2nd problem set

Problem 1. Let G be a directed graph. Prove or disprove the following implications:

1. If for each $x \in V(G)$ there exists exactly one terminal vertex $NF(x)$ such that $x \to^\star NF(x)$ then G is finitely terminating.
2. If for each $x \in V(G)$ there exists exactly one terminal vertex $NF(x)$ such that $x \to^\star NF(x)$ then G is normal.

Problem 2. Let $\mathcal{E}$ be the system of equations consisting of one equation "$x \approx t(x, x, x, x, x)$" in the signature with one 5-ary term t. Is $D(\mathcal{E})$ convergent?

Problem 3. Let $\mathcal{E}$ be the system of equations $t(x, t(x, y)) \approx g(x), t(y, y) \approx h(y)$ in the signature with one binary term t and two unary terms g, h. Is $D(\mathcal{E})$ convergent?

Problem 4. Consider the system $\mathcal{E}$ of equations $w(x, x, y) \approx w(x, y, x)$, $w(x, y, x) \approx w(y, x, x)$, $w(x, x, x) \approx x$ in the signature of one ternary term w. Prove that the graph $D(\mathcal{E})$ is convergent. How do normal forms of terms from Σ look like?

Problem 5. Let G be a directed graph. We say that G is

1. *confluent* if for every $x, y, z \in V(G)$ such that $x \to^\star y, z$ there exists $u \in V(G)$ such that $y, z \to^\star u$.
2. *locally confluent* if if for every $x, y, z \in V(G)$ such that $x \to y, z$ (no star here!) there exists $u \in V(G)$ such that $y, z \to^\star u$.

Give an example of a graph that is locally confluent, but not confluent. (Note that at the lecture, Libor shows that *for finitely terminating graphs*, being confluent and locally confluent are the equivalent.)

Problem 6. Let Σ be a signature, $\mathcal{E}$ a system of equations and let α be the relation

$$\{(s, t) \colon s, t \in F_\Sigma(x_1, x_2, \dots), \mathcal{E} \vdash s \approx t\}.$$

Finish the proof from the lecture that α is an invariant congruence of $F_\Sigma(x_1, x_2, \dots)$.