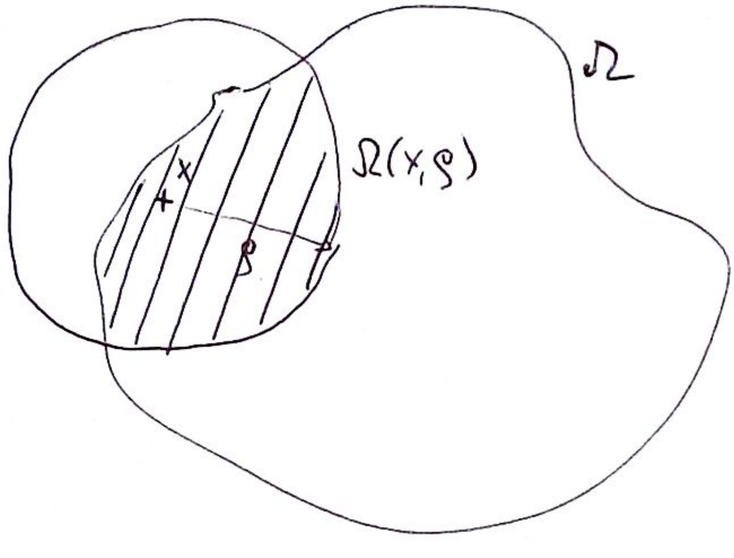
12.10.

Cíl: $-\Delta u = f \in C^{0,\alpha}(\bar\Omega) \Rightarrow \nabla u \in C^{0,\alpha}(\bar\Omega)$

+ odhad: $\|\nabla^2 u\|_{C^{0,\alpha}(\bar\Omega)} \le C \|f\|_{C^{0,\alpha}(\bar\Omega)}$.

Morrey:



$$\sup_{\substack{x\in\Omega \\ 0<\varrho<\operatorname{diam}\Omega}} \left(\frac{1}{\varrho^\lambda} \int_{\Omega(x,\varrho)} |u|^p \right) = \|u\|_{L^{p,\lambda}(\Omega)}$$

Campanato: • $p=2$: minimalizuji $\int_{\Omega(x,\varrho)} |u-c|^2$ přes $c \in \mathbb{R}$

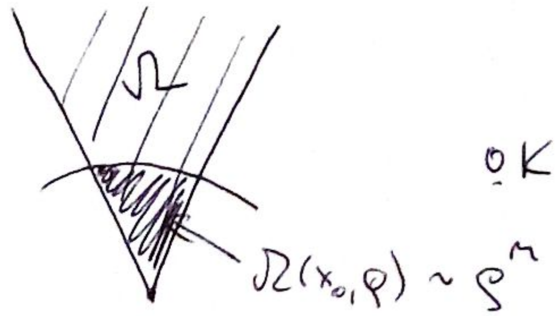
$$\Rightarrow c = (u)_{x_0,\varrho} \qquad \left| \int_{\Omega(x_0,\varrho)} (u-c) \right|$$

• $$\int_{\Omega(x_0,\varrho)} |u-(u)_{x_0,\varrho}|^p \overset{2^{p-1}}{\le} C \int_{\Omega(x_0,\varrho)} \left(|u-c|^p + |(u)_{x_0,\varrho} - c|^p \right)$$

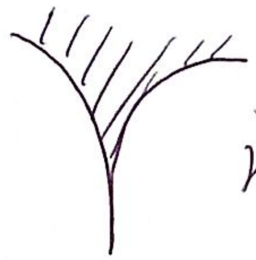
$$\le C \int_{\Omega(x_0,\varrho)} |u-c|^p$$

Jensen = Hölder

K definici 1.3:



OK.



jehla ne!

K Prop 1.1:

a) $\|u\|_{L^{p,0}(\Omega)} = \left(\sup_{x,\rho} \int_{\Omega(x,\rho)} |u|^p\right)^{1/p} = \left(\int_\Omega |u|^p\right)^{1/p} = \|u\|_{L^p(\Omega)}$

b) $\|u\|_{L^{p,n}(\Omega)} = \left(\sup_{x,\rho} \frac{1}{\rho^n}\int_{\Omega(x,\rho)} |u|^p\right)^{1/p}$

$u \in L^{p,n}(\Omega) \Rightarrow |u(x_0)| \le \left(\sup_{\operatorname{diam}\Omega \ge \rho > 0} ⨍_{\Omega(x_0,\rho)} |u|^p\right)^{1/p} \le C\,\|u\|_{L^{p,n}(\Omega)}$

$u \in L^\infty(\Omega) \Rightarrow u \in L^{p,n}(\Omega)$ + obráceně.

c) $\lambda > n \Rightarrow L^{p,n}(\Omega) = \{0\}$: $\underbrace{\frac{1}{\rho^{\lambda-n}}}_{\to +\infty} \cdot \underbrace{\frac{1}{\rho^n}\int_{\Omega(x_0,\rho)} |u|^p}_{\xrightarrow[\rho\to 0]{} |u(x_0)|^p \text{ s.v.}}$ je omezené!

$\Rightarrow |u(x_0)| = 0$ s.v.

d) ~~$u \in L^{p,\lambda}(\Omega)$~~ $u \in L^{q,\mu}(\Omega)$

$$\frac{1}{\rho^\lambda}\int_{\Omega(x_0,\rho)} |u|^p \le \frac{1}{\rho^\lambda}\Big(\int_{\Omega(x_0,\rho)} |u|^q\Big)^{\frac{p}{q}} |\Omega(x_0,\rho)|^{1-\frac{p}{q}} \overset{c}{\le} \Big[\Big(\int_{\Omega(x_0,\rho)} |u|^q\Big)^{1/q} \rho^{\frac{n}{p}-\frac{n}{q}-\frac{\lambda}{p}}\Big]^p$$

$$\frac{n}{p}-\frac{n}{q}-\frac{\lambda}{p}$$

$\le \|u\|^p_{L^{q,\mu}(\Omega)}$; pokud

$$n\frac{q}{p} - n - \lambda\frac{q}{p} \ge -\mu$$

$$\frac{n-\lambda}{p} \ge \frac{n-\mu}{q}$$

~~R~~ ~~$\ge$~~ nebo ~~$\not\le$~~ $\perp$

K Prop 1.2:

Jasné: $[u]_{p,\lambda} \le \Big\}\, \|u\|_{L^{p,\lambda}(\Omega)} \cdot C$

$\|u\|_p \le$

$$\Rightarrow \|u\|_{\mathcal{L}^{p,\lambda}(\Omega)} \le \|u\|_{L^{p,\lambda}(\Omega)} \cdot C$$

Pro obrácenou implikaci dokážeme lemma (viz další strana)

$$u \in \mathcal{L}^{p,\lambda}(\Omega) \overset{?}{\Rightarrow} u \in L^{p,\lambda}(\Omega)$$

$$\frac{1}{\rho^\lambda}\int_{\Omega(x_0,\rho)} |u|^p \le \frac{1}{\rho^\lambda}\int_{\Omega(x_0,\rho)}\big(|u-(u)_{x_0,\rho}|^p + |u|^p_{x_0,\rho}\big)$$

$$\le [u]^p_{p,\lambda} + C|(u)_{x_0,\rho}|^p \rho^{n-\lambda} \qquad \boxed{n > \lambda}$$

Lemma: $u \in \mathcal{L}^{p,\lambda}(\Omega)$, $x_0 \in \Omega$. Pak ex. $c_2 > 0$,

$\forall\, 0 < r < R < \operatorname{diam} \Omega$

$$|u_{x_0,R} - u_{x_0,r}| \le c_2 [u]_{p,\lambda} R^{\frac{\lambda}{p}} r^{-\frac{n}{p}}$$

Je-li navíc: $0<\lambda<n$: $\quad |u_{x_0,R} - u_{x_0,r}| \le c_2 [u]_{p,\lambda} r^{\frac{\lambda-n}{p}}$

$\lambda > n$: $\quad \le c_2 [u]_{p,\lambda} R^{\frac{\lambda-n}{p}}$

Dk: $|u_{x_0,R} - u_{x_0,r}|^p \le 2^{p-2}\left(|u(x) - u_{x_0,R}|^p + |u(x) - u_{x_0,r}|^p\right)$

$\pm u(x)$

$$R^{-\lambda+\lambda} \qquad r^{-\lambda+\lambda} ⨍_{\Omega(x_0,r)}$$

$$|u_{x_0,R} - u_{x_0,r}|^p \le 2^{p-2}\left(\frac{1}{|\Omega(x_0,R)|}\int_{\Omega(x_0,R)} |u(x) - u_{x_0,R}|^p + ⨍_{\Omega(x_0,r)} |u(x) - u(x_0,r)|^p\right)$$

$$\le C r^{-n}(R^{\lambda} + r^{\lambda})[u]_{p,\lambda} \le C r^{-n} R^{\lambda} [u]^p_{p,\lambda}$$

Odstraníme R: $0 < r < 2r < \dots < 2^{k} r < R \le 2^{k+1} r$.

$k \in \mathbb{N}$

$$* := |u_{x_0,R} - u_{x_0,r}| \le |u_{x_0,R} - u_{x_0,2^k r}| + |u_{x_0,2^k r} - u_{x_0,\dots}| + \dots + |u_{x_0,2r} - u_{x_0,r}|$$

$$\le C [u]_{p,\lambda} \sum_{k=1}^{k+1} (2^{k} r)^{\frac{\lambda}{p}} (2^{k-1} r)^{-\frac{n}{p}} = 2^{n/p} C [u]_{p,\lambda} r^{\frac{\lambda-n}{p}} \sum_{\ell=1}^{k+1} 2^{\ell\left(\frac{\lambda-n}{p}\right)}$$

- $\lambda < n$: $(*) \le C [u]_{p,\lambda} r^{\frac{\lambda-n}{p}} \dfrac{1}{1 - 2^{\frac{\lambda-n}{p}}}$

- $\lambda > n$: $(*) \le C [u]_{p,\lambda} r^{\frac{\lambda-n}{p}} \dfrac{2^{(k+2)\left(\frac{\lambda-n}{p}\right)}}{2^{\frac{\lambda-n}{p}} - 1}$

$$\le C [u]_{p,\lambda} (r 2^{k})^{\frac{\lambda-n}{p}} \cdot 2^{2\frac{\lambda-n}{p}} \le C [u]_{p,\lambda} R^{\frac{\lambda-n}{p}} \quad \perp$$

Dokončení dk. Prop 1.2:

$$c\,|(u)_{x_0,\varrho}|^p\,\varrho^{n-\lambda} \le c\,|(u)_{x_0,\varrho}-(u)_{x_0,R}|^p\,\varrho^{n-\lambda} + c\,|(u)_{x_0,R}|^p\,\varrho^{n-\lambda}$$

$R > \varrho$

Lemma

$\varrho < 1$

$$\le c\,[u]^p_{p,\lambda}\,\varrho^{\lambda-n}\cdot\varrho^{n-\lambda} + c\left|\int_\Omega u\right|^p$$

$$\le c\left([u]^p_{p,\lambda} + \|u\|^p_p\right) \le c\,\|u\|_{\mathcal{L}^{p,\lambda}(\Omega)}$$

$\perp$