DEL: 4.21 - Blow-up

Sporem: P_p: $\forall c_* > 0$, $\exists \tau \in (0,1)$, $\varepsilon_j \to 0$, $R_j \to 0$,

$u_j \in W^{1,2}(\Omega, \mathbb{R}^N)$, a_j koeficienty, $x_j \in \Omega$; $B_{R_j}(x_j) \subset \Omega$

$$E(u_j, x_j, R_j) = \varepsilon_j^2 \quad \& \quad E(u_j, x_j, \tau R_j) > c_* \tau^2 E(u_j, x_j, R_j).$$

Plán:
- Škálování - blow up
- limitní předpis
- spor!

- Škálování: $$v_j(y) = \left[u_j(x_j + R_j y) - (u_j)_{x_j, R_j}\right] \frac{1}{\varepsilon_j}$$

$$b_j(y) = a_j\left(x_j + R_j y,\ v_j(y)\cdot \varepsilon_j + (u_j)_{x_j, R_j}\right)$$

$$\nabla v_j(y) = (\nabla u_j)(x_j + R_j y)\frac{R_j}{\varepsilon_j}$$

$$b_j(y)\nabla v_j(y) = a_j\left(x_j + R_j y, u_j(x_j + R_j y)\right)\nabla u_j(x_j + R_j y)\cdot\frac{R_j}{\varepsilon_j}$$

$$= a_j(\cdot, u_j(\cdot))(\nabla u_j)(\cdot)\circ(x_j + R_j y)\cdot\frac{R_j}{\varepsilon_j}$$

$$\Rightarrow \boxed{v_j \text{ řeší } \operatorname{div}(b_j(y)\nabla v_j(y)) = 0 \text{ v } B_1(0).}$$

$$\boxed{⨍_{B(0,1)} v_j} = \frac{1}{\varepsilon_j} ⨍_{B(0,1)} \left(u_j(\underbrace{x_j + R_j y}_{x}) - (u_j)_{x_j, R_j}\right)$$

$$= \frac{1}{\varepsilon_j}\left[\left(⨍_{B(x_j, R_j)} u_j\right) - (u_j)_{x_j, R_j}\right] = \boxed{0}$$

$\Rightarrow$ " $⨍_{B(0,\tau)} v_j = \frac{1}{\varepsilon_j}\left((u_j)_{x_j, \tau R_j} - (u_j)_{x_j, R_j}\right.$

$$E(v_j, 0, \tau) = \fint_{B(0,\tau)} \Big| \frac{1}{\varepsilon_j}\big(u_j(x_j + R_j y) - \cancel{(u_j)_{x_j,R_j}}\big) -$$

$$- \frac{1}{\varepsilon_j}\Big((u_j)_{x_j,\tau R_j} - \cancel{(u_j)_{x_j,R_j}} \Big) \Big|^2$$

$$= \frac{1}{\varepsilon_j^2} \fint_{B(0,\tau)} \big| u_j(x_j + R_j y) - (u_j)_{x_j,\tau R_j} \big|^2$$

$$= \frac{1}{\varepsilon_j^2} \fint_{B(x_j,\tau R_j)} \big| u_j - (u_j)_{x_j,\tau R_j} \big|^2 = \frac{1}{\varepsilon_j^2} E(u_j, x_j, \tau R_j)$$

$$\boxed{E(v_j, 0, 1) = \fint_{B(0,1)} |v_j|^2 = 1}$$

$$\boxed{E(v_j, 0, \tau) > C_* \tau^2 E(v_j, 0, 1)} = C_* \tau^2 \quad (\varepsilon)$$

$\boxed{\text{Limitní přechod}}$

$$v_j \xrightarrow{\varepsilon_j} 0 \text{ v } L^2(B(0,1))$$

s.v. (~~blaze~~) $\boxed{\text{až na podposl.}}$

$$a_j(x_j, (u_j)_{x_j,R_j}) \longrightarrow b \text{ v } \mathbb{R}^{(n\times N)^2}$$

$$b_j(y) \longrightarrow b \text{ s.v. v } B(0,1)$$

$$|b_j(y) - b| \le |a_j(x_j + R_j y, v_j(y)\varepsilon_j + (u_j)_{x_j,R_j}) - a_j(x_j, (u_j)_{x_j,R_j})|$$

$$+ \underbrace{|a_j(x_j, (u_j)_{x_j,R_j}) - b|}_{\longrightarrow 0} = \mathrm{I} + \mathrm{II}$$

$$\mathrm{I} \le \omega\left(|R_j y| + |v_j(y)\varepsilon_j|\right) \longrightarrow 0 \text{ s.v.}$$

- $v_j \rightharpoonup v$ v $L^2(B(0,1))$
- Caccioppoli nerovnost: $\|\nabla v_j\|^2_{L^2(B(0,\varrho))} \le C(\varrho)\|v_j\|^2_{L^2(B(0,1))} \le C(\varrho)$, $\varrho \in (0,1)$
- $\nabla v_j \rightharpoonup \nabla v$ v $L^2(B(0,\varrho))$ $\forall \varrho \in (0,1)$
- $v_j \to v$ na $L^2(B(0,\varrho))$ $\forall \varrho \in (0,1)$

$\Rightarrow$ v řeší $\displaystyle\int_{B(0,1)} b \nabla v \nabla \varphi = 0 \quad \forall \varphi \in \mathcal{D}(B(0,1))$

$$\Big(b\nabla v \nabla\varphi = (b - b_j)\nabla v_j \nabla\varphi + b_j(\nabla v - \nabla v_j)\nabla\varphi + b_j \nabla v_j \nabla \varphi = \alpha_1 + \alpha_2 + \alpha_3$$

$\displaystyle\int_{B(0,1)} \alpha_3 = 0$, $\displaystyle\int_{B(0,1)} \alpha_2 \xrightarrow{j\to+\infty} 0$ (slabá konv. grad.)

$$\int_{B(0,1)} \alpha_1 \le \underbrace{\left(\int_{B(0,\varrho)} |\nabla v_j|^2\right)^{1/2}}_{< C} \cdot \underbrace{\left(\int \underbrace{|b - b_j|^2}_{\to 0 \text{ s.v.}} |\nabla\varphi|^2\right)^{1/2}}_{\to 0 \text{ Lebesgue}} \xrightarrow{j\to+\infty} 0 \Big)$$

• limitní přechod v (ε)

$$\underline{E}(v_j, 0, \tau) \xrightarrow[j \to +\infty]{} E(v, 0, \tau)$$

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ale v splňuje: $\exists\, c > 0,\ c = c(n, N, L)$

$$E(v, 0, \tau) \leq c\,\tau^2 E(v, 0, 1) \leq c\,\tau^2$$

$(\varepsilon) \Rightarrow E(v, 0, \tau) > c_* \tau^2$

Volil $c_* > c$ ↯ spor $\perp$