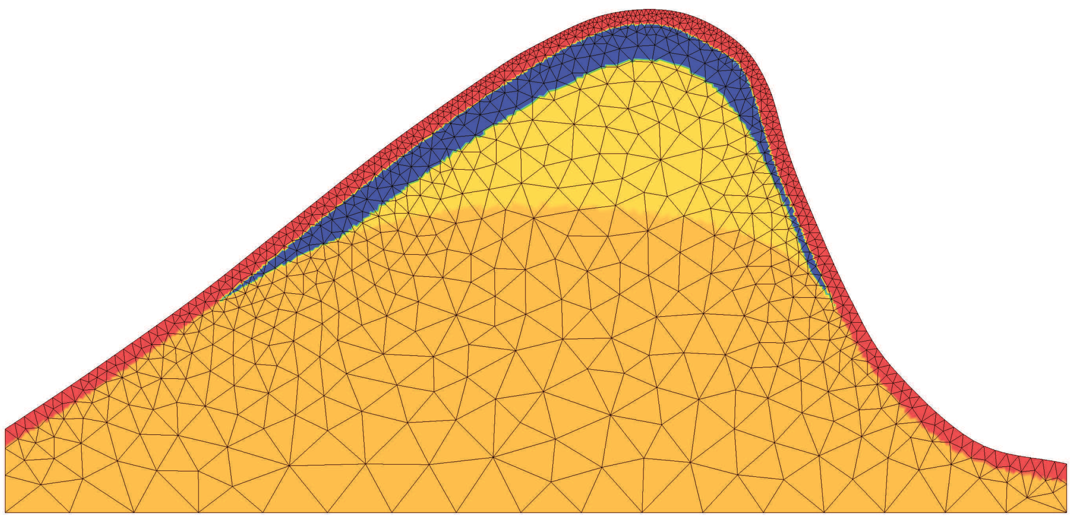

Figure 5.11: Nonhomogeneous model of vocal folds - layers with different Lamé parameters.

Further, the initial displacement and the initial deformation velocity are set to be zero. On the bottom, right and left straight parts of the boundary we prescribe homogeneous Dirichlet boundary condition (5.8) and on the curved part of the boundary the Neumann boundary condition (5.9). The damping coefficient c_M^b is set to 1.0 s^{-1}. For the solution of the dynamic elasticity problem we employ the NIPG version of the DGM, where the penalization constant is set to $C_W^b = 4 \cdot 10^6$.

The ALE mapping is determined as described in Section 5.1.4. For the solution of the static elasticity problem (5.24) we employ the NIPG version of the DGM,

layer	E^b	ν^b	λ^b	μ^b
1. layer (orange)	$12 \cdot 10^3$	0.4	17143	4285
2. layer (yellow)	$8 \cdot 10^3$	0.4	11430	2857
3. layer (blue)	$1 \cdot 10^3$	0.495	33110	335
4. layer (red)	$100 \cdot 10^3$	0.4	142857	35714

Table 5.9: Nonhomogeneous model of vocal folds - prescribed Young modulus, Poisson ratio and Lamé parameters for different layers, ordered from the lower layer to the upper layer. See Figure 5.11 for the visualization of the corresponding subdomains.

where the penalization constant is set to $C_W^b = 10^3$. Then the DG solution of the ALE discrete problem (5.24) is interpolated to a continuous approximation.

We use the strong coupling algorithm described in Section 5.2.3 with the prescribed tolerance 10^{-5}. Further, we use 5 coupling subiterations as the maximum, however the prescribed tolerance was usually reached after $2 - 3$ coupling subiterations.

In what follows we compare the linear strain tensor $\boldsymbol{e}$ and the nonlinear Green strain tensor $\boldsymbol{E} \in \mathbb{R}^{2\times 2}$, see [25], defined by (5.17) - (5.18).

In the case of the linear elasticity the stress tensor depends on the strain tensor $\boldsymbol{e} = (e_{ij})_{i,j=1}^2$ and in the case of nonlinear elasticity it depends on $\boldsymbol{E} = \boldsymbol{e} + \boldsymbol{E}^*$, where $\boldsymbol{E}^* = (E_{ij}^*)_{i,j=1}^2$.

The influence of the nonlinear part of the strain tensor is given by the ratio

$$R := \frac{\|\boldsymbol{e}\|}{\|\boldsymbol{E}\|} = \frac{\|\boldsymbol{e}\|}{\|\boldsymbol{e} + \boldsymbol{E}^*\|}. \tag{5.100}$$

If $R \approx 1$, then the nonlinear part of the strain tensor has no influence to the computation (the linear elasticity model is sufficient), but if $R \approx 0$, then the nonlinear part strongly takes effect and it is necessary to use a nonlinear elasticity model.

Comparing the linear and the neo-Hookean nonlinear elasticity model

Figure 5.12 shows numerical simulation of the vocal folds from the beginning of the FSI computation at 12 time instants. Figure 5.13 shows in detail the deformation of the vocal folds at 2 time instants for a maximal and minimal glottal gap during vocal folds oscillations. In Figure 5.12 and Figure 5.13 case $R \approx 1$ is depicted by white and case $R \approx 0$ by dark red color. It can be seen, that nonlinear part of the strain tensor takes effect in elements near to the boundary, therefore to correctly capture deformations of the vocal folds, it is necessary to use a nonlinear model of elasticity.

Comparing the linear and the St. Venant-Kirchhoff nonlinear elasticity model

Figure 5.14 shows numerical simulation of the vocal folds from the beginning of the FSI computation at 12 time instants. Figure 5.15 shows in detail the deformation of the vocal folds at 2 time instants for a maximal and minimal

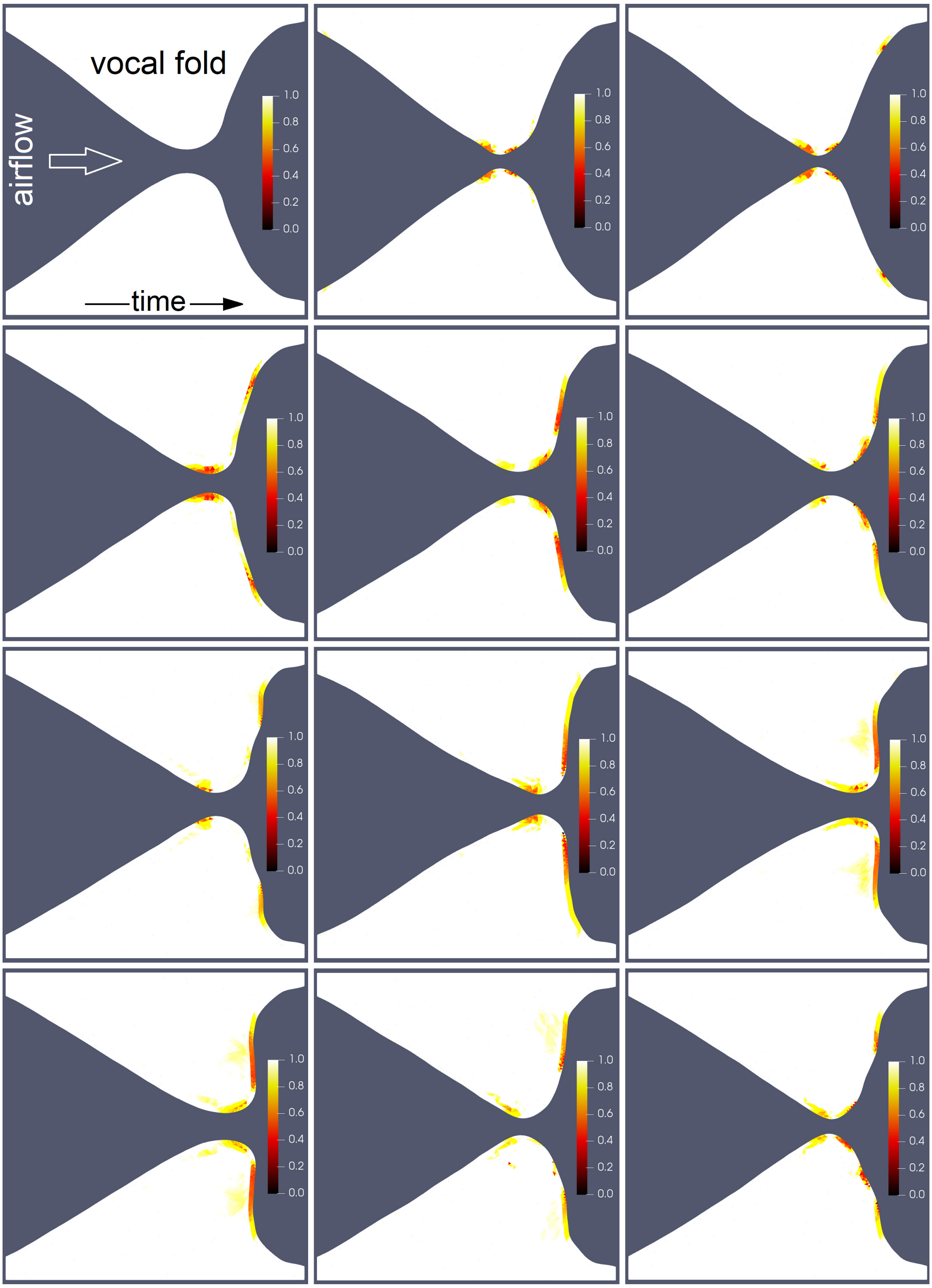

Figure 5.12: Deformation of vocal folds in dependence on time computed by the neo-Hookean model and the ratios of the norms of the linear strain tensor and the nonlinear Green strain tensor at different time instants.

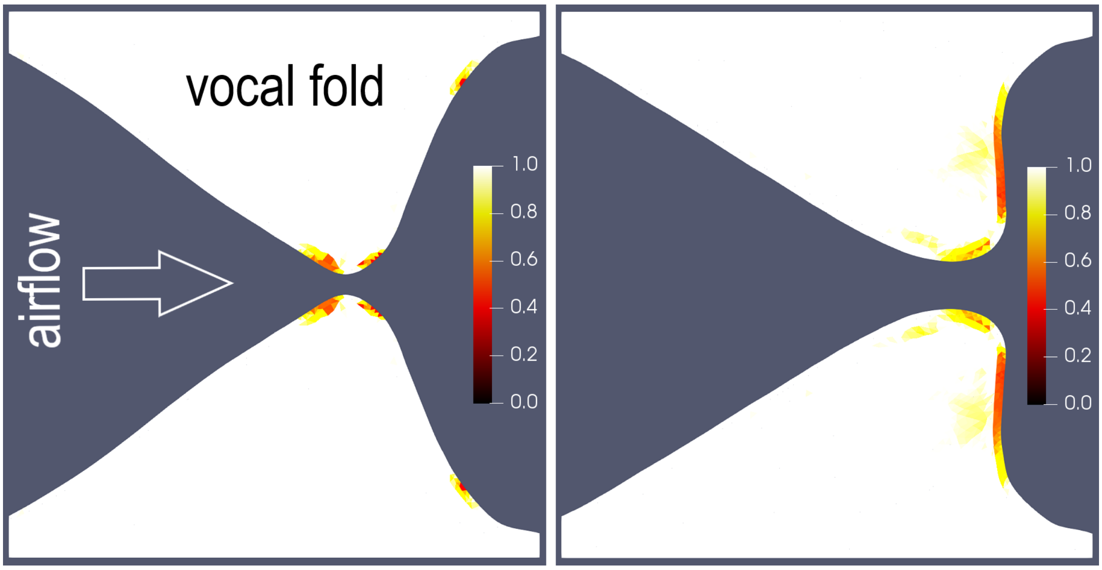

Figure 5.13: Deformation of vocal folds in dependence on time computed by the neo-Hookean model and the ratios of the norms of the linear strain tensor and the nonlinear Green strain tensor at different time instants - details for the smallest and the largest glottal gap between the vocal folds.

glottal gap during vocal folds oscillations. In Figure 5.14 and Figure 5.15 case $R \approx 1$ is depicted by gray and case $R \approx 0$ by dark red color. It can be seen, that nonlinear part of the strain tensor takes effect in elements near to the boundary, therefore to correctly capture deformations of the vocal folds, it is again necessary to use a nonlinear model of elasticity.

Figure 5.16 shows velocity field in the glottal region at two time instants of the vocal folds self-oscillation. In these time instants different jet declination behind the channel construction, i.e. the Coanda effect can be observed.

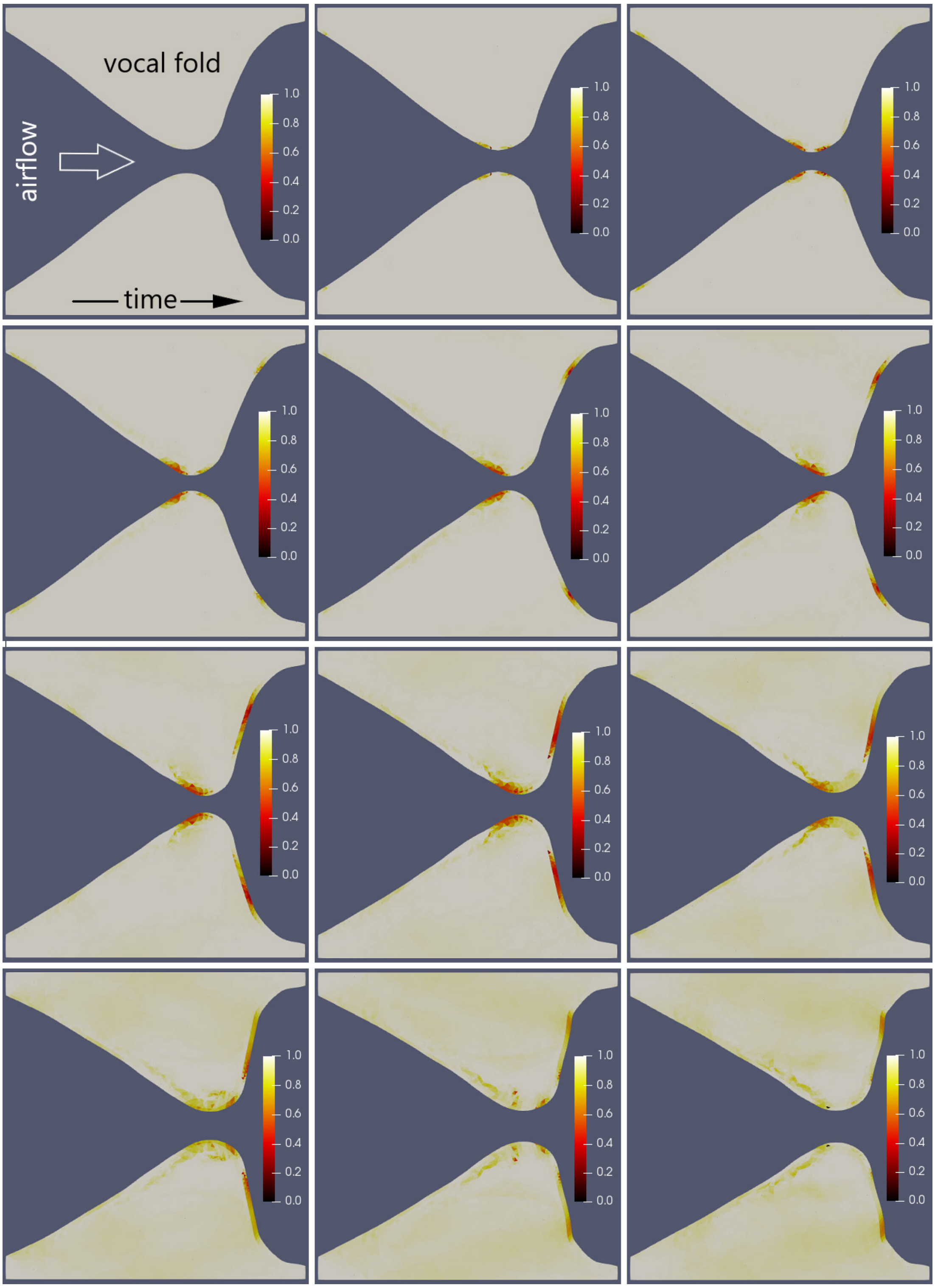

Figure 5.14: Deformation of vocal folds in dependence on time computed by the St. Venant-Kirchhoff model and the ratios of the norms of the linear strain tensor and the nonlinear Green strain tensor at different time instants.

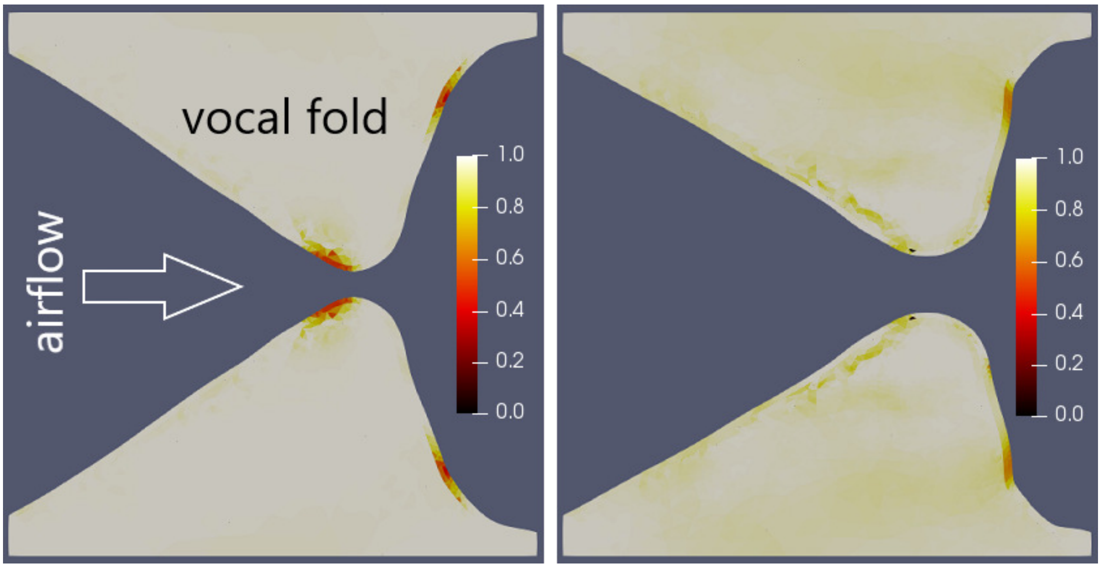

Figure 5.15: Deformation of vocal folds in dependence on time computed by the St. Venant-Kirchhoff model and the ratios of the norms of the linear strain tensor and the nonlinear Green strain tensor at different time instants - details for the smallest and the largest glottal gap between the vocal folds.

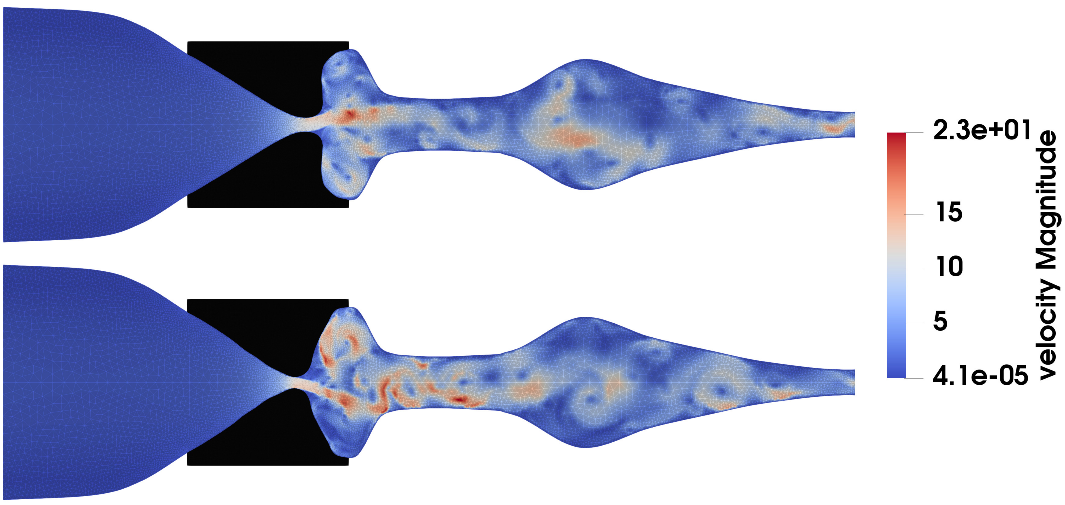

Figure 5.16: Velocity field in the glottal region at two time instants of the vocal folds self-oscillation.

Conclusion

In the first four chapters of this thesis, we have formulated and theoretically analyzed the space-time discontinuous Galerkin method for the solution of non-stationary nonlinear convection-diffusion problem in time-dependent domains. The problem was reformulated using the arbitrary Lagrangian-Eulerian (ALE) method, where the ALE mapping is constructed successively from one time slab to the next one. The problem is discretized with the aid of the ALE space-time discontinuous Galerkin method (ALE-STDGM). In the formulation of the numerical scheme we use the nonsymmetric, symmetric and incomplete versions of the space discretization of the diffusion terms and an interior and boundary penalty. The nonlinear convection terms are discretized with the aid of a numerical flux. The discontinuous Galerkin discretization uses piecewise polynomial approximation of degree $p \geq 1$ in space and $q \geq 1$ in time.

Chapter 3 is devoted to the stability analysis of the ALE space-time discontinuous Galerkin method. An important tool in this analysis was the discrete characteristic function, which was generalized for problems in time-dependent domains Ω_t. The key requirement for this function was the continuity with respect to the $\|\cdot\|_{L^2(\Omega_t)}$ and $\|\cdot\|_{DG,t}$ norms, which has been also proved. On the basis of a technical analysis we obtained unconditional stability of this method, which means that the approximate solution is bounded by terms of data, without any limitation of the time step in dependence on the size of the space mesh. For further research we shall investigate the ALE-STDGM for nonlinear convection-diffusion problems with prescribed Neumann or mixed boundary conditions in a time-dependent domain. An interesting, but very difficult further extension would be the stability analysis of the ALE-STDGM applied to singularly perturbed nonlinear problems.

In Chapter 4 we derived a priori error estimates for the ALE-STDGM, first in terms of the interpolation error η and then in terms of h (mesh size) and τ (time step). In the presented error analysis we used a simplification, namely we omitted expression containing $\|\{e\}_{j-1}\|^2_{\Omega_{t_{j-1}}}$, because it is not clear how to estimate expression $\|\{\eta\}_{j-1}\|_{\Omega_{t_{j-1}}}$ in terms of h and τ. Therefore further work must be invested in the derivation of more accurate a priori error estimates.

In the last chapter, we presented some numerical results. At first we have applied the STDGM to the solution of the nonlinear elasticity benchmark problem, which was proposed by Turek and Hron and originally was solved by the finite element method (FEM). Our results show a very good agreement with computational results obtained by the FEM. From this comparison we can conclude, that STDGM yields an accurate and robust method capable of solving elasticity problems. In the next part of this chapter we were focused on fluid-structure interaction in a time-dependent domain, namely on numerical solution of flow induced vocal folds vibration in a simplified human vocal tract. The fluid flow problem was described by the compressible Navier-Stokes equations in the conservative ALE form and it was coupled with the solution of the linear or nonlinear elasticity problem. The elastic structure domain was split into four subdomains with different material characteristics. The ALE mapping was determined on the basis of an artificial static linear elasticity problem in the domain occupied by the fluid. Our main goal was to compare linear and nonlinear elasticity models of vocal folds in the FSI simulation. From the obtained results we can conclude, that linear elasticity model is not sufficient to correctly capture deformation of the vocal folds and is better to use nonlinear elasticity models. There are many possible extensions for the future work. The most challenging would be the simulation of the complete closure of the channel between the vocal folds. This effect takes place during the voice creation process in human vocal folds.

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List of Abbreviations

ALE	arbitrary Lagrangian-Eulerian method
ALE-BDF-DGM	combination of the ALE method with BDF (in time) and DGM (in space discretization)
ALE-STDGM	combination of the ALE method with STDGM
BDF	backward difference formula
BDF-DGM	combination of the BDF (in time discretization) and DGM (in space discretization)
DG	discontinuous Galerkin
DGM	discontinuous Galerkin method
FSI	fluid-structure interaction
GMRES	generalized minimal residual method
IIPG	incomplete internal penalty Galerkin discretization
NIPG	nonsymmetric internal penalty Galerkin discretization
PDEs	partial differential equations
SIPG	symmetric internal penalty Galerkin discretization
STDGM	space-time discontinuous Galerkin method
UMFPACK	package of routines for solving sparse linear systems using the Unsymmetric MultiFrontal method

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