6th problem set

Problem 1. Let $B, C \trianglelefteq \mathbf{A}$ (by different terms). Show that then there is a common term t such that $B, C \trianglelefteq^t \mathbf{A}$.

Problem 2. Let $\mathbf{A}$ be a finite algebra such that every $\{a\}$ is an absorbing subuniverse of $\mathbf{A}$. Prove that $\mathbf{A}$ has a near unanimity term.

Problem 3. Let $\mathbb{G}$ be a digraph. The smooth part of $\mathbb{G}$, which we denote by smooth($\mathbb{G}$), is the largest $U \subseteq V(G)$ such that the digraph induced by U on $\mathbb{G}$ is smooth ("smooth" means that each vertex has at least one incoming and at least one outgoing edge). Find a primitive positive definition for smooth($\mathbb{G}$) (the definition might depend on $\mathbb{G}$).

Problem 4 (Walking)**.** In this problem, we use the following notation: Take an algebra $\mathbf{A}$ and $R \leq_{sd} \mathbf{A}^2$. Let $\mathbb{P}$ be an oriented path with a designated starting and ending vertex and let $B \subset A$. Denote by $B^{+\mathbb{P}}$ the set of $a \in A$ for which there exists a $b \in B$ from which there is a $\mathbb{P}$-shaped path from b to a (using edges from R). Prove that:

a) if $B \leq \mathbf{A}$, then $B^{+\mathbb{P}} \leq \mathbf{A}$,

b) if $B \trianglelefteq \mathbf{A}$, then $B^{+\mathbb{P}} \trianglelefteq \mathbf{A}$.

Problem 5. In this problem, we will show that the complete graph on 3 vertices does not have any nontrivial idempotent polymorphism.

1. Verify that idempotent polymorphisms of $\mathbb{K}_3$ are exactly the polymorphisms of the relational structure "$\mathbb{K}_3$ with constants" defined as

   $$\mathbb{K}_3^c = (\{1,2,3\}, E, c_1, c_2, c_3)$$

   where $E = \{(u,v) \in \{1,2,3\}^2 \colon u \neq v\}$ and $c_i = \{(i)\}$ for $i = 1,2,3$.

2. Verify that all polymorphisms of $\mathbb{K}_3^c$ of arity at most 3 are projections.

3. Let f be an n-ary polymorphism of $\mathbb{K}_3^c$. For $i = 1, 2, \ldots, n$, we define $f_i(x,y) = f(x,\ldots,x,y,x,\ldots,x)$, where the y is on the i-th place. Prove that each $f_i(x,y)$ is equal to either x or y and there exists at most one i such that $f_i(x,y) = y$.

4. Show that in the situation of the previous point we cannot have $f_i(x,y) = x$ for all i. (Hint: Absorption lives here.)

5. Show that if f is an n-ary polymorphism of $\mathbb{K}_3^c$ and i the uniqe number such that $f_i(x,y) = y$ then f is the projection to the i-th coordinate.

Problem 6. Let $\mathbf{G}$ be a commutative group, $\mathbf{H}$ be a proper subgroup of $\mathbf{G}$. Prove that H does not absorb $\mathbf{G}$.