5th problem set

Problem 1. Let $\mathbb{G}$ be the graph consisting of one (symmetric) edge. Show that there is an idempotent Taylor operation in $\mathrm{Pol}(\mathbb{G})$.

Problem 2. Prove that if a nontrivial idempotent algebra $\mathbf{A}$ does not have a Taylor term, then $HSP(\mathbf{A})$ contains a nontrivial algebra whose all operations are projections.

Note: For $\mathbf{A}$ finite, this result can be improved to just $HS(\mathbf{A})$ (this is due to Bulatov – if you are bored, try proving it on your own).

Problem 3. Prove that the algebra $\mathbf{A} = (\{0,1,2\}, \times)$ with the operation $\times$ defined by the following table does not have a Taylor term.

$\times$	0	1	2
0	0	0	2
1	2	1	2
2	0	2	2

Hint: What subalgebras does $\mathbf{A}$ have? Why should we care about them?

Problem 4. In this problem, we will show that the complete graph on 3 vertices does not have any nontrivial idempotent polymorphism.

1. Verify that idempotent polymorphisms of $\mathbb{K}_3$ are exactly the polymorphisms of the relational structure "$\mathbb{K}_3$ with constants" defined as

$$\mathbb{K}_3^c = (\{0,1,2\}, E, c_0, c_1, c_2)$$

where $E = \{(u,v) \in \{0,1,2\}^2 \colon u \neq v\}$ and $c_i = \{(i)\}$ for $i = 0,1,2$.

2. Verify that all polymorphisms of $\mathbb{K}_3^c$ of arity at most 3 are projections.

3. Let f be an n-ary polymorphism of $\mathbb{K}_3^c$. For $i = 1,2,\dots,n$, we define $f_i(x,y) = f(x,\dots,x,y,x,\dots,x)$, where the y is on the i-th place. Prove that each $f_i(x,y)$ is equal to either x or y and there exists at most one i such that $f_i(x,y) = y$.

4. Show that in the situation of the previous point we cannot have $f_i(x,y) = x$ for all i. (This is probably the hardest part of the proof. If you get stuck here, move on to the next point.)

5. Show that if f is an n-ary polymorphism of $\mathbb{K}_3^c$ and i the uniqe number such that $f_i(x,y) = y$ then f is the projection to the i-th coordinate.