nmma583

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Plan

Campanatovy a Morreyovy prostory

Elliptic systems with constant coefficients

Morrey space

Let Ω be a bounded connected open set in R^n and let us denote

$$\Omega(x,\rho) = \Omega \cap B(x,\rho)$$

$$\operatorname{diam} \Omega = \sup \{|x-y| : x, y \in \Omega\} .$$

DEFINITION 1.1 (Morrey spaces). Let $p \geq 1$ *and* $\lambda \geq 0$. *By* $L^{p,\lambda}(\Omega)$ *we denote the linear space of functions* $u \in L^p(\Omega)$ *such that*

$$(1.2) \qquad \|u\|_{L^{p,\lambda}(\Omega)} = \left\{ \sup_{\substack{x \in \Omega \\ 0<\rho<\operatorname{diam}\Omega}} \rho^{-\lambda} \int_{\Omega(x,\rho)} |u|^p \, dx \right\}^{\frac{1}{p}} < +\infty .$$

It is easy to see that $\|u\|_{p,\lambda}$ in (1.2) is a norm respect to which $L^{p,\lambda}(\Omega)$ is a Banach space.

Campanato space

Set

$$u_{x_0,\rho} = \frac{1}{|\Omega(x_0,\rho)|} \int_{\Omega(x_0,\rho)} u(x)\,dx\,.^{2)}$$

DEFINITION 1.2 (Campanato spaces). *Let* $p \geq 1$ *and* $\lambda \geq 0$. *By* $\mathcal{L}^{p,\lambda}(\Omega)$ *we denote the linear space of functions* $u \in L^p(\Omega)$ *such that*

$$(1.3)\quad [u]_{p,\lambda} = \left\{ \sup_{\substack{x_0 \in \Omega \\ 0<\rho<\operatorname{diam}\Omega}} \rho^{-\lambda} \int_{\Omega(x_0,\rho)} |u(x) - u_{x_0,\rho}|^p\,dx \right\}^{\frac{1}{p}} < +\infty\,.$$

$\mathcal{L}^{p,\lambda}(\Omega)$ are Banach spaces with the norm

$$\|u\|_{\mathcal{L}^{p,\lambda}(\Omega)} = \|u\|_{L^p(\Omega)} + [u]_{p,\lambda}$$

and one sees that $u \in \mathcal{L}^{p,\lambda}(\Omega)$ if and only if

$$\sup_{\substack{x \in \Omega \\ 0<\rho<\operatorname{diam}\Omega}} \rho^{-\lambda} \inf_{c \in \mathbb{R}} \int_{\Omega(x,\rho)} |u-c|^p\,dx < +\infty\,.$$

Properties of Morrey and Campanato spaces

DEFINITION 1.3. *Let* $A > 0$. *The bounded set* Ω *is said to be of type* (A) *if for all* $x_0 \in \Omega$ *and* $\rho < \operatorname{diam} \Omega$

$$|\Omega(x_0,\rho)| \geq A \rho^n .$$

Let Ω be a bounded connected open set in R^n and let us denote

$$\Omega(x,\rho) = \Omega \cap B(x,\rho)$$

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It is easy to see that $\|u\|_{p,\lambda}$ in (1.2) is a norm respect to which $L^{p,\lambda}(\Omega)$ is a Banach space.

PROPOSITION 1.1. *We have*

a) $L^{p,0}(\Omega) \simeq L^p(\Omega)$

b) $L^{p,n}(\Omega) \simeq L^\infty(\Omega)$

c) $L^{p,\lambda}(\Omega) = \{0\}$ *for* $\lambda > n$

d) $L^{q,\mu}(\Omega) \subset L^{p,\lambda}(\Omega)$ *if* $p \leq q$, $\frac{n-\lambda}{p} \leq \frac{n-\mu}{q}$.

For $\lambda \in [0, n)$ Morrey=Campanato

PROPOSITION 1.2. *Let* Ω *be of type* (A) *and* $0 \leq \lambda < n$. *Then* $\mathcal{L}^{p,\lambda}(\Omega)$ *is isomorphic to* $L^{p,\lambda}(\Omega)$.

DEFINITION 1.2 (Campanato spaces). *Let* $p \geq 1$ *and* $\lambda \geq 0$. *By* $\mathcal{L}^{p,\lambda}(\Omega)$ *we denote the linear space of functions* $u \in L^p(\Omega)$ *such that*

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For $\lambda \in (n, n+p]$ Campanato=Hölder

THEOREM 1.2 (An integral characterization of Hölder continuous functions). *Let* Ω *be of type* (A) *and* $n < \lambda \leq n+p$. *Then* $\mathcal{L}^{p,\lambda}(\Omega)$ *is isomorphic to the space* $C^{0,\alpha}(\Omega)$ *with* $\alpha = \frac{\lambda-n}{p}$. *Moreover if* $u \in \mathcal{L}^{p,\lambda}(\Omega)$ *with* $\lambda > n+p$, *then* u *is constant in* Ω.

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Corolaries

Corollary

If Ω has the extension property and $u \in W^{1,p}(\Omega)$ with $p > n$ then $u \in C^{0,1-n/p}(\overline{\Omega})$ and $(C = C(\Omega, p))$

$$\|u\|_{C^{0,1-n/p}} \leq C\|u\|_{W^{1,p}}.$$

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Corollary (Morrey)

Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, $\nabla u \in L^{p,n-p+p\epsilon}_{\mathrm{loc}}(\Omega)$, for some $\epsilon > 0$. Then $u \in C^{0,\epsilon}_{\mathrm{loc}}(\Omega)$.

Plan

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Elliptic systems with constant coefficients

Sobolev Poincare inequality

Theorem (Sobolev-Poincare)

For every bounded and connected domain Ω *with the extension property,* $p \geq 1$, $q \in [1, p^*)$ *there is a constant* $c = c(n, p, q, \Omega)$ *such that for each* $u \in W^{1,p}(\Omega)$ *we have*

$$\Big(\fint_\Omega |u - u_\Omega|^q\Big)^{\frac{1}{q}} \leq c\Big(\fint_\Omega |\nabla u|^p\Big)^{\frac{1}{p}}.$$

When $\Omega = B_r$ *or cube of sidelength* r *then* $c \leq c(n, p, q)r$

- $p^* = np/(n-p)$ if $n > p$ and $+\infty$ otherwise $\implies$ compactness of the embedding $W^{1,p} \hookrightarrow L^q$
- u_Ω is mean over Ω

Setting

leading part with constant coefficients[5)]

(2.1) $$- D_\alpha(A^{\alpha\beta}_{ij} D_\beta u^j) = 0 \qquad i = 1, \cdots, N .$$

Elliptic means that the coefficients satisfy the Legendre-Hadamard condition

(2.2) $$A^{\alpha\beta}_{ij} \xi_\alpha \xi_\beta \eta^i \eta^j \geq \nu |\xi|^2 |\eta|^2 \qquad \forall\, \eta, \xi ;\ \nu > 0$$

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Definition 3.36 *A matrix of coefficients* $\left(A^{\alpha\beta}_{ij}\right)^{1\le\alpha,\beta\le n}_{1\le i,j\le m}$ *is said to satisfy*

1. *the* very strong ellipticity *condition, or the* Legendre *condition, if there is a* $\lambda > 0$ *such that*

$$A^{\alpha\beta}_{ij} \xi^i_\alpha \xi^j_\beta \geq \lambda |\xi|^2, \quad \forall \xi \in \mathbb{R}^{m\times n}; \tag{3.16}$$

2. *the* strong ellipticity condition*, or the* Legendre-Hadamard *condition, if there is a* $\lambda > 0$ *such that*

$$A^{\alpha\beta}_{ij} \xi_\alpha \xi_\beta \eta^i \eta^j \geq \lambda |\xi|^2 |\eta|^2, \quad \forall \xi \in \mathbb{R}^n, \forall \eta \in \mathbb{R}^m. \tag{3.17}$$

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Definition 3.41 *A bilinear form* $\mathcal{B}$ *on* $W_0^{1,2}(\Omega, \mathbb{R}^m)$ *is said to be* weakly coercive *if there exist* $\lambda_0 > 0$ *and* $\lambda_1 \geq 0$ *such that*

$$\mathcal{B}(u,u) \geq \lambda_0 \int_\Omega |Du|^2 dx - \lambda_1 \int_\Omega |u|^2 dx. \tag{3.23}$$

Theorem 3.42 (Gårding's inequality) *Assume that* $A_{ij}^{\alpha\beta}$ *are uniformly continuous on* Ω *and that they satisfy the Legendre-Hadamard condition (3.17) for some* $\lambda > 0$ *independent of* $x \in \Omega$*. Then the bilinear form on* $W_0^{1,2}(\Omega, \mathbb{R}^m)$ *defined by*

$$\mathcal{B}(u,v) := \int_\Omega A_{ij}^{\alpha\beta} D_\alpha u^i D_\beta v^j dx$$

is weakly coercive. If $A_{ij}^{\alpha\beta}$ *are constant then* $\mathcal{B}$ *is in fact coercive.*

Caccioppoli inequality

PROPOSITION 2.1. *Let* $u \in H^1(\Omega, R^N)$ *be a weak solution to system (2.1), i.e.*

(2.3)
$$\int_\Omega A_{ij}^{\alpha\beta} D_\beta u^j D_\alpha \phi^i \, dx = 0 \qquad \forall \phi \in H_0^1(\Omega, R^N) .$$

Then for all $x_0 \in \Omega$ *and all* $R < \frac{1}{2} \operatorname{dist}(x_0, \partial\Omega)$ *the following inequality holds*

(2.4)
$$\int_{B_R(x_0)} |\nabla u|^2 \, dx \leq \frac{c}{R^2} \int_{B_{2R}(x_0)} |u|^2 \, dx .$$

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if u *is a solution to system (2.1), then for all* $x_0 \in \Omega$ *and for all* $\rho < R < \operatorname{dist}(x_0, \partial\Omega)$ *the following estimate holds:*

(2.5) $$\int_{B_\rho(x_0)} |\nabla u|^2 dx \le \frac{c}{(R-\rho)^2} \int_{B_R \setminus B_\rho} |u - \lambda|^2 dx \, .$$

Basic estimate

THEOREM 2.1. *Let* u *be a weak solution to system (2.1). Then there exists a constant* c *depending on the constants of the system such that for each* $x_0 \in \Omega$ *and* $0 < \rho \le R \le \operatorname{dist}(x_0, \partial\Omega)$ *the following estimates hold*

$$\int_{B_\rho(x_0)} |u|^2 dx \le c\left(\frac{\rho}{R}\right)^n \int_{B_R(x_0)} |u|^2 dx \tag{2.7}$$

$$\int_{B_\rho(x_0)} |u - u_{x_0,\rho}|^2 dx \le c\left(\frac{\rho}{R}\right)^{n+2} \int_{B_R(x_0)} |u - u_{x_0,R}|^2 dx \,. \tag{2.8}$$

Liouville

Theorem 5.12 *Let* $u : \mathbb{R}^n \to \mathbb{R}^m$ *be an entire solution to the elliptic system* (5.12), *and assume that there exists a constant* $M > 0$ *and an integer* $k \geq 0$ *such that*

$$|u(x)| \leq M(1 + |x|^k), \quad \forall x \in \mathbb{R}^n.$$

Then u *is a polynomial of degree at most* k.

Regularity with constant coefficients

Theorem 5.14 *Let* $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ *be a solution to*

$$D_\alpha(A^{\alpha\beta}_{ij} D_\beta u^j) = -D_\alpha F^\alpha_i, \tag{5.19}$$

with $A^{\alpha\beta}_{ij}$ *constant and satisfying the Legendre-Hadamard condition* (3.17). *If* $F^\alpha_i \in \mathcal{L}^{2,\mu}_{\mathrm{loc}}(\Omega)$, $0 \le \mu < n+2$, *then* $Du \in \mathcal{L}^{2,\mu}_{\mathrm{loc}}(\Omega)$, *and*

$$\|Du\|_{\mathcal{L}^{2,\mu}(K)} \le c\Big(\|Du\|_{L^2(\Omega)} + [F]_{\mathcal{L}^{2,\mu}(\widetilde{\Omega})}\Big), \tag{5.20}$$

for every compact $K \Subset \widetilde{\Omega} \Subset \Omega$, *with* $c = c(n, m, K, \widetilde{\Omega}, \lambda, \Lambda, \mu)$.

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for every compact $K \Subset \widetilde{\Omega} \Subset \Omega$, *with* $c = c(n, m, K, \widetilde{\Omega}, \lambda, \Lambda, \mu)$.

Lemma 5.13 *Let* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *be a non-negative and non-decreasing function satisfying*

$$\phi(\rho) \le A\Big[\Big(\frac{\rho}{R}\Big)^\alpha + \varepsilon\Big]\phi(R) + BR^\beta,$$

for some $A, \alpha, \beta > 0$, *with* $\alpha > \beta$ *and for all* $0 < \rho \le R \le R_0$, *where* $R_0 > 0$ *is given. Then there exist constants* $\varepsilon_0 = \varepsilon_0(A, \alpha, \beta)$ *and* $c = c(A, \alpha, \beta)$ *such that if* $\varepsilon \le \varepsilon_0$, *we have*

$$\phi(\rho) \le c\Big[\frac{\phi(R)}{R^\beta} + B\Big]\rho^\beta. \tag{5.17}$$

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for all $0 \le \rho \le R \le R_0$.

Corollary 5.15 *In the hypothesis of the theorem, if* $F^\alpha_i \in C^{k,\sigma}(\overline{\Omega})$, $k \ge 1$, *then* $u \in C^{k+1,\sigma}_{\mathrm{loc}}(\Omega)$ *and*

$$\|u\|_{C^{k+1,\sigma}(K)} \le c\Big(\|Du\|_{L^2(\Omega)} + \|F\|_{C^{k,\sigma}(\overline{\Omega})}\Big),$$

with $c = c(n, m, K, \Omega, \lambda, \Lambda, \sigma)$

Regularity with continuous coefficients

Theorem 5.17 *Let* $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ *be a solution to*

$$D_\alpha(A^{\alpha\beta}_{ij} D_\beta u^j) = -D_\alpha F^\alpha_i, \tag{5.25}$$

with $A^{\alpha\beta}_{ij} \in C^0_{\mathrm{loc}}(\Omega)$ *satisfying the Legendre-Hadamard condition* (3.17)*. Then, if* $F^\alpha_i \in L^{2,\lambda}_{\mathrm{loc}}(\Omega)$ *for some* $0 \le \lambda < n$*, we have* $Du \in L^{2,\lambda}_{\mathrm{loc}}(\Omega)$ *and the following estimate*

$$\|Du\|_{L^{2,\lambda}(K)} \le c\Big(\|Du\|_{L^2(\widetilde{\Omega})} + \|F\|^2_{L^{2,\lambda}(\widetilde{\Omega})}\Big) \tag{5.26}$$

holds for every compact $K \Subset \widetilde{\Omega} \Subset \Omega$*, where* $c = c(n, m, \lambda, \Lambda, K, \widetilde{\Omega}, \omega)$ *and* ω *is the modulus of continuity of* $(A^{\alpha\beta}_{ij})$ *in* $\widetilde{\Omega}$ *:*

$$\omega(R) := \sup_{\substack{x,y\in\widetilde{\Omega} \\ |x-y|\le R}} |A(x) - A(y)|,$$

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holds for every compact $K \Subset \widetilde{\Omega} \Subset \Omega$, *where* $c = c(n, m, \lambda, \Lambda, K, \widetilde{\Omega}, \omega)$ *and* ω *is the modulus of continuity of* $(A^{\alpha\beta}_{ij})$ *in* $\widetilde{\Omega}$:

$$\omega(R) := \sup_{\substack{x,y\in\widetilde{\Omega} \\ |x-y|\le R}} |A(x) - A(y)|,$$

Corollary 5.18 *In the same hypothesis of the theorem, if* $\lambda > n-2$, *then* $u \in C^{0,\sigma}_{\text{loc}}(\Omega, \mathbb{R}^m)$, $\sigma = \frac{\lambda-n+2}{2}$.

Regularity with Hölder continuous coefficients

Theorem 5.19 *Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ be a solution to*

$$D_\alpha(A^{\alpha\beta}_{ij} D_\beta u^j) = -D_\alpha F^\alpha_i, \tag{5.29}$$

with $A^{\alpha\beta}_{ij} \in C^{0,\sigma}_{\mathrm{loc}}(\Omega)$ satisfying the Legendre-Hadamard condition (3.17) for some $\sigma \in (0,1)$. If $F^\alpha_i \in C^{0,\sigma}_{\mathrm{loc}}(\Omega)$, then we have $Du \in C^{0,\sigma}_{\mathrm{loc}}(\Omega)$. Moreover for every compact $K \Subset \widetilde{\Omega} \Subset \Omega$

$$\|Du\|_{C^{0,\sigma}(K)} \le c\Big(\|Du\|_{L^2(\widetilde{\Omega})} + \|F\|_{C^{0,\sigma}(\widetilde{\Omega})}\Big), \tag{5.30}$$

c depending on K, $\widetilde{\Omega}$, the ellipticity and the Hölder norm of the coefficients $A^{\alpha\beta}_{ij}$.

Remark

We will show $F \in \mathcal{L}^{2,\lambda} \implies \nabla u \in \mathcal{L}^{2,\lambda}$ for $\lambda \in [0, n+2)$.

Counterexamples to full regularity

From now on we follow [Beck, 2016].

$$u(\alpha, x) := |x|^{-\alpha} x$$

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$$A_{ij}^{\kappa\lambda}(b_1, b_2, x) = \delta_{\kappa\lambda}\delta_{ij} + \left(b_1\delta_{i\kappa} + b_2\frac{x_i x_\kappa}{|x|^2}\right)\left(b_1\delta_{j\lambda} + b_2\frac{x_j x_\lambda}{|x|^2}\right)$$

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Example 4.1 (De Giorgi) *Assume $n \geq 3$ and let $u\colon \mathbb{R}^n \supset B_1 \to \mathbb{R}^n$ be given by*

$$u(\alpha, x) = |x|^{-\alpha}x \qquad \textit{for } \alpha := \frac{n}{2}\left(1 - ((2n-2)^2 + 1)^{-1/2}\right).$$

Then $u \in W^{1,2}(B_1, \mathbb{R}^n)$ is an unbounded weak solution of the elliptic system

$$\operatorname{div}\left(A(n-2, n, x)Du(\alpha)\right) = 0 \qquad \textit{in } B_1\,.$$

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$$\tilde{A}_{ij}^{\kappa\lambda}(u) = \delta_{\kappa\lambda}\delta_{ij} + \Big(\delta_{i\kappa} + \frac{4}{n-2}\frac{u_i u_\kappa}{1+|u|^2}\Big)\Big(\delta_{j\lambda} + \frac{4}{n-2}\frac{u_j u_\lambda}{1+|u|^2}\Big),$$

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From now on we follow [Beck, 2016].

$$u(\alpha, x) := |x|^{-\alpha} x$$

$$A_{ij}^{\kappa\lambda}(b_1, b_2, x) = \delta_{\kappa\lambda}\delta_{ij} + \Big(b_1\delta_{i\kappa} + b_2\frac{x_i x_\kappa}{|x|^2}\Big)\Big(b_1\delta_{j\lambda} + b_2\frac{x_j x_\lambda}{|x|^2}\Big)$$

Example 4.1 (De Giorgi) *Assume $n \geq 3$ and let $u\colon \mathbb{R}^n \supset B_1 \to \mathbb{R}^n$ be given by*

$$u(\alpha, x) = |x|^{-\alpha} x \qquad \textit{for } \alpha := \frac{n}{2}\big(1 - ((2n-2)^2 + 1)^{-1/2}\big)\,.$$

Then $u \in W^{1,2}(B_1, \mathbb{R}^n)$ is an unbounded weak solution of the elliptic system

$$\operatorname{div}\big(A(n-2, n, x)Du(\alpha)\big) = 0 \qquad \textit{in } B_1\,.$$

$$\tilde{A}_{ij}^{\kappa\lambda}(u) = \delta_{\kappa\lambda}\delta_{ij} + \Big(\delta_{i\kappa} + \frac{4}{n-2}\frac{u_i u_\kappa}{1+|u|^2}\Big)\Big(\delta_{j\lambda} + \frac{4}{n-2}\frac{u_j u_\lambda}{1+|u|^2}\Big)\,,$$

Example 4.3 (Giusti and Miranda) *Assume $n \geq 3$ and let $u\colon \mathbb{R}^n \supset B_1 \to \mathbb{R}^n$ be given by $u(x) = x/|x|$. Then $u \in W^{1,2}(B_1, \mathbb{R}^n) \cap L^\infty(B_1, \mathbb{R}^n)$, and u is a discontinuous weak solution of the elliptic system*

$$\operatorname{div}(\tilde{A}(u)Du) = 0 \qquad \textit{in } B_1\,. \tag{4.1}$$

Partial regularity—assumptions

$$\operatorname{div}\big(a(x,u)Du\big) = 0 \qquad \text{in } \Omega\,. \tag{4.13}$$

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$$a(x,u)\xi \cdot \tilde{\xi} \leq L|\xi||\tilde{\xi}| \tag{4.7}$$

for almost every $x \in \Omega$, all $u \in \mathbb{R}^N$, all $\xi, \tilde{\xi} \in \mathbb{R}^{Nn}$, and some $L \geq 1$.

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---

there exists

or a modulus of continuity $\omega\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ (concave and monotonically non-decreasing) satisfying $\lim_{t\searrow 0}\omega(t) = \omega(0) = 0$ such that

$$|a(x,u) - a(\tilde{x},\tilde{u})| \leq \omega(|x-\tilde{x}| + |u-\tilde{u}|) \tag{4.14}$$

for all $x, \tilde{x} \in \Omega$ and all $u, \tilde{u} \in \mathbb{R}^N$.

Partial regularity—basic concepts

introduce the (open) α-regular set of a measurable function $f\colon \Omega \to \mathbb{R}^N$ via

$$\mathrm{Reg}_\alpha(f) := \{x_0 \in \Omega\colon f \text{ is locally continuous near } x_0 \text{ with Hölder exponent } \alpha\}$$

for $\alpha \in [0,1]$, and the singular set of f as its complement in Ω, i.e.

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$$\mathrm{Sing}_\alpha(f) := \Omega \setminus \mathrm{Reg}_\alpha(f)\,.$$

$$E(u; x_0, \varrho) := \fint_{B_\varrho(x_0)} |u - (u)_{B_\varrho(x_0)}|^2 \, dx \tag{4.15}$$

Lemma 4.21 (Excess decay estimate via blow up; [41], Lemma 4) *For every $\tau \in (0,1)$ there exist two positive constants ε_0, R_0 depending only on n, N, L, ω, and τ such that the following statement is true: if $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ is a weak solution to the system (4.13) with continuous coefficients $a \colon \Omega \times \mathbb{R}^N \to \mathbb{R}^{Nn \times Nn}$ satisfying (4.6), (4.7) and (4.14), and if for some ball $B_R(x_0) \subset \Omega$ with $R \leq R_0$ there holds*

$$E(u; x_0, R) < \varepsilon_0^2 \,, \tag{4.16}$$

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Theorem 4.23 (Giusti and Miranda, Morrey) *Let $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be a weak solution to the system (4.13) with continuous coefficients $a\colon \Omega \times \mathbb{R}^N \to \mathbb{R}^{Nn \times Nn}$ satisfying (4.6), (4.7) and (4.14). Then we have the characterization of the singular set via*

$$\mathrm{Sing}_0(u) = \Big\{ x_0 \in \Omega\colon \liminf_{\varrho \searrow 0} \fint_{\Omega(x_0,\varrho)} |u - (u)_{\Omega(x_0,\varrho)}|^2\, dx > 0 \Big\}$$

and in particular $\mathcal{L}^n(\mathrm{Sing}_0(u)) = 0$. Moreover, for every $\alpha \in (0,1)$ there holds $\mathrm{Reg}_0(u) = \mathrm{Reg}_\alpha(u)$, i.e. $u \in C^{0,\alpha}(\mathrm{Reg}_0(u), \mathbb{R}^N)$.

Definition 9.19 *For* $k > 0$ integer, *define* ω_k *to be the volume of the unit ball in* $\mathbb{R}^k$, *given by*

$$\omega_k = \frac{2\pi^{\frac{k}{2}}}{k\Gamma(\frac{k}{2})}, \tag{9.37}$$

where Γ *is the* Euler function

$$\Gamma(t) := \int_0^{+\infty} x^{t-1} e^{-x} dx, \quad t \geq 0. \tag{9.38}$$

Since Γ *is defined for every positive number we shall use* (9.37) *to define* ω_k *for any real number* $k > 0$.

Given a set $A \subset \mathbb{R}^n$ and $k, \delta > 0$, define

$$\mathcal{H}^k_\delta(A) := \inf\left\{ \sum_{j=0}^{\infty} \omega_k \rho_j^k : A \subset \bigcup_{j=0}^{\infty} B_{\rho_j}(x_j),\ \rho_j \leq \delta,\ x_j \in \mathbb{R}^n \right\}.$$

Hausdorff measure and dimension

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Definition 9.20 *The* k*-dimensional Hausdorff measure* $\mathcal{H}^k(A)$ *of a set* $A \subset \mathbb{R}^n$ *is defined as*

$$\mathcal{H}^k(A) := \sup_{\delta > 0} \mathcal{H}^k_\delta(A).$$

The Hausdorff dimension of A *is defined as*

$$\dim^{\mathcal{H}}(A) := \inf \left\{ k \geq 0 : \mathcal{H}^k(A) = 0 \right\}.$$

We also recall that for every $k > \dim^{\mathcal{H}}(A)$, we have $\mathcal{H}^k(A) = 0$, and for every $k < \dim^{\mathcal{H}}(A)$, $\mathcal{H}^k(A) = +\infty$.

Proposition 9.21 *Let* $\Omega \subset \mathbb{R}^n$ *be an open set and* $f \in L^1_{\mathrm{loc}}(\Omega)$, $0 \leq \alpha < n$. *Define*

$$\Sigma_\alpha := \left\{ x \in \Omega : \limsup_{\rho \to 0} \frac{1}{\rho^\alpha} \int_{B_\rho(x)} |f| dx > 0 \right\}.$$

Then $\mathcal{H}^\alpha(\Sigma_\alpha) = 0$. *In particular* $\dim^{\mathcal{H}}(\Sigma_\alpha) \leq \alpha$.

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Lemma 1.72 (Vitali covering lemma) *Let $\mathcal{G}$ be an arbitrary family of closed balls B in $\mathbb{R}^n$ with radius $r(B) \in (0, R]$ for some uniform constant $R < \infty$. There exists an at most countable subfamily $\mathcal{G}'$ of pairwise disjoint balls such that*

$$\bigcup_{B \in \mathcal{G}} B \subset \bigcup_{B \in \mathcal{G}'} \widehat{B} \qquad \text{with } \widehat{B} = B_{5r}(x_0) \text{ if } B = B_r(x_0)\,.$$

Hausdorff measure of singular set

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and in particular $\mathcal{L}^n(\mathrm{Sing}_0(u)) = 0$. *Moreover, for every* $\alpha \in (0,1)$ *there holds* $\mathrm{Reg}_0(u) = \mathrm{Reg}_\alpha(u)$, *i.e.* $u \in C^{0,\alpha}(\mathrm{Reg}_0(u), \mathbb{R}^N)$.

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Corollary 4.25 *Let $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be a weak solution to the system (4.13) under the assumptions of Theorem 4.23. Then we have* $\dim_{\mathcal{H}}(\operatorname{Sing}_0(u)) \le n - 2$.

Approaches to proof of the decay estimate

Lemma 4.21 (Excess decay estimate via blow up; [41], Lemma 4) *For every* $\tau \in (0,1)$ *there exist two positive constants* ε_0, R_0 *depending only on* n, N, L, ω, *and* τ *such that the following statement is true: if* $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ *is a weak solution to the system* (4.13) *with continuous coefficients* $a \colon \Omega \times \mathbb{R}^N \to \mathbb{R}^{Nn \times Nn}$ *satisfying* (4.6), (4.7) *and* (4.14), *and if for some ball* $B_R(x_0) \subset \Omega$ *with* $R \leq R_0$ *there holds*

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- Blow-up
- A harmonic approximation
- direct approach

Decay estimate via blow-up

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then we have the excess decay estimate

$$E(u; x_0, \tau R) \le c_*(n, N, L)\tau^2 E(u; x_0, R)\,. \tag{4.17}$$

Lemma 4.20 ([41], Lemma 2) *Let $(b_j)_{j\in\mathbb{N}}$ be a sequence of bilinear forms such that for every $j \in \mathbb{N}$ the functions $b_j\colon B_1 \to \mathbb{R}^{Nn\times Nn}$ are measurable, bounded and elliptic in the sense of*

$$b_j(x)\xi\cdot\xi \ge |\xi|^2$$

$$b_j(x)\xi\cdot\tilde{\xi} \le L|\xi||\tilde{\xi}|$$

for almost every $x \in B_1$, all $\xi, \tilde{\xi} \in \mathbb{R}^{Nn}$ and some $L \ge 1$. Suppose that b_j converges pointwise almost everywhere in B_1 to some bilinear form $b\colon B_1 \to \mathbb{R}^{Nn\times Nn}$. Let further $(u_j)_{j\in\mathbb{N}}$ be a sequence in $W^{1,2}(B_1, \mathbb{R}^N)$ such that u_j solves the system $\operatorname{div}(b_j(x)Du_j) = 0$ in B_1 in the weak sense for every $j \in \mathbb{N}$, and which converges weakly in $L^2(B_1, \mathbb{R}^N)$ to a function $u \in L^2(B_1, \mathbb{R}^N)$. Then $u \in W^{1,2}_{\mathrm{loc}}(B_1, \mathbb{R}^N)$, and we have

- (i) *$u_j \to u$ strongly in $L^2(B_\varrho, \mathbb{R}^N)$, $Du_j \rightharpoonup Du$ weakly in $L^2(B_\varrho, \mathbb{R}^{Nn})$ for every $\varrho < 1$;*
- (ii) *u solves the system $\operatorname{div}(b(x)Du) = 0$ in B_1 in the weak sense.*

Decay estimate via A-harmonic approximation

Definition 4.26 Let $\mathcal{A} \in \mathbb{R}^{Nn \times Nn}$. A function $h \in W^{1,1}(\Omega, \mathbb{R}^N)$ is called $\mathcal{A}$-harmonic if it satisfies

$$\int_\Omega \mathcal{A} Dh \cdot D\varphi \, dx = 0 \qquad \text{for all } \varphi \in C_0^1(\Omega, \mathbb{R}^N) .$$

Lemma 4.27 (De Giorgi; Duzaar and Grotowski) *Let $L \geq 1$ be a fixed constant, $n, N \in \mathbb{N}$ with $n \geq 2$ and $B_\varrho(x_0) \subset \mathbb{R}^n$. For every $\varepsilon > 0$ there exists $\delta = \delta(n, N, L, \varepsilon) > 0$ with the following property: if $\mathcal{A}$ is a constant bilinear form on $\mathbb{R}^{Nn}$ which is elliptic with (4.3) and bounded by L with (4.4), and if $u \in W^{1,2}(B_\varrho(x_0), \mathbb{R}^N)$ satisfies*

$$\varrho^{2\gamma - n} \int_{B_\varrho(x_0)} |Du|^2 \, dx \leq 1$$

(for some $\gamma \in \mathbb{R}$) and is approximately $\mathcal{A}$-harmonic in the sense of

$$\left| \varrho^{\gamma - n} \int_{B_\varrho(x_0)} \mathcal{A} Du \cdot D\varphi \, dx \right| \leq \delta \sup_{B_\varrho(x_0)} |D\varphi| \quad \text{for all } \varphi \in C_0^1(B_\varrho(x_0), \mathbb{R}^N) ,$$

then there exists an $\mathcal{A}$-harmonic function $h \in W^{1,2}(B_\varrho(x_0), \mathbb{R}^N)$ which satisfies

$$\varrho^{2\gamma - n - 2} \int_{B_\varrho(x_0)} |u - h|^2 \, dx \leq \varepsilon \quad \text{and} \quad \varrho^{2\gamma - n} \int_{B_\varrho(x_0)} |Dh|^2 \, dx \leq 1 . \quad (4.21)$$

Lemma 4.21 (Excess decay estimate via blow up; [41], Lemma 4) *For every $\tau \in (0,1)$ there exist two positive constants ε_0, R_0 depending only on n, N, L, ω, and τ such that the following statement is true: if $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ is a weak solution to the system (4.13) with continuous coefficients $a \colon \Omega \times \mathbb{R}^N \to \mathbb{R}^{Nn \times Nn}$ satisfying (4.6), (4.7) and (4.14), and if for some ball $B_R(x_0) \subset \Omega$ with $R \leq R_0$ there holds*

$$E(u; x_0, R) < \varepsilon_0^2 \,, \tag{4.16}$$

then we have the excess decay estimate

$$E(u; x_0, \tau R) \leq c_*(n, N, L) \tau^2 E(u; x_0, R) \,. \tag{4.17}$$

Lemma 4.28 (Approximate $\mathcal{A}$-harmonicity I) *Let $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be a weak solution to the system (4.13) with continuous coefficients $a \colon \Omega \times \mathbb{R}^N \to \mathbb{R}^{Nn \times Nn}$ satisfying (4.7) and (4.14). Then, for every $B_\varrho(x_0) \subset \Omega$ and all $u_0 \in \mathbb{R}^N$, we have*

$$\begin{aligned} &\Big| \varrho^{1-n} \int_{B_\varrho(x_0)} a(x_0, u_0) Du \cdot D\varphi \, dx \Big| \\ &\qquad \leq c(n, L) \omega^{1/2} \Big(\varrho + \Big(⨍_{B_\varrho(x_0)} |u - u_0|^2 \, dx \Big)^{\frac{1}{2}} \Big) \\ &\qquad\qquad \times \Big(\varrho^{2-n} \int_{B_\varrho(x_0)} |Du|^2 \, dx \Big)^{\frac{1}{2}} \sup_{B_\varrho(x_0)} |D\varphi| \end{aligned}$$

for all $\varphi \in C_0^1(B_\varrho(x_0), \mathbb{R}^N)$.

Decay estimate — direct approach

Lemma 4.21 (Excess decay estimate via blow up; [41], Lemma 4) *For every $\tau \in (0,1)$ there exist two positive constants ε_0, R_0 depending only on n, N, L, ω, and τ such that the following statement is true: if $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ is a weak solution to the system (4.13) with continuous coefficients $a\colon \Omega \times \mathbb{R}^N \to \mathbb{R}^{Nn\times Nn}$ satisfying (4.6), (4.7) and (4.14), and if for some ball $B_R(x_0) \subset \Omega$ with $R \le R_0$ there holds*

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then we have the excess decay estimate

$$E(u; x_0, \tau R) \le c_*(n, N, L)\tau^2 E(u; x_0, R)\,. \tag{4.17}$$

Theorem 1.22 (Gehring; Giaquinta and Modica) *Let $f \in L^1(B_R(x_0))$, $\sigma \in (0,1)$, and $m \in (0,1)$. Suppose that there exist a constant A and a function $g \in L^q(B_R(x_0))$ for some $q > 1$ such that for all balls $B_\varrho(y) \Subset B_R(x_0)$ there holds*

$$\fint_{B_{\sigma\varrho}(y)} |f|\,dx \le A\Big(\fint_{B_\varrho(y)} |f|^m\,dx\Big)^{\frac{1}{m}} + \fint_{B_\varrho(y)} |g|\,dx\,.$$

Then there exists an exponent $p \in (1, q]$ depending only on A, m and n such that $f \in L^p_{\mathrm{loc}}(B_R(x_0))$. Moreover, for every $\tau \in (0,1)$ we have

$$\Big(\fint_{B_{\tau R}(x_0)} |f|^p\,dx\Big)^{\frac{1}{p}} \le K(A, m, n, \tau)\Big[\fint_{B_R(x_0)} |f|\,dx + \Big(\fint_{B_R(x_0)} |g|^p\,dx\Big)^{\frac{1}{p}}\Big]\,.$$

Gehring theorem

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$$\fint_{B_{\sigma\varrho}(y)} |f|\,dx \le A\Big(\fint_{B_\varrho(y)} |f|^m\,dx\Big)^{\frac{1}{m}} + \fint_{B_\varrho(y)} |g|\,dx\,.$$

Then there exists an exponent $p \in (1,q]$ depending only on A, m and n such that $f \in L^p_{\mathrm{loc}}(B_R(x_0))$. Moreover, for every $\tau \in (0,1)$ we have

$$\Big(\fint_{B_{\tau R}(x_0)} |f|^p\,dx\Big)^{\frac{1}{p}} \le K(A,m,n,\tau)\Big[\fint_{B_R(x_0)} |f|\,dx + \Big(\fint_{B_R(x_0)} |g|^p\,dx\Big)^{\frac{1}{p}}\Big]\,.$$

[Beck, 2016]

Proposition 6.1. *Let Ω be a cube in $\mathbb{R}^n$ and let $g, h \in L^p(\Omega)$, $1 < p < \infty$, be nonnegative functions satisfying:*

$$\Big(\fint_Q g^p\Big)^{\frac{1}{p}} \le K \fint_{2Q} g + \Big(\fint_{2Q} h^p\Big)^{\frac{1}{p}} \tag{6.1}$$

for all cubes $Q \subset 2Q \subset \Omega$. Then for each $0 < \sigma < 1$ and $p < s < p + \frac{p-1}{10^{n+p}4^n K^p}$ we have

$$\Big(\fint_{\sigma\Omega} g^s\Big)^{\frac{1}{s}} \le \frac{100^n}{\sigma^{\frac{n}{s}}(1-\sigma)^{\frac{n}{p}}}\Big[\Big(\fint_\Omega g^p\Big)^{\frac{1}{p}} + \Big(\fint_\Omega h^s\Big)^{\frac{1}{s}}\Big] \tag{6.2}$$

[Iwaniec, 1998]

Gehring theorem

Lemma 3.2. *Suppose g and h are nonnegative functions of class $L^p(\mathbb{R}^n)$, with $1 < p < \infty$, and satisfy*

$$\left(\fint_Q g^p \right)^{\frac{1}{p}} \leq K \fint_{2Q} g + \left(\fint_{2Q} h^p \right)^{\frac{1}{p}} \tag{3.14}$$

for all cubes $Q \subset \mathbb{R}^n$. Then there exist a new exponent $s = s(n, p, K) > p$ and a constant $C = C(n, p, K)$ such that

$$\int_{\mathbb{R}^n} g^s \leq C \int_{\mathbb{R}^n} h^s \tag{3.15}$$

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