4.1. - mmm 583

Dokončení Lemma 3.2 - Gehring

Víme: (3.15) $\left(\fint_Q g^p\right)^{1/p} \leq K \fint_{2Q} g + \left(\fint_{2Q} h^p\right)^{1/p}$

$\Rightarrow \exists \alpha, \beta > 0, \forall t > 0:$

$$(*) \qquad \int_{\{g>t\}} g^p \leq \alpha K t^{p-1} \int_{\{g>t\}} g + \beta \int_{\{h>t\}} h^p$$

Násob $(*)\ (t^\varepsilon)'$ a $\int_0^G dt$, kde G velké.

$$(LS): \int_0^G (t^\varepsilon)' \left(\int_{\{g>t\}} g^p \, dx \right) dt = \int_0^G \int_{\mathbb{R}^n} (t^\varepsilon)' g^p(x) \chi_{\{g(x)>t\}}(x) \, dx \, dt$$

$$= \int_{\mathbb{R}^n} \int_0^G (t^\varepsilon)' \chi_{\{g(x)>t\}} \, dt \; g^p(x) \, dx = \left| \begin{array}{l} 0 < t < g(x) \\ t < G \end{array} \right.$$

$$= \int_{\{g>G\}} \int_0^G (t^\varepsilon)' dt \; g^p(x) \, dx + \int_{\{g<G\}} \int_0^{g(x)} (t^\varepsilon)' dt \; g^p(x) \, dx$$

$$= \int_{\{g>G\}} G^\varepsilon g^p(x) \, dx + \int_{\{g<G\}} g^{p+\varepsilon}(x) \, dx$$

• $\int_0^G \underbrace{\varepsilon t^{\varepsilon-1} t^{p-1}}\Big(\int_{\{g>t\}} g\Big)\,dt = \int_0^G \frac{\varepsilon}{p-1+\varepsilon}\big(t^{p-1+\varepsilon}\big)'\Big(\int_{\{g>t\}} g\Big)\,dt$

$$t^{p-2+\varepsilon} = \frac{\big(t^{p-1+\varepsilon}\big)'}{p-1+\varepsilon}$$

$$\Rightarrow \frac{\varepsilon}{p-1+\varepsilon}\left[\int_{\{g>G\}} G^{p-1+\varepsilon} g(x)\,dx + \int_{\{g<G\}} g^{p+\varepsilon}(x)\,dx\right]$$

$$\int_0^G (t^{\varepsilon})'\Big(\int_{\{h>t\}} h^p\Big)\,dt \le \int_{\mathbb{R}^n} h^{p+\varepsilon}$$

<u>Dohromady:</u>

odhad: $G < g$

$$\underbrace{\int_{\{g>G\}} G^{\varepsilon} g^p(x)\,dx}_{\substack{<+\infty \\ \Leftarrow g\in L^p(\mathbb{R}^n)}} + \underbrace{\int_{\{g<G\}} g^{p+\varepsilon}(x)\,dx}_{<+\infty} \le \frac{\alpha\varepsilon K^p}{p-1+\varepsilon}\left[\int_{\{g>G\}} G^{\varepsilon} g^p(x)\,dx + \int_{\{g<G\}} g^{p+\varepsilon}(x)\,dx\right] + \beta\int_{\mathbb{R}} h^{p+\varepsilon}$$

$$\left(< \int_{\{g\in[0,1]\}} g^{p+\varepsilon} + \int_{\{g\in(1,G)\}} g^{p+\varepsilon} \le \int_{\{g\in[0,1]\}} g^p + G^{p+\varepsilon}\,|\{g>1\}| < +\infty \right)$$

$$\cancel{1 - \frac{\alpha \varepsilon K^p}{p-1+\varepsilon}}$$

Stačí volit $\varepsilon > 0$ tak, aby $\dfrac{\alpha \varepsilon K^p}{p-1+\varepsilon} < \dfrac{1}{2}$,

$$\Rightarrow \int_{\{g<\sigma\}} g^{p+\varepsilon} \leq 2\beta \int_{\mathbb{R}^n} h^{p+\varepsilon} \quad \forall \sigma > 1$$

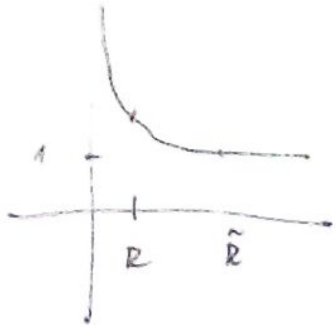
$$\overset{\text{Leb.}}{\Rightarrow} \int_{\mathbb{R}^n} g^{p+\varepsilon} \leq 2\beta \int_{\mathbb{R}^n} h^{p+\varepsilon}$$

$$\varepsilon \sim c \frac{1}{\alpha K^p}.$$

$\perp$.

<u>Dk: Proposition 6.1</u>

Def $\varrho(x) := \left[\operatorname{dist}(x, \mathbb{R}^n \setminus \Omega)\right]^{n/p}$



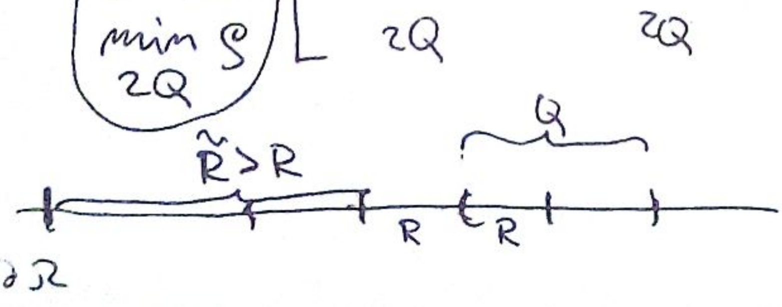
Cíl: Ověřit pp. Lem 3.2 pro $\varrho \cdot g$.

Fix $Q \cap \Omega \neq \emptyset$ (jinak triv)

A) $3Q \subset \Omega$:

$$\left(⨍_Q (\varrho g)^p\right)^{1/p} \leq \max_Q \varrho \left(⨍_Q g^p\right)^{1/p} \overset{(6.1)}{\leq} \max_Q \varrho \left(K ⨍_{2Q} g + \left(⨍_{2Q} h^p\right)^{1/p}\right)$$

$$\leq \underbrace{\frac{\max_Q \varrho}{\min_{2Q} \varrho}}^{< C} \left[K ⨍_{2Q} (\varrho g) + \left(⨍_{2Q} (\varrho h)^p\right)^{1/p}\right]$$



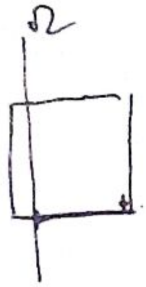
$\tilde{R} > R$

$$\frac{\tilde{R} + 3R}{\tilde{R}} \leq 4$$

B) $3Q \not\subset \Omega$

$$\fint_Q (\rho g)^p \le \int_Q g^p \cdot \frac{\max \rho^p}{|Q|} \le c \int_\Omega g^p \fint_{2Q} \chi_\Omega^p$$

$$\to \max_Q \rho^p = \max_Q \operatorname{dist}(x, \mathbb{R}^n \setminus \Omega)^n \le |\Omega \cap Q|$$



Dohromady: $$\left(\fint_Q (\rho g)^p\right)^{1/p} \le cK \fint_{2Q} \rho g + c \Big(\underbrace{\fint_{2Q} \Big(\rho h + c \int_\Omega g^p \chi_\Omega\Big)^p}_{H^p} \Big)^{1/p}$$

$$H^p = \left[\rho h + c\left(\int_\Omega g^p\right)^{1/p} \chi_\Omega\right]^p ; \quad \sigma = \rho g$$

L3.2 : $\exists \varepsilon \sim \frac{c}{KP}$; $s \in (p, p+\varepsilon)$

$\Rightarrow$

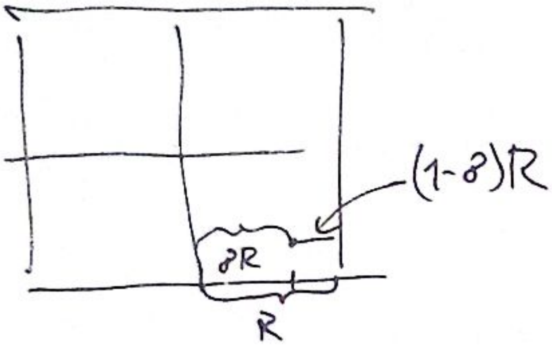
$$\int_{\mathbb{R}^n} \sigma^s \le c\left[\int_{\mathbb{R}^n} (\rho h)^s + c\left(\int_\Omega g^p\right)^{s/p} \chi_\Omega^s\right]$$

$\sigma = \rho g$

$$\int_{\delta\Omega} g^s \le \int_{\delta\Omega \to \mathbb{R}^n} (\rho g)^s \frac{1}{(\min_{\delta\Omega} \rho)^s} \le c\left[\left(\frac{\max \rho}{\min_{\delta\Omega} \rho}\right)^s \int_\Omega h^s + \left(\int_\Omega g^p\right)^{s/p} \frac{|\Omega|}{(\min_{\delta\Omega} \rho)^s}\right]$$

Délka hrany Ω : $2R$ $\quad \max\limits \varrho = \left(\frac{R}{\cancel{\ }}\right)^{n/p}$

$$\min_{\delta\Omega} \varrho = \left((1-\delta)R\right)^{n/p}$$



$$|\delta\Omega| = \delta^n |\Omega|$$

$$\fint_{\delta\Omega} g^s \le c \frac{1}{\delta^n} \frac{1}{(1-\delta)^{ns/p}} \fint_{\Omega} h^s + \left(\int_{\Omega} g^p\right)^{s/p} \frac{1}{\delta^n} \frac{1}{(1-\delta)^{ns/p} R^{ns/p}}$$

$$\le c \frac{1}{\delta^n (1-\delta)^{ns/p}} \left[\fint_{\Omega} h^s + \left(\fint_{\Omega} g^p\right)^{s/p} \right]$$

$\perp$

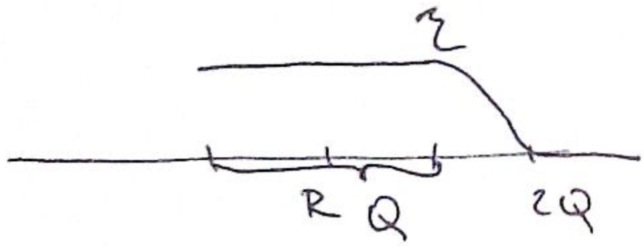
Aplikace Gehringova lemmatu na kvazilineární problém:

$$-\operatorname{div}\left(a(x,u)\nabla u\right) = 0 \quad \left(= \operatorname{div} f\right)$$

Test: $\left(u-(u)_{2Q}\right)\xi^2$; kde ξ je uřezávací fce

$$\chi_Q \le \xi \le \chi_{2Q} \; ; \; \xi \in C^\infty(\mathbb{R}^n)$$

$$|\nabla \xi| \le C\cdot\frac{1}{R} \; ; \; R \ldots \text{hrana } Q$$



$$\int a(x,u)\nabla u\nabla u\,\xi^2 \le \left|\int \underbrace{a(x,u)}_{\le L}\nabla u\,\left(u-(u)_{2Q}\right)\nabla\xi\,\xi\right|$$

$$c\int_{\Lambda} |\nabla u|^2\xi^2 \le C\int |\nabla u|\,\xi\,\frac{|u-(u)_{2Q}|}{R}$$

$$\underset{\text{Hölder}}{\le} \varepsilon\int |\nabla u|^2\xi^2 + C\underbrace{\int}_{\{\operatorname{spt}\xi\}=2Q}\left(\frac{|u-(u)_{2Q}|}{R}\right)^2$$

Sobolev – Poincaré

$$\Longrightarrow \left(\fint_Q |\nabla u|^2\right)^{1/2} \le C\left(\fint_{2Q}\left|\frac{u-(u)_{2Q}}{R}\right|^2\right)^{1/2} \le C\left(\fint_{2Q}|\nabla u|^r\right)^{1/r}$$

$$W^{1,r} \hookrightarrow L^2, \quad r<2$$

Gehring
$\Rightarrow$ $$\exists s>2: \left(⨍_{Q} |\nabla u|^s\right)^{1/s} \le C\left(⨍_{2Q} |\nabla u|^2\right)^{1/2} \quad \perp.$$

$$\left(+ \left(⨍_{2Q} f^s\right)^{1/s}\right)$$

Odhad členu s f:

$$\left|\int_{2Q} f\, \nabla\left((u-(u)_{2Q})\, \xi^2\right)\right| \le \left|\int_{2Q} f\left(\nabla u \cdot \xi^2 + (u-(u)_{2Q}) \nabla \xi^{2}\, \xi\right)\right|$$

$$\le \varepsilon \int_{2Q} |\nabla u|^2 \xi^2 + c\int_{2Q} |f|^2 + C_\varepsilon \int_{2Q} \left|\frac{u-(u)_{2Q}}{R}\right|^2$$

Sobolev-Poincaré
$$\le \quad \varepsilon \int_{2Q} |\nabla u|^2 + c\int_{2Q} |f|^2 + c\left(\int_{2Q} |\nabla u|^r\right)^{2/r} |2Q|^{1-\frac{2}{r}}$$