Lecture Notes on the Course
"Entropy Methods
and Related Functional Inequalities"

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Introduction

This manuscript contains the enriched lecture notes of my six-lesson short course for graduate students on entropy methods, held at the Università di Pavia (Italy) around Christmas 2007.

Outline. These lecture notes are intended as a brief introduction to the various application of entropy methods in the theory of nonlinear evolution equations. The emphasis is on the equilibration properties of the respective solutions, but it is also shown how entropies are used to derive nonlinear functional inequalities and establish qualitative properties of the solutions. The first lecture provides a brief sketch of the treatment of several easy but typical examples, like the *homogeneous Boltzmann* and the *radiative transfer equation*. The second and third lecture, respectively, cover Toscani's approach to the Bakry-Émery method for *linear and non-linear second-order diffusions*. The remaining three lectures are entirely devoted to a selection of results on the one-dimensional *thin film equation*. Positivity properties are the topic of lectures four and five, while lecture six deals with the equilibration of classical solutions.
In their current form, these notes contain noticeably more material than what has been covered in the original material lectures at Pavia. Moreover, the problem sheets for the exercise classes are included; most of the solutions have been incorporated into the main text.

Disclaimers. I do not claim completeness of these notes in any respect! The following pages simply represent one (biased and narrow) selection of material from a research field that has been heavily investigated for about a hundred years by now. I have decided to cover entropy methods from the PDE point of view, thus more or less ignoring the numerous important applications in probability theory. Likewise, kinetic theory — the cradle of entropy methods — is only briefly touched. Also two of the "canonical" applications of entropy methods are excluded here: the derivation of a priori estimates and the proof of smoothness properties. I found both subjects are too technical to enter in these short notes.
Apart from the above, two more warnings are due:

- Most of the calculations in these notes are formal. As an excuse, let me point out that the rigorous part of the proofs is usually both a painful and a boring issue. However, I cite the original papers in which the full proofs can be found.
- Some of the results presented in the first half of the notes are a bit outdated in the sense that there exist more elegant (and often more general) proofs by now, using Wasserstein techniques that have been developed only recently. The whole topic of Wasserstein methods is not touched here; the interested reader might want to have a look at [**18, 1**].

Additional Reading. The most complete and up-to-date collection of references for entropy methods is most probably found in the still unpublished "encyclopedia" by Villani [**35**]. Moreover, apart from numerous review articles on specific aspects of entropy methods, there are some short lecture notes [**2, 26**] available on the web. The latter are comparable in their view on the topic, but are much shorter and more focussed on a particular sub-topic.

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CHAPTER 1

Historical and Pedagogical Examples

1. A definition of entropies

Generally speaking, an entropy is a Lyapunov functional of a specific form. It is however hard (and even somewhat artificial) to give a formal narrow definition of entropies that distinguishes them from, say, energies.

In this lecture, we will be concerned with evolution equations

$$\partial_t u(t;x) = F\big(u(t;x)\big) \tag{1.1}$$

that describe — in a wide sense of the word — the behavior of a particle density $u(t;x)$ in some domain $\Omega \subset \mathbb{R}^d$. The natural spaces to work with are subsets U of $L^1_+(\Omega)$, the set of non-negative integrable functions, representing the particle density at a given time. These subsets U may be specified by additional integrability assumptions or constraints on moments. In particular, U should be chosen so that there exists exactly one stationary density u_∞ for (1.1). For instance, we shall frequently use $U = \mathcal{P}(\Omega)$, the space of probability densities on Ω.

There are two principal types of entropies which are considered here in detail. The first are *absolute* entropies, which are defined through a function $\psi : \mathbb{R}_{\geq 0} \times \Omega \to \mathbb{R}$ by

$$H[u] = H_\psi[u] = \int_\Omega \psi\big(u(x), x\big)\, dx. \tag{1.2}$$

Naturally, ψ is chosen so that u_∞ is the unique minimizer of H_ψ on U. This kind of entropies will be used e.g. in the context of nonlinear diffusion in lecture 3.

The second kind of entropies are *relative* ones, which are a little more special. Instead of the Lebesgue measure on Ω, now the steady state u_∞ is taken as reference measure. For a given convex function $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}$,

$$H[u] = H_\phi[u] = \int_\Omega \phi\big(\rho(x)\big)\, u_\infty(x) dx, \quad \rho(x) = \frac{u(x)}{u_\infty(x)}. \tag{1.3}$$

This definition, of course, is sensible only if the steady state is everywhere positive. Applications of this entropy are found e.g. in the linear Bakry-Emery theory, see lecture 2.

Notice that any relative entropy corresponds to an absolute one via the identification

$$\psi(s,x) := \phi\big(s/u_\infty(x)\big) u_\infty(x),$$

but not the other way around. Obviously, both types of entropy agree if Ω is some bounded domain and u_∞ is a constant. This, in fact, is the typical situation for the second part of the lecture.

Apart from the principle types of entropies (1.2) and (1.3), we will occasinally consider further functionals in the context of the thin film equation. These functionals are integrals which contain *spatial derivatives* of u; however, we shall refer to them as *energies* rather than entropies.

Here are the two properties which we shall require in order to call the H in (1.2) or (1.3), respectively, an entropy.

(1) **Lyapunov Property** $H[u(t)]$ is non-increasing along solutions $u(t)$ to (1.1), and the *entropy production*

$$D_H[u] := -\frac{d}{dt} H[u]$$

is positive unless $u(t) = u_\infty$.

(2) **Equilibration Property** H is convex as a functional on $L^1(\Omega)$, and there is a constant $C > 0$ such that

$$\|u - u_\infty\|_{L^1} \le C \cdot \big(H[u] - H[u_\infty]\big)^{1/2} \text{for all } u \in U.$$

Finally, we introduce a concept of convergence that will frequently appear in our considerations. An entropy functional H and D_H, respectively, are said to *converge exponentially at rate* $\mu > 0$ if

$$t \mapsto e^{\mu t}\big(H[u(t)] - H[u_0]\big), \text{ and } t \mapsto e^{\mu t} D[u(t)]$$

are non-increasing with respect to time $t \ge 0$ along all (sufficiently regular) solutions $u : [0, \infty) \to U$ to equation (1.1).
The rest of this introductionary lecture is devoted to examples, which are supposed to shed light on this somewhat abstract definition.

2. Example: Gradient flow with convex potential

Here is a finite-dimensional toy model, which is simply included for pedagogical reasons. Let an open domain $U \subset \mathbb{R}^n$ be given, with a smooth potential $H : U \to \mathbb{R}$ on it. Assume that H possesses a unique minimum $u_\infty \in U$, and is λ-convex, i.e.

$$\nabla^2 H \ge \lambda \mathbf{1}$$

uniformly on U, with some positive number $\lambda > 0$.

THEOREM 1.1. *H is an entropy for its own gradient flow $\dot{u} = -\nabla H(u)$. Moreover, H and D_H converge exponentially at rate 2λ.*

PROOF. The entropy production amounts to

$$D_H[u(t)] = -\frac{d}{dt} H[u(t)] = \|\nabla H[u(t)]\|^2 \ge 0,$$

with equality only at the $u = u_\infty$. A Taylor expansion yields

$$H[u] - H[u_\infty] = \underbrace{\nabla H[u_\infty]}_{=0} \cdot (u - u_\infty) + \frac{1}{2} \underbrace{(u - u_\infty) \cdot \nabla^2 H(\tilde{u}) \cdot (u - u_\infty)}_{\ge \lambda \|u - u_\infty\|^2},$$

so that

$$\|u - u_\infty\|^2 \le \frac{2}{\lambda}(H[u] - H[u_\infty]).$$

This obviously proves the equilibration property of H on $U \subset \mathbb{R}^n$ (recall that there is no distinction between the L^1 and the L^2-norm in finite dimensions).
In order to prove that H converges exponentially at rate 2λ, observe that by convexity,

$$\begin{aligned} H[u] - H[u_\infty] &\le \nabla H[u] \cdot (u - u_\infty) - \frac{\lambda}{2}\|u - u_\infty\|^2 \\ &\le \frac{1}{2\lambda}\|\nabla H[u]\|^2 = \frac{1}{2\lambda} D_H[u]. \end{aligned}$$

By Gronwall's inequality, we conclude that $\exp(2\lambda t)(H[u(t)] - H[u_\infty])$ is non-increasing in time. On the other hand,

$$\frac{d}{dt} D_H[u] = -2\nabla H(u) \cdot \nabla^2 H(u) \cdot \nabla H(u) \le -2\lambda \big\|\nabla H(u)\big\|^2 = -2\lambda D[u],$$

proving monotonicity of $\exp(2\lambda t) D[u(t)]$. □

A trivial consequence of the proof is the a priori estimate

$$\frac{\lambda}{2}\|u - u_\infty\|^2 \le H[u(t)] - H[u_\infty] \le (H[u_0] - H[u_\infty]) \exp(-2\lambda t).$$

Moreover, this toy example shows that in general, entropies are not unique (even after identifying trivial deformations of an existing entropy). For instance, also $\tilde{H}[u] = \frac{1}{2}\|u - u_\infty\|^2$ is an entropy, since $\tilde{H}$ is strictly convex along rays through the fixed point:

$$D_{\tilde{H}} = -\frac{d}{dt}\tilde{H}[u(t)] = -\nabla H(u) \cdot (u - u_\infty) = H[u] - H[u_\infty] + \frac{\lambda}{2}\|u - u_\infty\|^2 \geq 0,$$

with equality exactly for $u = u_\infty$.

However, we stress that the existence of an entropy *does not* imply that the flow is of gradient type in a suitable metric. Consider, for example, on $\mathbb{R}^2$

$$\dot{u} = F(u) = -u + \lambda J u, \qquad \lambda \geq 0, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1.4}$$

Obviously, $H[u] = u_1^2 + u_2^2$ is an entropy, but there is no (positive) metric which generates $F(u)$ in (1.4) as a gradient flow unless $\lambda = 0$. Indeed, assume that there exists some metric tensor $G(u)$ such that $G(u) \cdot F(u) = -\nabla\Phi(u)$ for a suitable potential Φ. It follows that $\nabla \times (G(u) \cdot F(u)) = -\nabla \times (\nabla\Phi(u)) = 0$, but

$$\begin{aligned}\nabla \times \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} \cdot \begin{pmatrix} u_1 + \lambda u_2 \\ u_2 - \lambda u_1 \end{pmatrix} &= (\lambda g_{11} + g_{12}) - (g_{12} - \lambda g_{22}) + O(\|u\|) \\ &= \lambda(g_{11} + g_{22}) + O(\|u\|\ ).\end{aligned}$$

For $\|u\|$ small enough, the last expression is positive if $\lambda > 0$.

3. Example: The Boltzmann equation and the H-Theorem

The next example is more serious and is actually a simplification of the situation which gave birth to the whole concept of entropies. Consider a d-dimensional tank with a well-mixed, mono-atomic gas. The homogeneous Boltzmann equation describes the temporal change in the probability to find molecules of a given velocity $v \in \mathbb{R}^d$ in the tank. The derivation of the Boltzmann equation is based on the assumption that the molecules move freely, and exchange momentum and energy in binary collisions. More precisely, when two atomes with (pre-collisional) velocities v and w, respectively, collide with contact line parallel to $n \in \mathbb{S}^{d-1}$, then the post-collisional velocities v^* and w^* are given by

$$v^* = \frac{1}{2}(v + w + |v - w|n), \quad w^* = \frac{1}{2}(v + w - |v - w|n). \tag{1.5}$$

These formulas follow from elementary geometric considerations under the further assumption that the total momentum and the total energy are conserved in each individual interaction.

Denote by $f(t; v)$ the probability density at time $t > 0$ to find a molecule with velocity $v \in \mathbb{R}^d$. Elementary considerations about the balance of gain and loss of atoms lead to the weak form of the homogeneous Boltzmann equation,

$$\frac{d}{dt}\int_{\mathbb{R}^d} \Phi(v) f(v)\, dv = \iiint_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} B(\nu, |v - w|)\big(\Phi(v_*) + \Phi(w_*) - \Phi(v) - \Phi(w)\big) \tag{1.6}$$

$$\big(f(v_*)f(w_*) - f(v)f(w)\big)\, dv\, dw\, dn \tag{1.7}$$

which holds for any sufficiently regular test function $\Phi : \mathbb{R}^d \to \mathbb{R}$. Here v, w and v^*, w^* are the pre- and post-collisional velocities, respectively, for a collision with contact line $n \in \mathbb{S}^{d-1}$; see (1.5). The collision kernel B cannot be determined from the previous considerations; here we assume that it only depends on the collision angle via

$$\nu = \frac{(v - w) \cdot n}{|v - w|}$$

and the modulus of the velocity difference.

A priori, it is not clear if solutions $f(t)$ to (1.6) tend to some limit f_∞ as $t \to \infty$. As individual microscopic particle interactions are reversible, it is not clear that $f(t)$ develops any trend at all. However, Boltzmann's H-Theorem states that (1.6) possesses an entropy.

Before stating the H-Theorem, a comment is in place which parameters of the initial condition f_0 determine the shape of f_∞. It is immediate to conclude that (1.6) preserves the total mass, momentum and energy of the gas. Simply use the test functions $\Phi \equiv 1$, $\Phi(v) = v$ and $\Phi(v) = |v|^2$, for which the right-hand side of (1.6) is identically zero. And in fact, it is exactly these three quantities which determine the shape of f_∞. For convenience, we assume unit mass, zero total momentum and temperature one (meaning that the second moment equals the number of spatial dimensions d).

THEOREM 1.2 (Boltzmann's H-Theorem). *The H-functional*

$$H[f] := \int f(v) \log f(v)\, dv \tag{1.8}$$

is an entropy for the Boltzmann equation on the domain $U \subset \mathcal{P}(\mathbb{R}^d)$ of probability densities f with vanishing first moment and unit temperature.

PROOF. Use $\Phi(v) = \log f(v)$ as test function in (1.6):

$$D_H[f(t)] = -\frac{d}{dt} H[f(t)] = \iiint B(\nu, |v-w|)\big(\log(f(v_*)f(w_*)) - \log(f(v)f(w))\big)\cdot \\ \big(f(v_*)f(w_*) - f(v)f(w)\big)\, dv\, dw\, dn.$$

Since $x \mapsto \log x$ is a strictly increasing function, the expression under the integral is non-negative. The integral vanishes iff $f(v^*)f(w^*) = f(v)f(w)$ a.e.

Next, one needs to identify the unique stationary solution f_∞. An initial guess (in agreement with physical intuition) is provided by the Gaussian,

$$f_\infty(v) = (2\pi)^{-d/2} \exp\big(-\frac{1}{2}|v|^2\big),$$

since it is formally a critical point of $H[f]$ under the given constraint. We show that f_∞ is indeed the unique minimizer.

To this end, we show that the H-functional is a relative entropy (1.3), defined by the convex function $\phi(s) = s \log s$ (up to an additive constant). To see this, observe that for $f \in U$,

$$\int f(v) \log f_\infty(v)\, dv = -\int f(v)(\log\sqrt{2\pi} + \frac{1}{2}|v|^2)\, dv = -\frac{1}{2} - \log\sqrt{2\pi} \\ = \int f_\infty(v) \log f_\infty(v)\, dv = H[f_\infty].$$

Now introduce $\rho = f/f_\infty$, which satisfies $\int \rho(v)\, f_\infty(v) dv = 1$. Then

$$H[f] - H[f_\infty] = \int_{\mathbb{R}^d} \rho(v) \log f(v)\, f_\infty(v) dv - \int_{\mathbb{R}^d} \rho(v) \log f_\infty(v)\, f_\infty(v) dv \\ = \int_{\mathbb{R}^d} \rho(v) \log \rho(v)\, f_\infty(v) dv.$$

By Jensen's inequality,

$$H[f] - H[f_\infty] \geq \Big(\int \rho(v)\, f_\infty(v) dv\Big) \log\Big(\int_{\mathbb{R}^d} \rho(v)\, f_\infty(v) dv\Big) = 0.$$

with equality exactly for $\rho \equiv 1$, i.e. for $f \equiv f_\infty \equiv M$. Finally, the relation

$$\|f - f_\infty\|_{L^1} \leq C \cdot \big(H[f] - H[f_\infty]\big)^{1/2} \tag{1.9}$$

is a particular case of the Csiszar-Kullback-inequality, which is proven in a more general setting in the following Proposition 1.1. □

PROPOSITION 1.1. *Let $\Omega \subset \mathbb{R}^d$ and $u_\infty : \Omega \to \mathbb{R}_+$ be a strictly positive probability density on Ω. Assume $\phi : \mathbb{R}_+ \to \mathbb{R}$ is smooth and convex with $\phi''(1) > 0$. For the relative entropy functional*

$$H_\phi[u] := \int_\Omega \phi\big(\frac{u}{u_\infty}\big)\, u_\infty dx \tag{1.10}$$

the Csiszar-Kullback inequality *holds*

$$\|u - u_\infty\|_{L^1(\Omega)} \le C\big(H_\phi[u] - H_\phi[u_\infty]\big)^{1/2} \tag{1.11}$$

for all probability densities $u \in \mathcal{P}(\Omega)$.

Proof. First observe that smoothness, convexity, and $\phi''(1) > 0$ imply that

$$\phi(s) - \phi(1) \ge \phi'(1)(s-1) + c(1-s)^2$$

for all $0 < s < 1$ with a suitable $c > 0$. Thus, since $u_\infty(x)dx$ defines a probability measure on Ω,

$$H_\phi[u] - H_\phi[u_\infty] = \int_\Omega \Big(\phi(\frac{u}{u_\infty}) - \phi(1)\Big)\, u_\infty dx \tag{1.12}$$

$$\ge \phi'(1) \underbrace{\int_\Omega (u - u_\infty)\, dx}_{=0} + c\int_\Omega \big(\frac{u}{u_\infty} - 1\big)^2 u_\infty dx. \tag{1.13}$$

The first integral above vanishes because u and u_∞ have the same mass. A further consequence of the latter fact is

$$\|u - u_\infty\|_{L^1} = 2\int_{u<u_\infty} |u - u_\infty|\, dx = 2\int_{u<u_\infty} \big|\frac{u}{u_\infty} - 1\big|\, dx \tag{1.14}$$

$$\le 2\Big(\int_\Omega \big(\frac{u}{u_\infty} - 1\big)^2 dx\Big)^{1/2}. \tag{1.15}$$

A combination of these two estimates gives (1.11), with $C = 1/c$. □

In the particular case of Boltzmann's H-functional, one has $\phi(s) = s\log s$ with $\phi''(1) = 0$, so Proposition 1.1 applies.
A question of big interest is how fast $H[f(t)]$ decays along a generic solution. This question, however, is suprisingly hard to answer. In general, one cannot expect exponential convergence of $f(t)$. Even in the easy situation where $B(\nu, |v-w|)$ only depends on ν, Bobylev constructed initial data f_0 such that the relaxation is exponential, but with an arbitrarily slow rate dependent on f_0. Later, he extended his result to basically all physically relevent kernels B; and his "violating" initial conditions were even such that they have all moments bounded.

4. Example: The radiative transfer equation

After sufficiently many simplifications, the transfer of a radiative density $u(x) \ge 0$ in a medium $\Omega \subset \mathbb{R}^d$ can be described by [**24**]

$$\frac{d}{dt}u(x) = -u(x) + \int_\Omega u(y)\,\mu(dy), \tag{1.16}$$

where μ is a given probability measure on Ω. Since (1.16) is linear in u and conserves the total mass, there is no loss in generality to restrict attention to $U = \mathcal{P}(\Omega) \subset L^1_+(\Omega)$.
The explicit solution for a given initial density u_0 is easily found:

$$u(t;x) = e^{-t}u_0(x) + (1 - e^{-t}), \tag{1.17}$$

which converges exponentially to $u_\infty \equiv 1$.

Theorem 1.3. *Any smooth, strictly convex function $\phi : \mathbb{R}_{\ge 0} \to \mathbb{R}$ with $\phi''(1) > 0$ gives rise to an entropy*

$$H_\phi[u] = \int_\Omega \phi(u(x))\,\mu(dx) \tag{1.18}$$

for the radiative transfer equation (1.16). *Moreover, each H_ϕ converges at rate one, and thus*

$$\|u(t) - u_\infty\|_{L^1} \le Ce^{-t/2}.$$

PROOF. First notice that by Jensen's inequality,

$$H_\phi[u] - H_\phi[u_\infty] = \int_\Omega \phi(u(x))\,\mu(dx) - \int \phi(1)\,\mu(dx) \ge \phi\Big(\int u(x)\,\mu(dx)\Big) - \phi(1) = 0.$$

The respective entropy production is

$$\begin{aligned} D_\phi[u] &= -\frac{d}{dt}E_\phi[u] = -\int_\Omega \phi'(u(x))\partial_t u(x)\,\mu(dx) \\ &= \iint_{\Omega\times\Omega} \phi'(u(x))(u(x)-u(y))\,\mu(dx)\mu(dy) \\ &= \frac{1}{2}\iint_{\Omega\times\Omega} \big(\phi'(u(x)) - \phi'(u(y))\big)(u(x)-u(y))\,\mu(dx)\mu(dy). \end{aligned}$$

By convexity of ϕ, the r.h.s., the integrand is non-negative, and by strict convexity, the integral is zero iff $u(x) = u(y)$ a.e. In fact, one can prove that

$$H_\phi[u(t)] - H_\phi[u_\infty] \le e^{-t}\big(H_\phi[u_0] - H_\phi[u_\infty]\big) \tag{1.19}$$

directly from here, without using the explicit formula (1.17). However, we shall take another route here. Convexity of ϕ implies for arbitrary $a, b, s \ge 0$

$$\begin{aligned} (\phi'(a) - \phi'(b))(a-b) &= (\phi'(a) - \phi'(b))(a-s) - (\phi'(a) - \phi'(b))(b-s) \\ &\ge \phi(a) - \phi(s) - \phi'(b)(a-s) - \phi'(a)(b-s) + \phi(b) - \phi(s). \end{aligned}$$

Now choose $a = u(x)$, $b = u(y)$ and $s = 1$, then integrate w.r.t. $\mu(dx)$ and $\mu(dy)$. This gives

$$\iint_{\Omega\times\Omega} \big(\phi'(u(x)) - \phi'(u(y))\big)(u(x)-u(y))\,\mu(dx)\mu(dy) \ge 2\int_\Omega \phi(u(x))\,\mu(dx) - 2\phi(1) \tag{1.20}$$

$$= 2\big(H_\phi[u] - H_\phi[u_\infty]\big). \tag{1.21}$$

Thus immediately implies that H converges exponentially at unit rate, and further implies (1.19) via Proposition 1.1. □

The preceeding proof, using the relation (1.20) rather than the explicit solution (1.17), seems circumstantial. Nonetheless, there are at least two reasons to take this long way. The first is that the radiative transfer equation (1.16) was indeed *invented* in [**24**] as a toy model to make the entropy techniques completely explicit in one easy example. The second reason is this approach allows for the following non-trivial generalization.

There is an equation closely related to (1.16), which is still easy but no longer explicitly solvable. Typically, its solutions decay only sub-exponentially in general. For a given positive function $\lambda : \Omega \to \mathbb{R}$, consider

$$\partial_t u(x) = -\lambda(x)u(x) + \int_\Omega \lambda(y)u(y)\,\mu(dy).$$

We shall assume that $\int_\Omega \lambda(x)^{-1}\mu(dx) = 1$ and that $M := \int_\Omega \lambda(x)^{-2}\mu(dx) < \infty$. Also, we only consider solutions with $\lambda(x)u(x) \le K$ (notice that this property is propagated from the initial condition to any time $t > 0$). As above, we find that for an arbitrary smooth and strictly convex function ϕ,

$$\tilde E_\phi[u] = \int_\Omega \phi\big(\lambda(x)u(x)\big)\,\frac{\mu(dx)}{\lambda(x)}$$

is an entropy. In fact,

$$\begin{aligned} \tilde D_\phi[u] = -\frac{d}{dt}\tilde E_\phi[u] &= \iint_{\Omega\times\Omega} \phi'\big(\lambda(x)u(x)\big)\big[\lambda(x)u(x) - \lambda(y)u(y)\big]\,\mu(dx)\mu(dy) \\ &= \frac{1}{2}\iint_{\Omega\times\Omega} \big[\phi'\big(\lambda(x)u(x)\big) - \phi'\big(\lambda(y)u(y)\big)\big]\cdot\big[\lambda(x)u(x) - \lambda(y)u(y)\big]\,\mu(dx)\mu(dy). \end{aligned}$$

Replace u in (1.20) by λu, and μ by μ/λ. Now split the integral over $\Omega\times\Omega$ into two parts I_1+I_2, where I_1 corresponds to $\lambda(x)\lambda(y)<\epsilon$, and I_2 is the remainder. Then

$$\begin{aligned} I_1 &\le \iint_{\lambda(x)\lambda(y)<\epsilon} \phi'(K)\cdot K\frac{\mu(dx)\mu(dy)}{\lambda(x)\lambda(y)} \\ &\le K\phi'(K)\epsilon^2\Big(\int_\Omega \lambda(x)^{-2}\mu(dx)\Big)^2 = MK\phi'(K)\epsilon^2. \end{aligned}$$

On the other hand,

$$\begin{aligned} I_2 &\le \iint_{\lambda(x)\lambda(y)\ge\epsilon} \big[\phi'\big(\lambda(x)u(x)\big)-\phi'\big(\lambda(y)u(y)\big)\big]\cdot\big[\lambda(x)u(x)-\lambda(y)u(y)\big]\,\frac{\mu(dx)\mu(dy)}{\lambda(x)\lambda(y)} \\ &\le \frac{2}{\epsilon}\tilde D_\phi[u]. \end{aligned}$$

So, altogether, (1.20) leads to

$$\tilde H_\phi[u] \le \frac12(I_1+I_2) \le \frac12 MK\phi'(K)\epsilon^2 + \frac1\epsilon\tilde D_\phi[u].$$

With a choice $\epsilon=\tilde D_\phi[u]^{1/3}$, we arrive at

$$\tilde H_\phi[u] \le [MK\phi'(K)/2+1]\tilde D_\phi[u]^{2/3}.$$

This last line allows to relate $\tilde H_\phi$ to its time derivative, thus proving an algebraic-in-time decay,

$$\tilde H_\phi[u] \le (A+Bt)^{-2},$$

with suitable constants $A>0$, $B>0$. By the same arguments as in the proof above, L^1-decay at the rate t^{-1} follows.

5. Example: The Fokker-Planck equation

The study of entropies reached a new quality when it was observed that an analogue of Boltzmann's H-Theorem also holds for certain diffusion equations. Consider the simplest case, the heat equation,

$$\partial_t u(t;x) = \Delta u(t;x). \tag{1.22}$$

It is easy to verify that Boltzmann's H-functional also decays along solutions to (1.22). However, one finds that $H[u(t)]\to-\infty$ as $t\to\infty$. This is in accordance with the fact that the only "steady state" of the heat equation on $\mathbb{R}^d$ is $v_\infty\equiv 0$, which is not a probability density. Hence the H-functional does not capture any interesting behavior of the free heat equation. Surprisingly, it becomes extremely useful in the study of the fine asymptotics in the vicinity of the self-similar solution

$$U(t;x) = (2\pi(2t+1))^{-d/2}\exp(-\frac12 x^2/(2t+1)).$$

In order to capture such fine asymptotics, we move to a coordinate frame in which U becomes stationary. Introduce the scaling factor $\sigma(t)=\sqrt{2t+1}$ and

$$y=x/\sigma,\quad s=\log\sigma,\quad v(s;y)=\sigma^d u(t;x).$$

Then (1.22) turns into the Fokker-Planck equation,

$$\partial_s v(s;y) = \Delta_y u(s;y) + \nabla_y\cdot(yv(s;y)). \tag{1.23}$$

This equation is mass preserving; we choose as domain for v the set of probability distributions with finite temperature and entropy. The unique positive steady state is then given by the Gaussian $v_\infty = M$. In contrast to the Boltzmann equation, the Fokker-Planck equation preserves neither momentum nor energy in general. This forces us to work immediately with the *relative* version of the entropy established in the proof of Theorem 1.2, i.e. define

$$\begin{aligned} H[v] &:= \int \rho(y)\log\rho(y)\,M(y)\,dy, \quad \rho(y) = M(y)^{-1}u(y), \quad M(y) = (2\pi)^{-d/2}e^{-|y|^2/2} \\ &= \int v(y)\log v(y)\,dy + \frac{1}{2}\int \big(|y|^2 + d\log(2\pi)\big)v(y)\,dy. \end{aligned}$$

For this, one has

THEOREM 1.4. *The just defined functional H is an entropy for the Fokker-Planck equation* (1.23). *Moreover, H and D_H converge exponentially at rate equal to two, and*

$$\|v(s) - M\|_{L^1} \le Ce^{-t}, \tag{1.24}$$

where C depends only on $H[u_0]$.

PROOF. Recall that $H[v] \ge 0$ by Jensen's inequality, and $H[v] = 0$ exactly iff $v = M$. The entropy production gives

$$\begin{aligned} D_H[v] = -\frac{d}{ds}H[v(s)] &= -\int \partial_s v(y)\log v(y)\,dy - \frac{1}{2}\int |y|^2\partial_s v(y)\,dy \\ &= \int v(y)^{-1}\nabla v(y)\cdot\big(\nabla v + yv(y)\big)\,dx + \int y\cdot(\nabla v + yv(y))\,dy \\ &= \int v(y)^{-1}\nabla v(y)\,dy + 2\int y\cdot\nabla v(y)\,dy + \int |y|^2 v(y)\,dy \\ &= \int v(y)^{-1}|\nabla v(y) + yv(y)|^2\,dy, \end{aligned}$$

which is obviously non-negative and zero exactly for $v = v_\infty$. Moreover, one can write

$$D_H[v] = 4\int |\nabla\sqrt{v(y)}|^2\,dy - 2d\int v(y)\,dy + \int |y|^2 u(y)\,dy.$$

The clue is that there exists a nice relation between $H[v]$ and $D_H[v]$, the celebrated logarithmic Sobolev inequality. The latter states that for any probability density u on $\mathbb{R}^d$,

$$\int u(x)\log u(x)\,dx + d\big(1 + \log\sqrt{2\pi}\big) \le 2\int |\nabla\sqrt{u(x)}|^2\,dx. \tag{1.25}$$

The proof of this inequality is one of the main issues in Lecture 2, see Corrollary 2.2. Substituting $u(x) = v(x)$ into (1.25), one establishes

$$H[v] \le \frac{1}{2}D_H[v],$$

leading to the exponential decay of H. The L^1-decay in (1.24) follows by Proposition 1.1. $\square$

What about the equation (1.22) in the original variables? Undoing the scalings, one finds that

$$\|u(t) - U(t)\|_{L^1} \le \sqrt{\frac{8E[u_0]}{2t+1}}$$

We note that the worst case is already achieved by comparing two self-similar solutions centered at different points in space.

All of the above, however, is only the beginning of a long story. What happens if not an arbitrary self-similar solution is taken for comparison, but one which is more adapted to the initial condition? There are two parameters to play with: the center of mass and the shift in time. Adjusting those, the convergence rate can be improved [**3**].

Furthermore, it was observed by McKean that also the main contribution of the entropy production, i.e. the relative Fisher information

$$F[u] := D_H[v] = 4\int |\nabla\sqrt{v(y)}|^2\,dy - d,$$

constitutes a Lyapunov functional for (1.23). In principle, it can be used to estimate the convergence of $v(s)$ to v_∞ in $H^1(\mathbb{R}^d)$. The entropy production of F is explicit

$$D_F[v] = \int \big(v^{-1}\nabla^2 v - v^{-2}\nabla v \otimes \nabla v\big)^2 v\, dy - F[v] + d,$$

and one finds

$$F[v] \le 2D_F[v],$$

hence also $D_H[v(s)]$ is exponentially decaying in time, at rate $\exp(-2s)$. What about the entropy production of D_H? It seems like one cannot go further, but so far, no counterexamples are known.

6. (Counter-)Example: Quantum Diffusion

Since the discussion in these notes is extremely formal, I feel obliged to point out that occasionally one runs into analytical problems with the formal calculations. A very intricate example is provided by the Quantum Diffusion equation [**28**],

$$\partial_t u(x) = -\big(u(x)(\log u(x))_{xx}\big)_{xx}, \quad x \in \mathbb{T}. \tag{1.26}$$

A straight-forward formal calculation reveals that

$$H[u] := \int_{\mathbb{T}} u(x) \log u(x)\, dx$$

is an entropy functional for (1.26). In fact, one can prove that for any initial condition $u_0 \ge 0$ with $E[u_0] < \infty$, there exists a corresponding weak solution to (1.26) which dissipates H exponentially in time.

In particular, there exists an entropy dissipating weak solution for the initial condition

$$\tilde{u}_0(x) = \sin^2 x.$$

On the other hand, $\tilde{u}_0$ constitutes a stationary solution to (1.26) in the sense that

$$\tilde{u}_0(x)(\log \tilde{u}_0(x))_{xx} = -2$$

for all $x \notin \pi\mathbb{Z}$. Hence, if the concept of solution is taken too weak, then entropies might not behave as expected.

7. Problems

PROBLEM 1.1. *Consider the ordinary differential equations $\dot{x} = f(x)$ on $\mathbb{R}^2$.*

(a) *Assume $f(x) = \nabla H(x)$ is the gradient of a strictly convex function $H : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ with $\nabla^2 H \geq \lambda \mathbf{1}$ for some $\lambda > 0$ and $H(0) = 0$. Show that H is an entropy for the corresponding flow, and that both H and its entropy production D_H decay like $\approx \exp(-2\lambda t)$ along arbitrary solutions $x(t)$. Does D_H necessarily constitute an entropy by itself?*

(b) *Show that the flow for the vector field f defined by $f(x_1, x_2) = -(x_1 + x_2, x_2 - x_1)$ possesses an entropy, but is* not *the gradient flow of some smooth potential $\Phi : \mathbb{R}^2 \to \mathbb{R}$ in a suitable smooth metric on $\mathbb{R}^2$.*

PROBLEM 1.2. *The so-called* BGK model *is a linear version of the homogeneous Boltzmann equation. A solution is a time-dependent probability density $f(t; x)$ on $\mathbb{R}$ satisfying*

$$\partial_t f(t; x) = -\frac{1}{\tau}\big(f(t; x) - P[f(t)](x)\big).$$

Here $\tau > 0$ is the relaxation time, and P is the projection on the Gaussians,

$$P[f](x) = (2\pi T)^{-1/2} \exp\big((x - \bar{x})^2/(2T)\big),$$

where $\bar{x} \in \mathbb{R}$, $T > 0$ are chosen s.t. f and $P[f]$ have the same first and second momentum. Determine a suitable domain U on which Boltzmann's H-functional constitutes an entropy for the BGK model. Calculate the decay rate.

PROBLEM 1.3. *Consider the following extension of the radiative transfer equation for a time-dependent probability density $f(t; x)$ on the domain $\Omega \subset \mathbb{R}^d$:*

$$\partial_t f(t; x) = -\lambda(x) f(t; x) + \int_\Omega \lambda(y) f(t; y)\, \mu(dy),$$

where μ is a fixed probability measure on Ω, and $\lambda : \Omega \to \mathbb{R}_+$ is a given function with $\int_\Omega \lambda(x)^{-1} \mu(dx) = 1$.

(a) *Show that each smooth, strictly convex function $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}$ with $\phi(1) = 0$ and $\phi'(1) = 0$ gives rise to an entropy via $E_\phi[f] = \int_\Omega \phi\big(\lambda(x) f(x)\big)\, \lambda(x)^{-1} \mu(dx)$.*

(b$\star$) *Under the additional assumption that $\int_\Omega \lambda(x)^{-2} \mu(dx) < +\infty$, show that each of the previously defined entropies decays like $\approx t^{-3}$ along arbitrary solutions $f(t)$, which are bounded initially.*

CHAPTER 2

Linear Diffusion

The topic of this lecture are estimates on the speed of equilibration for solutions to linear scalar diffusion equation of second order. The notes for this lecture are divided into two parts: first, the essentials of the original Bakry-Émery-method [**7**] are presented, and second, Toscani's more direct approach to the method is described. Only the second part was discussed in the course.
To have a rough idea of the method described in the following, recall Example 5 of the rescaled heat equation from Lecture 1. The strategy for the proof of equilibration has been the following:

(1) Starting from the relative entropy H, calculate the dissiation D_H.
(2) Relate D_H to H by means of the logarithmic Sobolev inequality (1.25).
(3) Use Gronwall's inequality to conclude decay of H, and hence equilibration of the solution.

The catch is that the logarithmic Sobolev inequality needs to be given a priori. This is a general problem with this straightforward approach: the relation between the entropy functional and its production, usually a non-linear and highly non-trivial functional inequality, needs to be known from some other source. The ground-breaking idea of Bakry and Émery is a variation of the scheme above, that delivers a proof of the correct functional inequality as a by-product.

(1) Given H, calculate the dissiation D_H of H, and also the dissipation R_H of D_H.
(2) Relate R_H to D_H by means of some elementary (usually pointwise) functional inequality.
(3) Relate D_H to H by integrating up (in time) the latter relation.
(4) Use the Gronwall argument to deduce decay of H and equilibration of the solution.

There are certain similarities of this procedure to the very basic example discussed in section 2. In effect, the Bakry-Émery allows to decide if the functionals H in a certain class are λ-convex along all solution trajectories of the given diffusion equation.

1. Functional inequalities on an interval — a warm up

The probably easiest equation for which the Bakry-Émery method leads to non-trivial results is the heat equation

$$\partial_t u = u_{xx}, \qquad u(0;x) = \hat{u}(x), \tag{2.1}$$

on the interval $[0,1]$ with homogeneous Neumann boundary conditions.

THEOREM 2.1. *Assume that* $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *is convex and s.t.* $(\phi'')^{-1/2}$ *is concave, and let* ψ *be such that*

$$\psi'(s)^2 = \phi''(s).$$

Then the following convex Sobolev inequality

$$\int_0^1 \phi(\hat{u})\,dx - \phi\Big(\int_0^1 \hat{u}\,dx\Big) \leq \frac{1}{2\pi^2}\int_0^1 \psi(\hat{u})_x^2\,dx, \tag{2.2}$$

holds for all smooth, positive functions $\hat{u}$ *on* $[0,1]$.

PROOF. Let us start from the special case of (2.2) with $\phi(s) = \frac{1}{2}s^2$ and $\psi(s) = s$,

$$\int_0^1 \hat{u}^2\,dx - \Big(\int_0^1 \hat{u}\,dx\Big)^2 \leq \frac{1}{\pi^2}\int_0^1 \hat{u}_x^2\,dx. \tag{2.3}$$

This is the Poincaré inequality which we shall not prove again. Instead, we shall now generalize it to other convex functions $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$.

Define the l.h.s of (2.2) as $H_\phi[u]$, and let u be the unique solution to (2.1). One finds

$$D_\phi[u] := -\frac{d}{dt}H_\phi[u] = -\int \phi'(u)u_t\,dx = -\int \phi'(u)u_{xx}\,dx = \int \phi''(u)u_x^2\,dx = \int \psi(u)_x^2\,dx$$

as first time derivative and

$$\begin{aligned} R_\phi[u] &:= -\frac{1}{2}\frac{d}{dt}D_\phi[u] = -\int \psi(u)_x\psi(u)_{xt}\,dx = \int \psi(u)_{xx}\psi'(u)u_{xx}\,dx \\ &= \int \psi'(u)^2u_{xx}^2\,dx + \int \psi''(u)\psi'(u)u_x^2u_{xx}\,dx \end{aligned}$$

as second. The crucial step is to relate $R_\phi[u]$ to $D_\phi[u]$. To this end, the expression

$$\begin{aligned} 0 &= \frac{1}{3}\int (\psi'(u)\psi''(u)u_x^3)_x\,dx \\ &= \int \psi''(u)\psi'(u)u_x^2u_{xx}\,dx + \frac{1}{3}\int (\psi''(u)^2 + \psi'(u)\psi'''(u))u_x^4\,dx. \end{aligned}$$

is added to $R_\phi[u]$, obviously without changing the value of the latter. Hence

$$R_\phi[u] = \int \psi'(u)^2u_{xx}^2\,dx + 2\int \psi''(u)\psi'(u)u_{xx}u_x^2\,dx + \frac{1}{3}\int (\psi''(u)^2 + \psi'(u)\psi'''(u))u_x^4\,dx.$$

On the other hand, one has

$$0 \le (\psi(u))_{xx}^2 = \psi'(u)^2u_{xx}^2 + 2\psi''(u)\psi'(u)u_{xx}u_x^2 + \psi''(u)^2u_x^4.$$

In combination with Poincaré's inquality (2.3), one concludes

$$D_\phi[u] \le \frac{1}{\pi^2}\,R_\phi[u], \tag{2.4}$$

provided that

$$\frac{1}{3}\big((\psi'')^2 + \psi'\psi'''\big) \ge \big(\psi''\big)^2. \tag{2.5}$$

Since $\psi' = (\phi'')^{1/2} > 0$, it is easy to see that (2.5) is equivalent to the concavity of $(\psi')^{-1} = (\phi'')^{-1/2}$.
To finish the argument, rewrite (2.4) as

$$-\frac{d}{dt}H_\phi[u(t)] \le -\frac{1}{2\pi^2}\,\frac{d}{dt}D_\phi[u(t)],$$

and integrate both sides from $t = +\infty$ to $t = 0$. This yields

$$H_\phi[u_0] - \lim_{t\to+\infty} H_\phi[u(t)] \le \frac{1}{2\pi^2}\,\big(D_\phi[u_0] - \lim_{t\to\infty} D_\phi[u(t)]\big). \tag{2.6}$$

By standard theory, the solution $u(t)$ to (2.1) converges to the homogeneous steady state $u_\infty \equiv \int_0^1 \hat{u}(x)dx$ in C^∞, implying that $D_\phi[u(t)] \to 0$ and $H_\phi[u(t)] \to H_\phi[u_\infty]$ as $t \to \infty$. Substituting these limits, (2.6) becomes (2.2). □

For example, $\phi(s) = s\log s$ with $\psi(s) = 2s^{1/2}$ is a possible choice, leading to a logarithmic Sobolev inequality,

$$0 \le \int_0^1 \hat{u}\log\hat{u}\,dx - \Big(\int_0^1 \hat{u}\,dx\Big)\log\Big(\int_0^1 \hat{u}\,dx\Big) \le \frac{2}{\pi^2}\int_0^1 \sqrt{\hat{u}}_x^2\,dx. \tag{2.7}$$

Moreover, the pairs $\phi(s) = s^\alpha/(\alpha-1)$ and $\psi(s) = 2s^{\alpha/2}/\sqrt{\alpha}$ are also allowed, when $1 < \alpha < 2$. These yield Beckner's interpolation inequalities,

$$0 \le \frac{1}{\alpha-1}\Big[\int_0^1 \hat{u}^\alpha\,dx - \Big(\int_0^1 \hat{u}\,dx\Big)^\alpha\Big] \le \frac{2}{\alpha\pi^2}\int_0^1 \big(\sqrt{\hat{u}^\alpha}\big)_x^2\,dx. \tag{2.8}$$

Notice that from (2.8), one obtains both the Poincare inequality (2.3) for $\alpha \nearrow 2$, as well as the logarithmic Sobolev inequality (2.7) for $\alpha \searrow 1$ as limit cases.

2. The Carré du Champ

The setting for the original method by Bakry and Émery has been a probabilistic one, which we quickly review now. Everything below is very formal — but the situation in the original setting [**7**] is hardly any better: the calculations in [**7**] are based on the "algebra assumption", excluding more or less all examples of practical interest. Fortunately, the formal ideas could be made rigorous with a certain amount of effort [**5**].

Let a continuous, stationary Markov process be defined on the set $C^\infty_+(\Omega)$ of non-negative smooth functions over a domain $\Omega \subset \mathbb{R}^d$. Essentially, this means that a semi-group P_t (with $t \geq 0$) of linear operators on $C^\infty_+(\Omega)$ is given,

$$P_0[f] = f, \qquad P_s \circ P_t = P_{s+t},$$

which can be represented in terms of probability transition kernels K_t,

$$P_t[f](x) = \int_\Omega f(y) K_t(x, dy).$$

Notice that P_t preserves the non-negativity of f, and leaves constant functions invariant. We assume that the process allows for a *unique invariant measure*, i.e. there is precisely one probability measure μ_∞ such that

$$\int_\Omega f(x)\, \mu_\infty(dx) = \int_\Omega P_t(f(x))\, \mu_\infty(dx) \text{ for all } t \geq 0. \tag{2.9}$$

Introduce accordingly for $f \in L^1(\mu_\infty)$ and g, $h \in L^2(\mu_\infty)$ the notations

$$\langle f \rangle := \int f(x)\, \mu_\infty(dx), \qquad \langle f, g \rangle := \langle fg \rangle.$$

Also, we assume ergodicity in the sense that

$$\lim_{t\to\infty} P_t(f) = f_\infty \equiv \langle f \rangle, \tag{2.10}$$

where we avoid to specify the precise meaning of convergence.

Moreover, there are two assumptions on the infinitesimal generator,

$$L[f] := \lim_{t \searrow 0} \frac{1}{t}\big(P_t[f] - f\big).$$

First, L is symmetric,

$$\langle Lf, g \rangle = \langle f, Lg \rangle. \tag{2.11}$$

And second, L acts like a second order diffusion operator,

$$L[\phi(f)] = \phi'(f) Lf + \phi''(f) \Gamma(f, f), \tag{2.12}$$

where Γ is the celebrated *carré du champ*,

$$\Gamma(f, g) = \frac{1}{2}\big(L[fg] - f(Lg) - (Lf)g\big). \tag{2.13}$$

PROPOSITION 2.1. *The* carre du champ *operator* Γ *has the following properties:*

- Γ *is bilinear and symmetric.*
- Γ *satisfies the Leibniz rule,*

$$\Gamma(fg, h) = f\Gamma(g, h) + g\Gamma(f, h). \tag{2.14}$$

- Γ *satisfies a chain rule in each argument,*

$$\Gamma(\phi(f), g) = \phi'(f)\Gamma(f, g). \tag{2.15}$$

- *The scalar product* $\langle \cdot \rangle$ *relates* Γ *and* L *by*

$$\langle \phi(f), Lf \rangle = -\langle \Gamma(\phi(f), f) \rangle. \tag{2.16}$$

Proof. Bilinearity and symmetry are obvious from the definition (2.13) and the properties of L. In order to derive (2.14), use (2.12) with $\phi(s) = s^3$ and replace f by $af + bg + ch$, with arbitrary functions f, g and h, and real numbers a, b and c. Then collect terms containing abc on both sides of (2.12); equating those yields (2.14) after some manipulations. The rule (2.15) follows at least for real analytic ϕ by using the previous rule (2.14) and a power series expansion of ϕ. The last property (2.16) is again a consequence of the definition (2.13), the symmetry property (2.11), and the fact that we integrate against the invariant measure μ_∞. $\square$

The canonical example to which the theory applies is the (generalized) Ornstein-Uhlenbeck process. The generator L of the Markov semigroup P_t acts on functions f like

$$Lf(x) := \nabla \cdot \big(D(x)\nabla f\big) - D(x)\nabla V(x) \cdot \nabla f(x),$$

with an everywhere positive definite diffusion matrix D and a confinement potential V. We assume $\Omega = \mathbb{R}^d$ in order to avoid the discussion of boundary conditions (which is a highly non-trivial matter, even in one spatial dimension, see e.g. [**23**]). By standard theory, the associated linear parabolic equation

$$\partial_t f(t;x) = Lf(t;x) \tag{2.17}$$

possesses (under mild assumptions on D and V) a unique solution for each initial condition f_0, thus defining a semi-group P_t. The formally $L^2(\mathbb{R}^d)$-adjoint generator L^* acts on the densities u of measures $\mu(dx) = u(x)dx$,

$$L^*u(x) = \nabla \cdot \big(D(x)(\nabla u(x) + u(x)\nabla V(x))\big).$$

Provided $\exp(-V)$ is integrable on $\mathbb{R}^d$, the ajoint parabolic equation

$$\partial_t u(t;x) = L^*u(t;x) \tag{2.18}$$

admits exactly one steady state u_∞ of unit mass,

$$u_\infty(x) = Z^{-1}\exp(-V(x)), \quad Z := \int_\Omega \exp(-V(x))\,dx,$$

corresponding to the unique invariant measure $\mu_\infty(dx) = u_\infty(dx)$. Direct computations allow to check both the symmetry property (2.11),

$$\begin{aligned}
\langle f, Lg\rangle &= \frac{1}{Z}\int_{\mathbb{R}^d} f\nabla \cdot \big(D\nabla g)e^{-V}\,dx - \frac{1}{Z}\int_{\mathbb{R}^d} fD(\nabla V \cdot \nabla g)e^{-V}\,dx \\
&= -\frac{1}{Z}\int_{\mathbb{R}^d} D(\nabla f \cdot \nabla g)e^{-V}\,dx \\
&= +\frac{1}{Z}\int_{\mathbb{R}^d} \nabla \cdot (D\nabla f)ge^{-V}\,dx - \int_{\mathbb{R}^d} D(\nabla f \cdot \nabla V)ge^{-V}\,dx = \langle Lf, g\rangle,
\end{aligned}$$

as well as the "diffusion-operator"-property (2.12),

$$\begin{aligned}
L\phi(f) &= \nabla \cdot \big(D\phi'(f)\nabla f\big) - D\phi'(f)\nabla V \cdot \nabla f \\
&= \phi'(f)\big(\nabla \cdot (D\nabla f) - D\nabla V \cdot \nabla f\big) + \phi''(f)D\nabla f \cdot \nabla f.
\end{aligned}$$

Thus, Proposition 2.1 applies to the associated carré du champ-operator

$$\begin{aligned}
\Gamma(f,g) &= \frac{1}{2}\Big(\nabla \cdot \big(D\nabla(fg)\big) - f\nabla \cdot \big(D\nabla g\big) - g\nabla \cdot \big(D\nabla f\big) - D(x)\nabla V \cdot \big(\nabla(fg) - f\nabla g - g\nabla f\big)\Big) \\
&= \frac{1}{2}\Big(\nabla \cdot \big(D(f\nabla g + g\nabla f)\big) - f\nabla \cdot \big(D\nabla g\big) - g\nabla \cdot \big(D\nabla f\big)\Big) \\
&= D\nabla f \cdot \nabla g.
\end{aligned}$$

In order to determine the equilibration properties of the Markov process P_t for (2.17), another operator needs to be studied.

3. Gamma-Deux

The gamma-operator is the first member in a hierarchy (the zeroth member being just the pointwise product of functions). In the next iteration, one obtains

$$\Gamma_2(f,g) = \frac{1}{2}\big(L\Gamma(f,g) - \Gamma(Lf,g) - \Gamma(f,Lg)\big). \tag{2.19}$$

In these terms, the celebrated result of Bakry and Émery plainly reads:

THEOREM 2.2. *Assume that there exists some $\lambda > 0$ such that*

$$\Gamma_2(h,h) \geq \frac{\lambda}{2}\,\Gamma(h,h) \tag{2.20}$$

for all non-negative functions h. Then the following convex Sobolev inequality holds w.r.t. the invariant measure μ_∞,

$$\int_\Omega \phi(g)\,\mu_\infty(dx) - \phi\Big(\int_\Omega g\,\mu_\infty(dx)\Big) \leq \lambda^{-1}\int_\Omega \phi''(g)\Gamma(g,g)\,\mu_\infty(dx), \tag{2.21}$$

provided ϕ is strictly convex and s.t. $1/\phi''$ is concave.

The left-hand side of (2.21) represents the difference of two relative entropies. In particular, choosing $\phi(s) = s\log s$ in (2.21) yields the log-Sobolev inequality,

$$\int_\Omega g\log g\,\mu_\infty(dx) - g_\infty \log g_\infty \leq \lambda^{-1}\int_\Omega \frac{\Gamma(g,g)}{g}\,\mu_\infty(dx),$$

whereas $\phi(s) = s^2$ leads to the Poincaré inequality,

$$\int_\Omega \big(g - g_\infty\big)^2\,\mu_\infty(dx) \leq \lambda^{-1}\int_\Omega \Gamma(g,g)\,\mu_\infty(dx).$$

Here $g_\infty \equiv \langle g\rangle$ as in (2.10).
The proof of Theorem 2.2 is very computational. The strategy, however, is very similar to the one used in the easy proof of Theorem 2.1. Denote by $f = f(t) = P_t g$ the time-dependent family of transformations of the given function $f(0) = g$ under the Markov semi-group. The key idea is to study the associated temporal evolution of the entropy

$$H_\phi[f] = \langle \phi(f)\rangle.$$

Using rules (2.15) and (2.16), the *first* dissipation (at any instant of time) is given by

$$D_\phi[f] = -\frac{d}{dt}\Big|_{t=0} H_\phi[P_t f] = -\langle \phi'(f)Lf\rangle = \langle \Gamma(\phi'(f),f)\rangle = \langle \phi''(f)\Gamma(f,f)\rangle,$$

and the *second* by

$$R_\phi[f] = \frac{1}{2}\frac{d^2}{dt^2}\Big|_{t=0} H_\phi[P_t f] = -\frac{1}{2}\langle \phi'''(f)Lf, \Gamma(f,f)\rangle - \langle \phi''(f)\Gamma(Lf,f)\rangle. \tag{2.22}$$

By a variety of formal manipulations — detailed below — it follows that

$$R_\phi[f] \geq \langle \phi''(f)^{-1}\Gamma_2(\phi'(f),\phi'(f))\rangle. \tag{2.23}$$

Assumption (2.20) allows to conclude that

$$R_\phi[f] \geq \frac{\lambda}{2}\langle \phi''(f)^{-1}\Gamma(\phi'(f),\phi'(f))\rangle = \frac{\lambda}{2}\langle \phi''(f)\Gamma(f,f)\rangle,$$

or, equivalently,

$$\lambda D_\phi[f] \leq -\frac{1}{2}\frac{d}{dt}D_\phi[f], \tag{2.24}$$

for all $t \geq 0$. Under the further hypothesis that the ergodicity-convergence in (2.10) is strong enough to conclude

$$\lim_{t\to\infty} D_\phi[P_t g] = 0, \quad \lim_{t\to\infty} H_\phi[P_t g] = H_\phi[g_\infty], \quad g_\infty = \langle g\rangle,$$

a time-integration of (2.24) on $t \in (0,\infty)$ reveals

$$H_\phi[g] - H_\phi[g_\infty] \leq \frac{1}{2\lambda} D_\phi[g], \tag{2.25}$$

which is nothing but (2.21). Hence, the main effort is the passage from the expression (2.22) for $R_\phi[f]$ to the other expression in (2.23). The basic tool is the following "change of variables formula".

Lemma 2.1. *For any sufficiently smooth function ψ, and all f*

$$\Gamma_2(\psi(f),\psi(f)) = \psi'(f)^2\Gamma_2(f,f) + \psi'(f)\psi''(f)\Gamma(f,\Gamma(f,f)) + \psi''(f)^2\Gamma(f,f)^2. \tag{2.26}$$

Proof. Applying the rule (2.12), (2.14) and (2.15) to the definition of Γ_2, one verifies

$$\begin{aligned}
\Gamma_2(\phi,\phi) &= \frac{1}{2}L\Gamma(\phi,\phi) - \Gamma(L\phi,\phi)\\
&= \frac{1}{2}L\big((\phi')^2\Gamma\big) - \phi'\Gamma(\phi' Lf, f) - \phi'\Gamma(\phi''\Gamma, f)\\
&= \Gamma\big((\phi')^2,\Gamma\big) + \frac{1}{2}(\phi')^2 L\Gamma + \frac{1}{2}L\Big((\phi')^2\Big)\Gamma\\
&\quad - (\phi')^2\Gamma(Lf,f) - \phi'(Lf)\Gamma(\phi',f) - \phi'\phi''\Gamma(\Gamma,f) - \phi'\Gamma(\phi'',f)\Gamma\\
&= 2\phi'\phi''\Gamma(f,\Gamma) + \frac{1}{2}(\phi')^2L\Gamma + \phi'\phi''(Lf)\Gamma + \big(\phi'\phi''' + (\phi'')^2\big)\Gamma^2\\
&\quad - (\phi')^2\Gamma(Lf,f) - \phi'\phi''(Lf)\Gamma - \phi'\phi''\Gamma(\Gamma,f) - \phi'\phi'''\Gamma^2.
\end{aligned}$$

This is exactly the claim. □

In particular, relation (2.26) with $\psi(s) = \phi'(s)$ gives

$$\langle \phi''(f)^{-1}\Gamma_2(\phi'(f),\phi'(f))\rangle = \langle \phi''(f)\Gamma_2(f,f)\rangle + \langle \phi'''(f)\Gamma(f,\Gamma(f,f))\rangle + \langle \phi''(f)^{-1}\phi'''(f)^2\Gamma(f,f)^2\rangle. \tag{2.27}$$

Now, applying the symmetry property (2.11) and rule (2.15) to (2.22),

$$\begin{aligned}
R[f] &= -\frac{1}{2}\langle Lf, \phi'''(f)\Gamma(f,f)\rangle - \langle \phi''(f)\Gamma(f,Lf)\rangle\\
&= \frac{1}{2}\langle \Gamma(f,\phi'''\Gamma(f,f))\rangle\\
&\quad + \langle \phi''\Gamma_2(f,f)\rangle - \frac{1}{2}\langle \phi''(f), L\Gamma(f,f)\rangle\\
&= \frac{1}{2}\langle \phi'''(f)\Gamma(f,\Gamma(f,f))\rangle + \frac{1}{2}\langle \Gamma(f,f)\Gamma(f,\phi'''(f))\rangle\\
&\quad + \langle \phi''(f)\Gamma_2(f,f)\rangle + \frac{1}{2}\langle \Gamma(\phi''(f),\Gamma(f,f))\rangle\\
&= \langle \phi''(f)\Gamma_2(f,f)\rangle + \langle \phi'''(f)\Gamma(f,\Gamma(f,f))\rangle + \frac{1}{2}\langle \phi^{IV}(f)\Gamma(f,f)^2\rangle.
\end{aligned}$$

Hence, in view of (2.27), relation (2.23) holds true provided

$$\frac{1}{2}\phi^{IV} \geq (\phi'')^{-1}(\phi''')^2. \tag{2.28}$$

But this condition is verified since ϕ is strictly convex and $1/\phi''$ is concave. This concludes the proof of Theorem 2.2.

4. Applications

4.1. Exponential convergence in entropy.

Corollary 2.1. *Under the hypotheses of Theorem 2.2, each entropy H_ϕ and its respective production D_ϕ converge exponentially at rate 2λ. Moreover,*

$$\big\|P_t[g] - g_\infty\big\|_{L^1(\mu_\infty)} \leq \Big(\int_\Omega (g-g_\infty)^2 d\mu_\infty\Big)^{1/2} e^{-\lambda t}. \tag{2.29}$$

Proof. The exponential convergence is an immediate consequence of the estimates (2.25) and (2.24), respectively. Moreover, choosing $\phi(s) = s^2$,

$$\int_\Omega \big|P_t[g] - g_\infty\big|\, d\mu_\infty \le \Big(\int_\Omega \big(P_t[g] - g_\infty\big)^2\, d\mu_\infty \Big)^{1/2} \Big(\int_\Omega d\mu_\infty \Big)^{1/2} = \big(H_\phi[P_t g] - H_\phi[g_\infty]\big)^{1/2},$$

which yields (2.29). □

4.2. Logarithmic Sobolev inequality. The probably most prominent application consists in the derivation of the log-Sobolev inequality (1.25).

Corollary 2.2. *All positive smooth functions h satisfy the following logarithmic Sobolev inequality*

$$\int_{\mathbb{R}^d} h \log\Big(\frac{h}{\int h\, dx}\Big)\, dx + d\big(1 + \sqrt{2\pi}\big) \int_{\mathbb{R}^d} h\, dx \le 2 \int_\Omega \big|\nabla \sqrt{h}\big|^2\, dx. \tag{2.30}$$

Proof. Apply Theorem 2.2 to the "classical" Ornstein-Uhlenbeck process, which is (2.17) with $D \equiv \mathbf{1}$ and $V(x) = |x|^2/2$, i.e.

$$Lf = \Delta f - x \cdot \nabla f.$$

The associated invariant measure $\mu_\infty(dx) = M(x)dx$ is the Gaussian,

$$M(x) = \frac{1}{Z} e^{-V(x)} = (2\pi)^{-d/2} \exp\big(-\frac{1}{2}|x|^2\big).$$

Direct calculations reveal

$$\Gamma(f, g) = \nabla f \cdot \nabla g$$

and

$$\begin{aligned}\Gamma_2(f,f) &= \frac{1}{2}\Delta|\nabla f|^2 - \nabla\Delta f \cdot \nabla f + \nabla f \cdot \nabla(x \cdot \nabla f) - \frac{1}{2} x \cdot \nabla|\nabla f|^2 \\ &= \sum_{i,j=1}^d \big(\partial_i\partial_j f\big)^2 + |\nabla f|^2.\end{aligned}$$

So (2.20) is satisfied with $\lambda = 2$. The inequality (2.21) with $\phi(s) = s\log s$ provides the estimate

$$\int_{\mathbb{R}^d} (gM) \log\Big(\frac{g}{\int gM\, dx}\Big)\, dx \le 2 \int_{\mathbb{R}^d} \big|\nabla\sqrt{g}\big|^2 M\, dx. \tag{2.31}$$

Now substitute $g = h/M$. The left-hand side of (2.31) becomes

$$\int_{\mathbb{R}^d} h \log\Big(\frac{h}{\int h\, dx}\Big)\, dx - \int_{\mathbb{R}^d} h \log M\, dx.$$

On the right-hand side, one finds

$$2\int_{\mathbb{R}^d} \big|\, \nabla\sqrt{h} - \frac{1}{2}\sqrt{h}\nabla \log M\big|^2\, dx = 2\int_{\mathbb{R}^d} \big|\nabla\sqrt{h}\big|^2\, dx + \frac{1}{2}\int_{\mathbb{R}^d} h\big|\nabla \log M\big|^2\, dx - \int_{\mathbb{R}^d} \nabla h \cdot \nabla \log M\, dx.$$

Apply integration by parts to the last term, yielding

$$-\int_{\mathbb{R}^d} \nabla h \cdot \nabla \log M\, dx = d \int_{\mathbb{R}^d} h\, dx.$$

Finally, observing that

$$\int_{\mathbb{R}^d} h\big(\log M\, dx + \frac{1}{2}\big|\nabla \log M\big|^2\big)\, dx = d \log\sqrt{2\pi} \int_{\mathbb{R}^d} h\, dx,$$

it is evident that (2.31) indeed implies (2.30). □

4.3. Convex Sobolev inequalities on the line.

Corollary 2.3. *Let $d = 1$, and assume that the smooth potential $V : \mathbb{R} \to \mathbb{R}$ satisfies*

$$\frac{1}{4}D^{-1}D_x^2 - \frac{1}{2}D_{xx} + \frac{1}{2}D_xV_x + DV_{xx} \geq \lambda, \tag{2.32}$$

with some $\lambda > 0$. Then the inequality

$$\int_{\mathbb{R}} g \log\Big(\frac{g}{\int ge^{-V}\,dx}\Big)e^{-V}\,dx \leq 2\lambda \int_{\mathbb{R}} \big(\sqrt{g}\big)_x^2 e^{-V}\,dx \tag{2.33}$$

holds for all smooth and positive $g : \mathbb{R} \to \mathbb{R}$.

The condition (2.32) — in its multi-dimensional generalization — has been used as characterization of entropy-dissipating diffusion processes in [**5**].

Proof. Consider the Ornstein-Uhlenbeck process on the real line $\Omega = \mathbb{R}$, with

$$Lf = (Df_x)_x - DV_xf_x.$$

By the preceeding calculations, $\Gamma(f,g) = Df_xg_x$. Furthermore,

$$\begin{aligned}\Gamma_2(f,f) &= \frac{1}{2}\big(D(Df_x^2)_x\big)_x - \frac{1}{2}DV_x(Df_x^2)_x - Df_x(Df_x)_{xx} + Df_x(DV_xf_x)_x \\ &= D^2f_{xx}^2 + DD_xf_xf_{xx} + \frac{1}{2}(D_x^2 - DD_{xx} + 2D^2V_{xx} + DD_xV_x)f_x^2.\end{aligned}$$

This is a quadratic form in f_x and f_{xx}; it is bounded from below by $(\lambda/2)\Gamma(f,f) = (\lambda/2)f_x^2$ iff

$$2D^2(D_x^2 - DD_{xx} + 2D^2V_{xx} + DD_xV_x - \lambda) \geq D^2D_x^2,$$

or, equivalently, iff (2.32) is true. Theorem 2.2 implies (2.33) with $\phi(s) = s\log s$. $\square$

4.4. Hypercontractivity. One application must be mentioned in this context, since it stood as the primary motivation at the very beginning of the theory: hypercontractivity estimates.

Corollary 2.4. *Assume* (2.20) *holds. Let $p > 1$, and define $q(t) = 1 + (p-1)\exp(\lambda t)$. Then*

$$\big\|P_tg\big\|_{L^{q(t)}} \leq \big\|g\big\|_{L^p} \tag{2.34}$$

holds for all non-negative functions $g \in L^p(\Omega)$.

Proof. Again, denote by $f = f(t) = P_tg$ the temporal transformations of g under the semi-group. We shall prove (2.34) by showing that

$$F(t) = \log\Big(\big\|f(t)\big\|_{q(t)}\Big) = \frac{1}{q(t)}\log\langle f(t)^{q(t)}\rangle$$

is non-increasing in time. Indeed, observe that

$$\begin{aligned}\frac{d}{dt}F(t) &= -\frac{\dot q}{q^2}\log\langle f^q\rangle + \frac{\dot q}{q}\frac{\langle f^q\log f\rangle}{\langle f^q\rangle} + \frac{\langle f^{q-1}Lf\rangle}{\langle f^q\rangle} \\ &= -\frac{\lambda(q-1)}{q^2\langle f^q\rangle}\Big(\langle f^q\rangle\log\langle f^q\rangle - \langle f^q\log f^q\rangle - \frac{q^2}{\lambda(q-1)}\langle f^{q-1}Lf\rangle\Big).\end{aligned}$$

Now substitute $f = h^{1/q}$. In view of (2.16) in combination with (2.15),

$$-\langle f^{q-1}Lf\rangle = (q-1)\langle f^{q-2}\Gamma(f,f)\rangle = \frac{q-1}{q^2}\langle \underbrace{h^{(q-2)/q}\big(h^{(1-q)/q}\big)^2}_{=h^{-1}}\Gamma(h,h)\rangle.$$

Altogether,

$$\frac{d}{dt}F(t) = -\frac{\lambda(q-1)}{q^2\langle h\rangle}\big(\langle h\rangle\log\langle h\rangle - \langle h\log h\rangle + \lambda^{-1}\langle h^{-1}\Gamma(h,h)\rangle\big),$$

and the term inside the round brackets is non-negative by (2.21) with $\phi(s) = s\log s$. $\square$

4.5. Extensions and limitation of the method. Since the discovery of its relation to hypercontractivity, the Bakry-Émery-criterion (2.20) has been improved and specialized in a huge number of publications since the late 80's until today. For instance, Theorem 2.2 remains still valid under certain perturbations of the strict requirement (2.20), see e.g. [**25**]. In fact, even the "original Bakry-Émery condition" from [**7**] is slightly weaker than (2.20). Moreover, in the context of diffusion on Riemannian manifolds, the connection between logarithmic Sobolev inequalities and curvature bounds has been exhaustively studied. A nice overview of available results can be obtained from [**35**].
There is at least one issue that deserves a quick discussion at this point. The Bakry-Émery method is designed to provide a strong logarithmic Sobolev inequality (which implies hypercontractivity of the semi-group P_t). From the point of equilibration estimates, however, the weaker Poincaré inequality is usually sufficient (since the latter implies a spectral gap of the generator L on $L^2(\Omega)$). There are indeed certain situations in which the log-Sobolev estimate (and thus the whole Bakry-Émery approach) is bound to fail, whereas a Poincaré inequality can still be proven. An example of great importance is provided by the linearized fast diffusion equation [**17**],

$$\partial_t u(t;x) = \nabla \cdot \big(B(x)\nabla(mB(x)^{m-2}u(t;x))\big), \quad B(x) = \Big(C + \frac{1-m}{2m}|x|^2\Big)^{-1/(1-m)} \tag{2.35}$$

for $1 - 2/d < m < 1$. Equation (2.35) is supposed to capture the behavior of solutions to the fast diffusion equation in a vicinity of the equilibrium point B.

5. Toscani's approach to the Bakry-Émery method

There exists another approach to proving logarithmic Sobolev inequalities and equilibration estimates, which is strongly related to the Bakry-Émery method, but attains the "adjoint" point of view. Instead of investigating the algebra of non-negative smooth functions on Ω, one directly investigates the time evolution of the underlying measure. This approach, which has been developed since the late 80's and was summarized in [**5**], has certain advantages over the original method. (In the actual lecture, only THIS approach has been presented.) Mainly, since the setup is much more restrictive, the calculations are more direct and can be made rigorous in the relevant function spaces with resonable effort. Naturally, much of the elegance and generality of the original method is lost.
For simplicity, we restrict attention to the following Fokker-Planck equation

$$\partial_t u(x) = \Delta u(x) + \nabla \cdot \big(u(x)\nabla V(x)\big), \tag{2.36}$$

posed for the probability density u on the whole space $\mathbb{R}^d$. Notice that equation (2.36) corresponds to the adjoint formulation (2.18) with $D \equiv \mathbf{1}$ in the context of Ornstein-Uhlenbeck processes.
The goal is to prove exponentially fast convergence of $u(t)$ to

$$u_\infty = \frac{1}{Z}e^{-V}, \quad Z = \int_{\mathbb{R}^d} e^{-V(x)}\,dx$$

in $L^1(\mathbb{R}^d)$, using entropy methods.

THEOREM 2.3. *Assume the Bakry-Émery condition*

$$\nabla^2 V(x) \geq \lambda\mathbf{1} \quad \textit{uniformly in } x \in \mathbb{R}^d\textit{, with some } \lambda > 0. \tag{2.37}$$

Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be such that ϕ is convex and $1/\phi''$ is concave. Then the associated relative entropy H_ϕ and its production D_ϕ w.r.t. (2.36) *satisfy the functional inequality*

$$H_\phi[u] - H_\phi[u_\infty] \leq \frac{1}{2\lambda}D_\phi[u]. \tag{2.38}$$

Entropy H_ϕ and entropy production D_ϕ converge exponentially at rate 2λ, and any solution $u(t)$ to (2.36) *with $H_\phi[u_0] < +\infty$ equilibrates exponentially fast,*

$$\|u(t) - u_\infty\|_{L^1} \leq C(u_0) \cdot e^{-\lambda t}. \tag{2.39}$$

The proof of Theorem 2.3 is divided into four steps.

5.1. Step one: first entropy production.

LEMMA 2.2. *The entropy production $D_\phi[u(t)]$ is non-negative and equal to zero iff $u(t) = u_\infty$.*

We shall be working with the ratio $\rho(x) = u(x)/u_\infty(x)$ rather than with $u(x)$ itself. The respective equation reads

$$\partial_t \rho(x) = u_\infty(x)^{-1}\nabla\cdot(u_\infty(x)\nabla\rho(x)) = \Delta\rho(x) - \nabla\rho(x)\cdot\nabla V(x). \tag{2.40}$$

To justify the manipulations below, we implicitly we assume that ρ is smooth in space and time, and that the quotient $\rho(x)$ is "sufficiently bounded" for $|x|\to\infty$.
The entropy production is given by

$$D_\phi[u] = -\frac{d}{dt}E_\phi[u(t)] = -\int \phi'(\rho(x))u_\infty(x)\partial_t\rho(x)\,dx \tag{2.41}$$

$$= -\int \phi'(\rho(x))\nabla\cdot(u_\infty(x)\nabla\rho(x))\,dx \tag{2.42}$$

$$= +\int \phi''(\rho(x))|\nabla\rho(x)|^2\,u_\infty(x)dx. \tag{2.43}$$

Obviously, $D_\phi[u]\geq 0$, and equality implies that $\nabla\rho\equiv 0$, which further implies $u\equiv u_\infty$.

5.2. Step two: second entropy production.

LEMMA 2.3. *The second order production*

$$R_\phi[u(t)] := -\frac{1}{2}\frac{d}{dt}D_\phi[u(t)]$$

satisfies the functional inequality

$$D_\phi[u] \leq \frac{1}{\lambda}R_\phi[u] \tag{2.44}$$

for all "sufficiently regular" probability densities u.

By definition,

$$R_\phi[u] = -\frac{1}{2}\int \partial_t\big(\phi''(\rho)\big)|\nabla\rho|^2\,u_\infty\,dx - \int \phi''(\rho)\nabla\rho\cdot\partial_t\nabla\rho\,u_\infty\,dx. \tag{2.45}$$

For the first integral in (2.45), we find

$$\begin{aligned}
-\int \partial_t\big(\phi''(\rho)\big)|\nabla\rho|^2\,u_\infty\,dx &= -\int \phi'''(\rho)|\nabla\rho|^2\,\nabla\cdot(u_\infty\nabla\rho)\,dx \\
&= +\int \nabla(\phi'''(\rho)|\nabla\rho|^2)\cdot\nabla\rho\,u_\infty dx \\
&= +\int \big(\phi^{IV}(\rho)|\nabla\rho|^4 + 2\phi'''(\rho)\nabla\rho\cdot\nabla^2\rho\cdot\nabla\rho\big)\,u_\infty dx.
\end{aligned}$$

The second integral in (2.45) can be rewritten using

$$\begin{aligned}
-\nabla\rho\cdot\partial_t\nabla\rho = -\nabla\rho\cdot\nabla(\partial_t\rho) &= -\Delta\nabla\rho\cdot\nabla\rho + \nabla\rho\cdot\nabla(\nabla V\cdot\nabla\rho) \\
&= -\nabla\cdot(\nabla^2\rho\cdot\nabla\rho) + \|\nabla^2\rho\|^2 + \underbrace{\nabla\rho\cdot\nabla^2 V\cdot\nabla\rho}_{\geq\lambda|\nabla\rho|^2} + \nabla\rho\cdot\nabla^2\rho\cdot\nabla V.
\end{aligned}$$

Putting this together yields

$$\begin{aligned}
R_\phi[u] \geq \lambda D_\phi[u] &+ \int \phi''(\rho)\big(\nabla\rho\cdot\nabla^2\rho\cdot\nabla V - \nabla\cdot(\nabla^2\rho\cdot\nabla\rho)\big)\,u_\infty dx \\
&+ \int \big(\frac{1}{2}\phi^{IV}(\rho)|\nabla\rho|^4 + \phi'''(\rho)\nabla\rho\cdot\nabla^2\rho\cdot\nabla\rho + \phi''(\rho)\|\nabla^2\rho\|^2\big)\,u_\infty dx \\
= \lambda D_\phi[u] &+ \int \big(\frac{1}{2}\phi^{IV}(\rho)|\nabla\rho|^4 + 2\phi'''(\rho)\nabla\rho\cdot\nabla^2\rho\cdot\nabla\rho + \phi''(\rho)\|\nabla^2\rho\|^2\big)\,u_\infty dx.
\end{aligned}$$

The following argument shows that the expression under the last integral is pointwise non-negative. By assumption, $\phi'' > 0$ and $\phi''\phi^{IV} \geq 2(\phi''')^2$, so that

$$Q(a,b) = \phi^{IV}a^2 + 4\phi'''ab + 2\phi''b^2$$

is a non-negative quadratic form. Since further $|\nabla\rho \cdot \nabla^2\rho\nabla\rho| \leq \|\nabla^2\rho\||\nabla\rho|^2$, one finds

$$\phi^{IV}(\rho)|\nabla\rho|^4 + 4\phi'''(\rho)\nabla\rho \cdot \nabla^2\rho \cdot \nabla\rho + 2\phi''(\rho)\|\nabla^2\rho\|^2 \geq Q(\|\nabla^2\rho\|, |\nabla\rho|^2) \geq 0.$$

Thus (2.44) is proven.

5.3. Step three: proof of the functional inequality.

LEMMA 2.4. *The inequality* (2.38) *holds.*

The second step provided us with inequality (2.44), which can be restated as

$$-\frac{d}{dt}D_\phi[u(t)] \geq -2\lambda\frac{d}{dt}H_\phi[u(t)], \tag{2.46}$$

provided $u(t)$ satisfies (2.36). Integrate (2.46) in time from $t = 0$ to $t = +\infty$ to obtain

$$D_\phi[u_0] - \lim_{t\to+\infty} D_\phi[u(t)] \geq 2\lambda\big(H_\phi[u_0] - \lim_{t\to+\infty} H_\phi[u(t)]\big).$$

This is very close to the desired inequality (2.38); it remains to investigate the limits. On the left-hand side, this is almost trivial: since D_ϕ is non-negative and non-increasing, and $\int_0^\infty D_\phi[u(t)]dt \leq H_\phi[u_0] < +\infty$, it follows that $D_\phi[u(t)] \to 0$ monotonically as $t \to \infty$. The formal argument for the limit on the right-hand side is also simple: Assuming that we can interchange the limit and the nonlinear functional D_ϕ, it follows

$$0 = \lim_{t\to+\infty} D_\phi[u(t)] = D_\phi[\lim_{t\to+\infty} u(t)]. \tag{2.47}$$

Since the entropy production attains zero exactly at the point u_∞, we conclude that $\lim_{t\to+\infty} u(t) = u_\infty$. Interchanging limits once again, we have

$$\lim_{t\to+\infty} H_\phi[u(t)] = H_\phi[\lim_{t\to+\infty} u(t)] = H_\phi[u_\infty]. \tag{2.48}$$

Unfortunately, some unpleasant density arguments are needed to make this argument rigorous [**5**]. The results, however, is that the limit vanishes for solutions corresponding to any sensible initial condition u_0.

5.4. Step four: proof of the equilibration estimate.

LEMMA 2.5. *The estimate* (2.39) *holds.*

Invoking the Gronwall argument, inequality (2.38) immediately implies that

$$H_\phi[u(t)] - H_\phi[u_\infty] \leq \big(H_\phi[u_0] - H_\phi[u_\infty]\big)e^{-2\lambda t}.$$

Since $1/\phi''$ is concave by assumption, it is continuous $\mathbb{R}$, and $\lim_{s\searrow 0} 1/\phi''(s) < +\infty$. Hence, ϕ'' has a positive lower bound on $[0, 1]$. Proposition 1.1 applies and yields the equilibration estimate (2.39).

6. Problems

PROBLEM 2.1. *Use the Bakry-Emery method for a (formal) derivation of the celebrated* logarithmic Sobolev inequality *on* $\mathbb{R}^d$,

$$\int_{\mathbb{R}^d} u(x)\log u(x)\,dx + d\log\big(\sqrt{2\pi}e\big) \leq 2\int_{\mathbb{R}^d} \big|\nabla\sqrt{u(x)}\big|^2\,dx. \tag{2.49}$$

Here u is a (strictly positive and sufficiently smooth) probability density.
Hint: *Apply the method to the Fokker-Planck equation with quadratic confinement potential $V(x) = \frac{1}{2}|x|^2$. The relevant relative entropy H_ϕ is generated by $\phi(s) = s\log s$.*

PROBLEM 2.2. *Calculate the value of the H-functional $H[u] = \int_{\mathbb{R}^d} u(x)\log u(x)\,dx$ as a function of time for the fundamental solution*

$$U(t;x) = \big(2\pi(t+1)\big)^{-d/2}\exp\Big(-\frac{|x|^2}{2(t+1)}\Big) \tag{2.50}$$

of the heat equation on $\mathbb{R}^d$,

$$\partial_t u(t;x) = \frac{1}{2}\Delta_x u(t;x). \tag{2.51}$$

What happens in the limit $t \to +\infty$, and why is this expected?

PROBLEM 2.3. *Let a (smooth, normalized, rapidly decaying) solution $u(t)$ to the heat equation* (2.51) *be given. Our mission is to estimate the decay of the* non-symmetric entropy

$$E\big[u(t)|U(t)\big] := H\big[u(t)\big] - \int_{\mathbb{R}^d} u(t;x)\log U(t;x)\,dx. \tag{2.52}$$

Here U is the fundamental solution from (2.50) *above.*
Proceed as follows:

- *Perform a change of variables,*

$$y = (1+t)^{-1/2}x, \quad s = \frac{1}{2}\log(1+t), \quad u(t;x) = (1+t)^{-d/2}v(s;y). \tag{2.53}$$

- *Verify that v satisfies the Fokker-Planck equation*

$$\partial_s v(s;y) = \Delta_y v(s;y) + \nabla_y\cdot\big(y\nabla_y v(s;y)\big). \tag{2.54}$$

  Notice the different coefficients in front of the Laplacians in (2.51) *and* (2.54).
- *Verify that V (the transformation of U) constitutes the steady state for* (2.54).
- *Prove that $E[u(t)|U(t)] = H_\phi[v(s)] - H_\phi[V]$, using a change of variables under the integral. Here H_ϕ is the relative entropy from the lecture, with $\phi(s) = s\log s$.*
- *Use inequality* (2.49) *to conclude convergence $H_\phi[v(s)]$ to $H_\phi[V]$, exponentially fast in s.*
- *Inteprete the result in terms of the original variables.*

CHAPTER 3

Nonlinear Diffusion

Having studied the *linear* Fokker-Planck equation (2.36) in great detail, we turn to investigate its nonlinear analogue,

$$\partial_t u = \Delta f(u) + \nabla \cdot (u \nabla V). \tag{3.1}$$

As before, V represents a confinement potential. The novelty is that the rate of diffusion is not constant anymore but depends on the solution $u(x)$ through $f'\big(u(x)\big)$ with a smooth function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$. Naturally, only non-negative solutions are considered.
This lecture is mainly concerned with the special case of the *rescaled porous medium equation* (3.2), which is (3.1) with

$$f(s) = s^m, \quad V(x) = \frac{\lambda}{2}|x|^2,$$

with some $m > 1$ and $\lambda > 0$. The nonlinear effect is the stronger, the larger m is, and the limit $m \searrow 1$ gives back the *linear* Fokker-Planck equation (2.36). Our main concern is the proof of equilibration — exponentially fast in time — of solutions u to (3.1). As a by-product of this, we shall obtain a version of the celebrated *Gagliardo-Nirenberg interpolation estimates.* This is in complete analogy to the derivation of the *logarithmic Sobolev inequality* (2.30) as a consequence of equilibration in the Ornstein-Uhlenbeck process. In the very end of this lecture, some comments are made about the situation with more general nonlinearities f.

1. The porous medium equation

1.1. The equation and its steady state. The *rescaled porous medium equation* reads

$$\partial_t u = \Delta u^m + \lambda \nabla \cdot (xu), \tag{3.2}$$

with $m > 1$ and $\lambda > 0$. The term "rescaled" is explained in section 1.2 below. In order to avoid the discussion of boundary integrals, we assume that

(1) *either* (3.2) is posed on a convex domain $\Omega \subset \mathbb{R}^d$ and the flux of u across the (smooth) boundary $\partial\Omega$ is zero, i.e. $\mathbf{n}(x) \cdot (\frac{m}{m-1}\nabla u(x)^{m-1} + \lambda x) = 0$ for $x \in \partial\Omega$,
(2) *or* (3.2) is posed on $\Omega = \mathbb{R}^d$, with the assumption that $u(x)$ decays rapidly as $|x| \to \infty$.

The stationary (weak) solutions to (3.2) are the celebrated *Barenblatt profiles*,

$$u_\infty(x) = \Big(\sigma - \lambda\frac{m-1}{2m}|x|^2\Big)_+^{1/(m-1)}. \tag{3.3}$$

We emphasize that u_∞ is compactly supported; u_∞ is positive on

$$B = \Big\{ x \,\Big|\, |x|^2 < R^2 := \frac{2m\sigma}{\lambda(m-1)} \Big\}.$$

The quantity $\sigma > 0$ is referred to as the mass parameter, which we assume to be arbitrary but fixed in the following (no simplification results from the restriction to unit mass).

1.2. Remarks on the "free" equation. The name *rescaled porous medium equation* originates from the fact that, for $\lambda = 1$ and $\Omega = \mathbb{R}^d$, the scaling

$$y = e^t x, \quad s = \frac{e^{\theta t} - 1}{\theta}, \quad u(t;x) = e^{dt} v(s;y) \quad \text{with} \quad \theta = 2 + d(m-1),$$

transforms (3.2) into the *free porous medium equation*,

$$\partial_s v = \Delta_y v^m. \tag{3.4}$$

Notice that u_∞ corresponds to a self-similar source-type solution v_s of the free equation (3.4),

$$v_s(s;y) = s^{-d\theta} u_\infty(s^{-\theta}x), \quad \theta = 2 + d(m-1). \tag{3.5}$$

Correspondingly, the self-similar solution v_s has a spreading support. Applying the maximum principle to (3.4), it is immediate to conclude that initially compactly supported solutions remain compactly supported for all positive times. Moreover, by the minimum principle, the support of any solution spreads out. The expansion of the support happens at the time scale s^θ.
Equation (3.4) has a somewhat unpleasant history. Its first derivation dates back to the times of the cold war, and the source-type solutions (3.5) to (3.4), first discovered by the russian mathematician Barenblatt, were supposed to describe the propagation of a heat front after the explosion of an atomic bomb. Today's applications of (3.4) are fortunately more restricted to the wetting of materials by a liquid.
The properties of weak solutions to (3.4) have been exhaustively studied, see for instance [**34**]. These translate word-by-word to properties of weak solutions to (3.2).

- Weak solutions exist for basically all sensible initial data, and they are unique.
- Mass and positivity are preserved.
- Positive solutions $u(t,x)$ are smooth (in fact classical) in space and time for $t > 0$, and non-negative solutions are Hölder-continuous in space.
- A variety of comparision principles (most important: maximum and minimum principle) and rearrangement inequalities are satisfied by weak solutions.

1.3. Entropy approach. From here on, we shall follow the strategy developed in [**20**], which is a non-linear version of (Toscani's approach to) the Bakry-Émery method. There are, however, some additional technical difficulties, even on the semi-rigorous level. One is that the elegant concept of relative entropy (1.3) cannot be used here, since u_∞ is zero outside of the bounded set B. Thus one has to resort to the more general but less convenient absolute entropies (1.2). More precisely, define

$$H[u] = \int_{\mathbb{R}^d} u\Big(\frac{u^{m-1}}{m-1} + \frac{\lambda}{2}|x|^2\Big)\,dx.$$

For convenience, we also introduce the difference

$$E[u] = H[u] - H[u_\infty].$$

The main results are summarized in

Theorem 3.1. *The following entropy production inequality holds for all $u \geq 0$ with $H[u] < \infty$,*

$$E[u] \leq \frac{1}{2\lambda}\int_\Omega u\Big|\frac{m}{m-1}\nabla u^{m-1} + \lambda x\Big|^2\,dx. \tag{3.6}$$

Consequently, assuming that $u(t)$ is a solution to (3.2) *with $H[u_0] < \infty$, then both the entropy H and its production D_H converge exponentially at rate 2λ. Moreover, solutions u equilibrate exponentially fast in $L^1(\Omega)$,*

$$\|u(t) - u_\infty\|_{L^1} \leq Ce^{-\lambda t}, \tag{3.7}$$

where C only depends on $H[u_0]$.

As in the linear case, the proof consists of four main steps. For brevity, introduce

$$Q = \frac{m}{m-1}u^{m-1} + \frac{\lambda}{2}|x|^2,$$

and notice that (3.2) can be written as

$$\partial_t u = \nabla\cdot(u\nabla Q).$$

Two word about the justification of the formal manipulations performed below are in place now. First, the porous medium equation does not have the strong regularizing effect of the linear

Fokker-Planck equation. Even at $t > 0$, the spatial profile $u(t;x)$ of a non-negative solution (like the stationary one) is only Hölder continuous in general. The common trick is to approximate non-negative solutions by strictly positive ones, which are smooth and classical. All relevant estimates pass to the non-negative limit. The second problem is the justification of integration by parts. Since the quadratic potential appears frequently under the integral, it is by no means trivial to conclude that the boundary terms indeed vanish. We leave it here to the reader to check that finiteness of an $2+\epsilon$-moment of the initial condition is indeed sufficient to proceed as formally shown.

1.4. Step one: First entropy production. The time derivative of $E[u(t)]$ amounts to

$$\begin{aligned} D[u] &= -\frac{d}{dt}E[u] \\ &= -\int \Big(\frac{m}{m-1}u^{m-1} + \frac{\lambda}{2}|x|^2\Big)\partial_t u \, dx \\ &= -\int Q\nabla\cdot(u\nabla Q)\,dx \\ &= \int u|\nabla Q|^2\,dx. \end{aligned}$$

This expression is obviously non-negative, and zero exactly if Q is constant on each connected component of the support of u. In view of the spatial Hölder-continuity of solutions and the boundary conditions, the latter implies that Q vanishes identically on Ω, and that u is a Barenblatt profile.

1.5. Step two: Second entropy production. For the second derivative, one finds

$$\begin{aligned} R[u] &= -\frac{1}{2}\frac{d}{dt}D[u] \\ &= -\frac{1}{2}\int_\Omega \partial_t u|\nabla Q|^2\,dx - \int_\Omega u\nabla Q\cdot\partial_t\nabla Q\,dx \\ &= -\frac{1}{2}\int_\Omega \nabla\cdot(u\nabla Q)|\nabla Q|^2\,dx + \int_\Omega \nabla\cdot(u\nabla Q)\partial_t Q\,dx \\ &= +\int_\Omega u\nabla Q\cdot\nabla^2 Q\cdot\nabla Q\,dx + m\int_\Omega u^{m-2}(\nabla\cdot(u\nabla Q))^2\,dx. \end{aligned}$$

Using the definition of Q, we realize that

$$\begin{aligned} R[u] = \lambda\int_\Omega u|\nabla Q|^2\,dx + \\ + \frac{m}{m-1}\underbrace{\Big(\int_\Omega u\nabla Q\cdot\nabla^2 u^{m-1}\cdot\nabla Q\,dx + (m-1)\int_\Omega u^{m-2}(\nabla\cdot(u\nabla Q))^2\,dx\Big)}_{=(*)}. \end{aligned}$$

Naturally, the goal is to prove that the sum of the terms in (*) is non-negative, leading immediately to

$$\lambda D[u(t)] \le R[u(t)]. \tag{3.8}$$

In order to prove non-negativity of (*), integrate the first contribution by parts, and expand the square $(u\Delta Q + \nabla u\cdot\nabla Q)^2$ in the second contribution. This yields

$$\begin{aligned} (*) = &-\int_\Omega (\nabla u\cdot\nabla Q)(\nabla Q\cdot\nabla u^{m-1})\,dx - \int_\Omega u\nabla Q\cdot\nabla^2 Q\cdot\nabla u^{m-1}\,dx - \int_\Omega u(\nabla Q\cdot\nabla u^{m-1})\Delta Q\,dx \\ &+ (m-1)\int_\Omega u^m(\Delta Q)^2\,dx + 2(m-1)\int_\Omega u^{m-1}(\nabla u\cdot\nabla Q)\Delta Q\,dx + (m-1)\int_\Omega u^{m-2}(\nabla u\cdot\nabla Q)^2\,dx \\ = &(m-1)\int_\Omega u^m(\Delta Q)^2\,dx + \frac{m-1}{m}\int_\Omega \nabla u^m\cdot\nabla Q(\Delta Q)\,dx - \frac{m-1}{m}\int_\Omega \nabla u^m\cdot\nabla^2 Q\cdot\nabla Q\,dx. \end{aligned}$$

Finally, integrate the last two terms by parts, removing the gradient from u^m. The third derivatives cancel, leaving

$$(*) = \frac{(m-1)^2}{m}\int u^m(\Delta Q)^2\,dx + \frac{m-1}{m}\int u^m\|\nabla^2 Q\|^2\,dx \geq 0.$$

1.6. Step three: Derivation of the functional inequality. Perform integration in time of the inequality (3.8) to obtain (3.6). It is probably needless to emphasize that a variety of technical obstacles need to be overcome in order to make this step rigorous. As in the linear case, the hardest part is to show that $E[u(t)] \to 0$ as $t \to \infty$.
In direct consequence of (3.6), one obtains the exponential convergence of H and D_H by a Gronwall argument.

1.7. Step four: Csiszar-Kullback inequality. The proof of the equilibration estimate is suprisingly tricky. Again, this is a technical difficulty caused by the lack of positivity of u_∞.

LEMMA 3.1. *Assume that the support of u is contained in $\{|x| \leq R\}$. Then $\|u-u_\infty\|_{L^1} \leq CE[u]^{1/2}$ for a suitable $C > 0$.*

PROOF. The idea is to perform a (pointwise) Taylor expansion w.r.t. $u(x)$ of the integrand, i.e. write

$$\frac{\lambda}{2}|x|^2(u-u_\infty) + \frac{1}{m-1}(u^m - u_\infty^m) = \underbrace{\Big(\frac{\lambda}{2}|x|^2 + \frac{m}{m-1}u_\infty^{m-1}\Big)}_{=\frac{m\sigma}{m-1}}(u-u_\infty) + \frac{m}{2}\tilde{u}^{m-2}(u-u_\infty)^2,$$

where $\tilde{u}(x)$ is an intermediate value between $u_\infty(x)$ and $u(x)$. Integrate this, using that u and u_∞ have the same mass, to find

$$E[u] = \frac{m}{2}\int_\Omega \tilde{u}^{m-2}(u-u_\infty)^2\,dx.$$

Suppose that $m \geq 2$. Then if $u(x) \geq u_\infty(x)$, also $\tilde{u}(x)^{m-2} \geq u_\infty(x)^{m-2}$, so

$$E[u] \geq \frac{m}{2}\int_{u \geq u_\infty} u_\infty^{m-2}(u-u_\infty)^2\,dx.$$

By the usual trick, exploiting equality of mass again, one finally obtains

$$\|u-u_\infty\|_{L^1} \leq 2\int_{\{u>u_\infty\}}|u-u_\infty|\,dx \leq 2\Big(\underbrace{\int_{\{u>u_\infty\}} u_\infty^{m-2}(u-u_\infty)^2\,dx}_{\leq \frac{2}{m}E[u]}\Big)^{1/2}\Big(\underbrace{\int_B u_\infty^{-(m-2)}\,dx}_{<+\infty}\Big)^{1/2}.$$

Finiteness of the lasr integral follows since $-(m-2)/(m-1) > -1$.
In the case where $1 < m < 2$, the same argument can be used, now considering the set $\{u < u_\infty\}$. □

It remains to be shown that $E[u]$ also controls the behavior of u outside the support of u_∞.

LEMMA 3.2. *For some $C > 0$, depending only on m, λ and d, the excess mass $\mu(u)$ satisfies*

$$\mu(u) := \int_{|x|\geq R} u\,dx \leq CE[u]^{1/2}. \tag{3.9}$$

As a by-product of the following proof, we find that $E[u] \geq 0$ with equality exactly for $u = u_\infty$.

PROOF. There are two steps in this proof. For the first, we use that $s \mapsto s^m$ is convex for $m > 1$, and so $u^m - u_\infty^m \geq m u_\infty^{m-1}(u - u_\infty)$. It follows

$$
\begin{aligned}
E[u] &\geq \int_{|x|<R} \underbrace{\Big(\frac{\lambda}{2}|x|^2 + \frac{m}{m-1}u_\infty^{m-1}\Big)}_{=\frac{m\sigma}{m-1}=(\lambda/2)R^2}(u - u_\infty)\,dx + \int_{|x|\geq R}\Big(\frac{\lambda}{2}|x|^2 u + \frac{u^m}{m-1}\Big)\,dx \\
&= -\frac{\lambda R^2}{2}\int_{|x|\geq R}(u - u_\infty)\,dx + \int_{|x|\geq R}\Big(\frac{\lambda}{2}|x|^2 u + \frac{u^m}{m-1}\Big)\,dx \\
&= \int_{|x|\geq R}\Big(\frac{\lambda}{2}\big(|x|^2 - R^2\big)u + \frac{u^m}{m-1}\Big)\,dx =: E_*[u].
\end{aligned}
$$

This finishes the first step. Next, the excess mass is estimated in terms of E_*. With $\rho > 0$ to be chosen later,

$$
\int_{|x|\geq R} u\,dx = \int_{R^2\leq |x|^2\leq R^2+\rho} u\,dx + \int_{|x|^2 > R^2+\rho} u\,dx \tag{3.10}
$$

$$
\leq \Big(\int_{|x|\geq R}\frac{u^m}{m-1}\,dx\Big)^{1/m} A\rho^{1-1/m} + \frac{1}{\lambda}\rho\int_{|x|\geq R}\frac{\lambda}{2}\big(|x|^2 - R^2\big)u\,dx \tag{3.11}
$$

$$
\leq A\rho^{1-1/m}E_*^{1/m} + \lambda^{-1}\rho^{-1}E_*. \tag{3.12}
$$

Here A is a function of λ, m and the dimension d only. The choice $\rho = E_*^{1/2}$ thus provides (3.9). $\square$

Finally, Lemma 3.1 and Lemma 3.2 need to be combined in order to obtain the equilibration estimate (3.7).
Introduce the function $\tilde{u} = \alpha u \mathbf{1}_{\{|x|\leq R\}}$, where $\alpha(M - \mu(u)) = M$, and M is the (conserved) mass of u. This function has the same mass and support as u_∞. By the triangle inequality,

$$
\|u - u_\infty\|_{L^1} \leq \|u - \tilde{u}\|_{L^1} + \|\tilde{u} - u_\infty\|_{L^1} \leq 2\mu(u) + \|\tilde{u} - u_\infty\|_{L^1}.
$$

The excess mass $\mu(u)$ is controlled by $E[u]^{1/2}$, see Lemma 3.2. And the second term is controlled by $E[\tilde{u}]^{1/2}$, see Lemma 3.1. Thus it remains to obtain control of $E[\tilde{u}]$ in terms of $E[u]$. One has

$$
\begin{aligned}
E[\tilde{u}] &= \int_{|x|\leq R}\Big(\frac{\lambda}{2}|x|^2(\alpha u - u_\infty) + \frac{\alpha^m u^m - u_\infty^m}{m-1}\Big)\,dx \\
&\leq E[u] + (\alpha - 1)\int\frac{\lambda}{2}|x|^2 u\,dx + (\alpha^m - 1)\int\frac{u^m}{m-1}\,dx \\
&\leq E[u] + (\alpha^m - 1)E[u].
\end{aligned}
$$

Observing that α remains bounded for $E[u] \to 0$ (in fact goes to one), the argument is finished, and (3.7) follows.

2. Gagliardo-Nirenberg estimates

Like the Bakry-Émery method, applied to a linear Fokker-Planck equation, provides a proof of the logarithmic Sobolev inequality (2.30), the non-linear method above, applied to the rescaled porous medium equation, delivers a proof of Gagliardo-Nirenberg inequalities (3.17). Below, we essentially follow [**21**], where the connection between optimal decay estimates for the porous medium equation and the Gagliardo-Nirenberg estimate has been nicely detailed.

LEMMA 3.3. *Given $m > 1$, there are constants A and B such that*

$$
\int_{\mathbb{R}^d} u(x)^m\,dx \leq A\int_{\mathbb{R}^d}\big|\nabla\big(u(x)^{m-1/2}\big)\big|^2\,dx + B\Big(\int_{\mathbb{R}^d} u(x)\,dx\Big)^\nu, \quad \nu = \frac{2m + d(m-1)}{2 + d(m-1)}. \tag{3.13}
$$

Notice that we do not assume unit mass of u anymore.

Proof. Inequality (3.13) is a consequence of (3.6) for $\lambda = 1$. In fact, rewrite the right-hand side of (3.6) as follows:

$$\begin{aligned}
&\int_{\mathbb{R}^d} u|\frac{m}{m-1}\nabla u^{m-1} + x|^2\,dx \\
&= \int_{\mathbb{R}^d} |\frac{m}{m-1}u^{1/2}\nabla u^{m-1}|^2\,dx + \int_{\mathbb{R}^d} |x|^2 u\,dx + \int_{\mathbb{R}^d} \frac{2m}{m-1} x\cdot u\nabla u^{m-1}\,dx \\
&= \Big(\frac{m}{m-1/2}\Big)^2 \int_{\mathbb{R}^d} |\nabla u^{m-1/2}|^2\,dx + \int_{\mathbb{R}^d} |x|^2 u\,dx - 2\underbrace{\int_{\mathbb{R}^d} x\cdot\nabla u^m\,dx}_{=d\int u^m\,dx}.
\end{aligned}$$

Joining corresponding terms on the left- and right-hand side, one obtains

$$(d + \frac{1}{m-1})\int_{\mathbb{R}^d} u^m\,dx \le \frac{1}{2}\Big(\frac{m}{m-1/2}\Big)^2 \int_{\mathbb{R}^d} |\nabla u^{m-1/2}|^2\,dx + H[u_\infty]. \tag{3.14}$$

Inequality (3.14) is almost in the shape of (3.13). It remains to be checked that the respective last terms in these formulas agree. To this end, observe that from the explicit formula for the Barenblatt profile,

$$u_\infty(x) = \Big(\sigma - \frac{m-1}{2m}|x|^2\Big)_+^{1/(m-1)} = \sigma^{1/(m-1)}\underbrace{\Big(1 - \frac{m-1}{2m}|y|^2\Big)_+^{1/(m-1)}}_{=U_m(y)}$$

with $\sqrt{\sigma}y = x$. Notice that the function U_m is completely determined by m. Now,

$$\begin{aligned}
H[u_\infty] &= \int_{\mathbb{R}^d} \frac{|x|^2}{2} u_\infty(x)\,dx + \int_{\mathbb{R}^d} \frac{u_\infty(x)^m}{m-1}\,dx \\
&= \sigma^{1/(m-1)+1-d}\int \frac{|y|^2}{2} U_m(y)\,dy + \sigma^{m/(m-1)-d}\int_{\mathbb{R}^d} \frac{U_m(y)^m}{m-1}\,dy \\
&= K_m \sigma^{m/(m-1)-d},
\end{aligned}$$

with a universal constant K_m. On the other hand, the mass of u_∞ is

$$\int_{\mathbb{R}^d} u_\infty(x)\,dx = \sigma^{1/(m-1)-d}\int_{\mathbb{R}^d} U_m(y)\,dx.$$

In combination, it is clear that there is some universal constant L_m such that

$$H[u_\infty] = L_m\Big(\int_{\mathbb{R}^d} u_\infty(x)\,dx\Big)^\nu,$$

where the ν agrees with the one given in (3.13). □

In the next step, (3.13) is optimized by scaling. Setting

$$u(x) = \lambda^{\frac{d}{m}}\tilde{u}(\lambda x), \quad \lambda > 0, \tag{3.15}$$

the left-hand side of (3.13) remains unchanged, while factors appear on the right-hand side. More precisely,

$$\int_{\mathbb{R}^d} \tilde{u}^m\,dy \le A\lambda^{\frac{2m+d(m-1)}{m}}\int_{\mathbb{R}^d} |\nabla_y \tilde{u}^{m-1/2}|^2\,dy + B\lambda^{-\nu\frac{d(m-1)}{m}}\Big(\int_{\mathbb{R}^d} \tilde{u}\,dy\Big)^\nu.$$

Optimality is achieved by choosing $\lambda > 0$ such that the right-hand side is minimal. Recall that the minimum of the expression $\lambda^\alpha x + \lambda^\beta y$ (with $\alpha\beta < 0$) is $cx^{\beta/(\beta-\alpha)}y^{\alpha/(\alpha-\beta)}$, with some $c > 0$. This eventually yields

$$\int_{\mathbb{R}^d} \tilde{u}^m\,dy \le C\Big(\int_{\mathbb{R}^d} |\nabla_y \tilde{u}^{m-1/2}|^2\,dy\Big)^{\frac{d(m-1)}{2(1+d(m-1))}}\Big(\int_{\mathbb{R}^d} \tilde{u}\,dy\Big)^{\frac{2m+d(m-1)}{2(1+d(m-1))}}. \tag{3.16}$$

Finally, introducing

$$w = u^{m-1/2}, \quad p = \frac{1}{2m-1} \in (0,1),$$

gives inequality (3.16) the following standard form.

COROLLARY 3.1. *Given an exponent $p \in (0,1)$, there exists a constant $C > 0$ such that*

$$\|w\|_{1+p} \leq C\|\nabla w\|_2^\theta \|w\|_{2p}^{1-\theta}, \quad \theta = \frac{d(1-p)}{(1+p)(2p+d(1-p))} \tag{3.17}$$

holds for all non-negative functions $w \in H^1(\mathbb{R}^d) \cap L^{2p}(\mathbb{R}^d)$.

3. Generalization

We close the discussion with a somewhat general theorem about entropy relaxation for (3.1). For a much deeper and also mathematically rigorous discussion, we refer to the article [**16**] (which covers the topic almost exhaustively). Indeed, serious analytical issues arise for general solutions to (3.1) as soon as one leaves the world of the porous medium equations (3.2). For instance, solutions the *fast diffusion equation* with $f(s) = s^m$ and $0 < m < 1$ tend to lose mass if $m < 1 - 2/d$, despite the divergence form of (3.1). Nontheless, entropy methods still apply and provide valuable information [**12**].

We avoid all these discussions by adopting once again a completely formal point of view, assuming positive and sufficiently smooth solutions to (3.1), which decay rapidly enough to justify all manipulations. For the sake of definiteness, let that the nonlinearity $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be smooth, with $f(0) = 0$ and $f'(s) > 0$ for $s > 0$. Hence, the only levelset on which the diffusion might degenerate is $\{u = 0\}$. In fact, the behavior of $f(s)$ for $s > 0$ is of little importance in the following; all the interesting features of the equation (3.1) are encoded in the behavior of f at $s = 0$. Moreover, the confinement potential should be convex in the strong sense that

$$\nabla^2 V \geq \lambda \mathbf{1} \tag{3.18}$$

with some $\lambda > 0$.

The starting point is to rewrite equation (3.1) in the form of a gradient flow type w.r.t. Wasserstein metric, i.e.

$$\partial_t u = \nabla \cdot (u\mathbf{v}),$$

with the "Wasserstein velocity"

$$\mathbf{v} = \nabla Q, \qquad Q = \theta(u) + V,$$

where Q is the variational derivative of the entropy functional

$$H[u] := \int \big(\Theta(u) + uV\big)\, dx.$$

The functions Θ, θ and f are related by

$$\Theta'(s) = \theta(s) \quad \text{and} \quad s\theta'(s) = f'(s).$$

The production term for H is given by

$$D[u] = -\frac{d}{dt}H[u] = \int u|\mathbf{v}|^2\, dx = \int u|\nabla Q|^2\, dx.$$

Clearly, this expression is non-negative. It is zero iff

$$\theta\big(\hat{u}(x)\big) = \sigma - V(x) \tag{3.19}$$

with some constant σ on each connected component of the support of $\hat{u}$. By convexity (3.18) of V, it is not hard to argue that there can be only one component (either a compact set or the whole space), and (3.19) holds with one global value of σ, which is determined by the mass of $\hat{u}$. The respective $\hat{u}$ defines the unique steady state u_∞ of (3.1) under the given mass constraint.

THEOREM 3.2. *Assume that the potential V is convex according to (3.18), and that $(1-1/d)f(s) \leq sf'(s)$ for all $s > 0$. Then entropy and entropy production satisfy the relation*

$$H[u] - H[u_\infty] \leq \frac{1}{2\lambda} D_H[u]. \tag{3.20}$$

Moreover, H and D_H converge exponentially with rate 2λ.

The equilibration property is hard to prove in this setting. See, however, the discussion in [**16**].

PROOF. A sketch of the formal part of the agument is given here. The proof for (3.20) is similar to the one leading to (3.6). Indeed, in complete analogy to step 1.5, calculate the second entropy production,

$$\begin{aligned} R[u(t)] &= -\frac{1}{2}\frac{d}{dt}D[u(t)] \\ &= \int_\Omega u\big(\mathbf{v}\cdot\nabla^2 V\cdot\mathbf{v}\big)\,dx + \int_\Omega u\big(\mathbf{v}\cdot\nabla^2 h(u)\cdot\mathbf{v}\big)\,dx + \int_\Omega h'(u)|\nabla\cdot(u\mathbf{v})|^2\,dx. \end{aligned} \tag{3.21}$$

By (3.18), the first integral controls the entropy production,

$$\int_\Omega u\big(\mathbf{v}\cdot\nabla^2 V\cdot\mathbf{v}\big)\,dx \geq \lambda D[u].$$

It remains to be shown that the remaining two integrals in (3.21) are non-negative. The contribution of the second integral amounts to

$$\begin{aligned} &\int_\Omega u\big(\mathbf{v}\cdot\nabla^2 h(u)\cdot\mathbf{v}\big)\,dx \\ &\quad= \int_\Omega (u\mathbf{v})\cdot\big((\mathbf{v}\cdot\nabla)\nabla h(u)\big)\,dx \\ &\quad= -\int_\Omega \nabla\cdot(u\mathbf{v})\big(\mathbf{v}\cdot\nabla h(u)\big)\,dx - \int_\Omega \nabla h(u)\cdot\big(u\mathbf{v}\cdot\nabla\mathbf{v}\big)\,dx \\ &\quad= -\int_\Omega \big(\mathbf{v}\cdot\nabla f(u)\big)(\nabla\cdot\mathbf{v})\,dx - \int_\Omega \big(\mathbf{v}\cdot\nabla h(u)\big)(\mathbf{v}\cdot\nabla u)\,dx - \int_\Omega \nabla f(u)\cdot\nabla^2 Q\cdot\mathbf{v}\,dx \end{aligned}$$

Now observe that

$$\begin{aligned} -\int_\Omega \nabla f(u)\cdot\nabla^2 Q\cdot\mathbf{v}\,dx &= \int_\Omega f(u)\nabla\cdot\big(\nabla^2 Q\cdot\nabla Q\big)\,dx - \int_{\partial\Omega} f(u)(\mathbf{n}\cdot\nabla^2 Q\cdot\nabla Q\,dx') \\ &\geq \int_\Omega f(u)\|\nabla^2 Q\|^2\,dx + \int_\Omega f(u)(\mathbf{v}\cdot\nabla\Delta Q)\,dx. \end{aligned}$$

Here we used convexity of Ω to estimate the boundary term. The third integral in (3.21) gives

$$\begin{aligned} &\int_\Omega h'(u)\big(\nabla\cdot(u\mathbf{v})\big)\,dx \\ &\quad= \int_\Omega h'(u)(u\nabla\cdot\mathbf{v}+\mathbf{v}\cdot\nabla u)^2\,dx \\ &\quad= \int_\Omega uf'(u)(\nabla\cdot\mathbf{v})^2\,dx + 2\int_\Omega \big(\mathbf{v}\cdot\nabla f(u)\big)(\mathbf{v}\cdot\nabla u)\,dx + \int_\Omega \big(\mathbf{v}\cdot\nabla h(u)\big)(\mathbf{v}\cdot\nabla u)\,dx. \end{aligned}$$

Summing up, we obtain

$$\begin{aligned} &\int_\Omega u\big(\mathbf{v}\cdot\nabla^2 h(u)\cdot\mathbf{v}\big)\,dx + \int_\Omega h'(u)\big(\nabla\cdot(u\mathbf{v})\big)\,dx \\ &\geq \int_\Omega \big(\mathbf{v}\cdot\nabla f(u)\big)(\nabla\cdot\mathbf{v})\,dx + \int_\Omega uf'(u)(\nabla\cdot\mathbf{v})^2\,dx + \int_\Omega f(u)\|\nabla^2 Q\|^2\,dx + \int_\Omega f(u)(\mathbf{v}\cdot\nabla\Delta Q)\,dx \\ &= \int_\Omega \big(uf'(u)-f(u)\big)(\nabla\cdot\mathbf{v})^2\,dx + \int_\Omega f(u)\|\nabla^2 Q\|^2\,dx. \end{aligned}$$

The elementary inequality

$$(\nabla\cdot\mathbf{v})^2 = \big(\mathrm{tr}(\nabla^2 Q)\big)^2 \leq d\|\nabla^2 Q\|^2$$

allows to conclude

$$-\frac{1}{2}\frac{d}{dt}D[u] \geq \lambda D[u] + \int_\Omega \big(uf'(u)-f(u)+d^{-1}f(u)\big)(\nabla\cdot\mathbf{v})^2\,dx.$$

In view of the hypothesis on f, this shows

$$-\frac{d}{dt}\big(H[u(t)] - H[u_\infty]\big) \leq -\frac{1}{2\lambda}\frac{d}{dt}D[u(t)].$$

Integration of the last line from $t = 0$ to $t = +\infty$ (ignoring all analytical difficulties that may arise) finally gives (3.20). $\square$

4. Problems

PROBLEM 3.1. *Consider the homogeneous raditive transfer equation*

$$\partial_t u(t;x) = -u(t;x) + \int_\Omega u(t;y)\,\mu(dy) \tag{3.22}$$

on a nice domain $\Omega \subset \mathbb{R}^d$ *with fixed probability measure* μ*. Let* $\psi : \mathbb{R} \to \mathbb{R}$ *be smooth and convex, and define the absolute entropy* $H_\psi[u] = \int \psi(u(x))\,\mu(dx)$*. Prove*

(3.23)
$$\int_\Omega \psi\big(v(x)\big)\mu(dx) - \psi\Big(\int_\Omega v(x)\mu(dx)\Big) \le \frac{1}{2}\int_{\Omega\times\Omega} \big[\psi'\big(v(x)\big) - \psi'\big(v(y)\big)\big]\big[v(x) - v(y)\big]\mu(dx\,dy)$$

for all $v \in L^1_\mu(\Omega)$ *by proceeding as follows:*

(1) *Find the explicit form of the solution* $u(t;x)$ *to* (3.22) *for* $u(0;x) = v(x)$*.*
(2) *Show that the left-hand side of* (3.23) *is* $H_\psi[u(0)] - H_\psi[u_\infty]$*.*
(3) *Show that the right-hand side of* (3.23) *is* $D_\psi[u(0)]$*.*
(4) *Show that* $H_\psi[u(t)] - H_\psi[u_\infty] \le -dH_\psi[u(t)]/dt$ *by using the explicit form of the solution.*

PROBLEM 3.2. *The Barenblatt solutions to the (rescaled) porous medium equation are given by*

$$U_{\sigma,m}(x) = \Big(\sigma - \frac{m-1}{2m}|x|^2\Big)_+^{1/(m-1)}. \tag{3.24}$$

The quantity $\sigma > 0$ *is the mass parameter, and* $m > 1$*. Show that there is exist an exponent* $\nu > 0$ *and a constant* K *(depending only on* m *and the dimension* d*) such that*

$$H[U_{\sigma,m}] = \int_{\mathbb{R}^d} \Big(\frac{|x|^2}{2}U_{\sigma,m} + \frac{U^m_{\sigma,m}(x)}{m-1}\Big)dx = K\Big(\int_{\mathbb{R}^d} U_{\sigma,m}(x)\,dx\Big)^\nu. \tag{3.25}$$

Calculate the value of ν*.*

PROBLEM 3.3. *Recall that the entropy production estimate for the linear Fokker-Planck equation lead to the famous* logarithmic Sobolev inequality*. We shall use the entropy production estimate for the rescaled porous medium equation to derive the following* Gagliardo-Nirenberg inequality*:*

$$\|w\|_{L^{p+1}} \le C\|\nabla w\|^\theta_{L^2}\|w\|^{1-\theta}_{L^{2p}}. \tag{3.26}$$

As a preliminary step, show that the entropy production estimate can be rephrased in terms of Dolbeault's inequality,

$$\int u(x)^m\,dx \le A\int \big|\nabla\big(u(x)^{m-1/2}\big)\big|^2\,dx + B\Big(\int u(x)\,dx\Big)^\nu, \tag{3.27}$$

where ν *is the exponent from* (3.25)*, and* A *and* B *are some constants (do* not *calculate those). Now optimize inequality* (3.13) *with respect to the scaling* $u(x) \mapsto \lambda^{d/m}u(\lambda x)$*. Finally, set* $w(x) = u(x)^{m-1/2}$ *and* $p = \frac{1}{2m-1}$ *to deduce* (3.26)*. Calculate the value of* θ*.*

CHAPTER 4

Introduction to the Thin Film Equation

Although entropy methods have been heavily investigated in the context of 2nd order equations, they usually provide only one of many possible approaches to solve a particular problem. In many cases, alternative tools are available, which may be based, for instance, on comparison principles. The situation changes dramatically when one moves to parabolic equations of higher order. Here, frequently entropies constitute the only (known) method to derive a priori estimates for existence proofs, calculate the large-time asymptotics etc.
This chapter is devoted to the family of one-dimensional thin film equations

$$\partial_t u = -(|u|^\beta u_{xxx})_x, \tag{4.1}$$

which frequently appear as limits of viscous fluid models. The real parameter β appearing in (4.1) may in principle take any value $\beta > 0$. The physically most relevant range lies between $\beta = 1$ (pinching of a neck in a Hele-Shaw cell) and $\beta = 3$ (viscous fluid moving with no slip); values of β between 1 and 3 correspond to non-zero slip conditions.
Some of the techniques reffered to as entropy methods today have been developed in the cause of studying solutions to (4.1) in the celebrated article [**9**] by Bernis and Friedman. (Though the notion "entropy" never appears in this work.)
The main feature of (4.1) is its degenerate structure, i.e. ellipticity of the operator on the right-hand side is lost where u vanishes. This causes difficulties in the existence theory. However, it is also the origin of an important property of solutions, namely the preservation of non-negativity. Loosely speaking, the degeneracy makes the x-axis impenetrable to solutions. In contrast, the linear equation (obtained in the limit $\beta \searrow 0$)

$$\partial_t u = -u_{xxxx} \tag{4.2}$$

does not have this property as is easily checked by considering the spatially periodic solution

$$u(t;x) = 3.1 - 4e^{-t}\cos x + 2e^{-16t}\cos 2x. \tag{4.3}$$

One has $u(0,x) \geq 0.1$ but $u(t,0) = 3.1 - 4e^{-t} + 2e^{-16t}$ becomes negative for $t \approx 0.2$.
The questions related to the phenomenon of non-negativity are still of great interest in the theory of thin-film equations. Even in one dimension, it is still not know which is the exact range of β such that *thin films do not rupture*. Here rupture means that the solution u vanishes at some point, $u(\hat{t};\hat{x}) = 0$, although it has been strictly postive at this location before, $u(\tau;\hat{x}) > 0$ for some $\tau < \hat{t}$. The by now classical argument that rupture cannot occur for $\beta \geq 4$ is given below, see Theorem 4.2. The proof can be extended to cover $\beta \geq 3.5$ by the results presented in the next lecture. However, it is widely conjectured that rupture is absent for all $\beta > 3$.

1. The initial boundary value problem

In these notes, we shall restrict ourselves mainly to solutions on the interval $\Omega = (0,1)$. The initial boundary value problem (IBVP) as formulated in [**9**] is (4.1) supplemented with boundary conditions

$$u_x = u_{xxx} = 0 \quad \text{on } \partial\Omega, \tag{4.4}$$

and an initial datum

$$u(0;x) = u_0(x), \quad u_0 \in H^1(\Omega). \tag{4.5}$$

A natural domain for solutions u is the Hölder space

$$X_T = C_{t,x}^{1/8,1/2}(\overline{\Omega_T}) \quad \text{with } \Omega_T = [0,T] \times \Omega. \tag{4.6}$$

(The Hölder norm is taken with exponent $1/2$ in x, and exponent $1/8$ in t.) The fundamental existence result reads

THEOREM 4.1. *For arbitrary $T > 0$, there exists a weak solution $u \in X_T$ satisfying the (IBVP) in the sense*

$$\iint_{\Omega_T} u\,\partial_t\phi\,dx + \iint_{\mathcal{P}} |u|^n u_{xxx}\,\phi_x\,dx = 0. \tag{4.7}$$

Here $\mathcal{P} = \overline{\Omega_T} \cap \{u \neq 0\} \cap \{t > 0\}$, and ϕ is a Lipschitz-continuous test function on $\overline{\Omega_T}$, vanishing near $t = 0$ and near $t = T$.
Moreover, the norm $\|u\|_{X_T}$ is controlled in terms of $\|u_0\|_{H^1}$, independently of $T > 0$, and in particular

$$\sup_{0<t<T} \int_\Omega u_x(t)^2\,dx + \iint_{\Omega_T} |u|^\beta u_{xxx}^2\,dx\,dt \le 2\|u_0\|_{H^1}. \tag{4.8}$$

Finally, this weak solution u is in fact a classical solution to (4.1)&(4.4) on any time interval (τ_1, τ_2) on which u is (strictly) positive in Ω.

The strategy of the proof is to work on the respective non-degenerate problem

$$\partial_t u_\epsilon = -\big((|u_\epsilon|^\beta + \epsilon)u_{\epsilon,xxx}\big)_x, \tag{4.9}$$

which possesses a local-in-time solution u_ϵ on some Ω_τ. Suitable a priori estimates for the Hölder-norm in X_τ, independent of τ, allow to extend these to solutions to X_T. By the Arzela-Ascoli theorem, for a suitable sequence $\epsilon_n \to 0$, the functions u_{ϵ_n} converge to a limit u in X_T. The a priori estimates guarantee that the integral equation (4.7) holds for u.
In the context of entropy methods, the interesting part of the proof is the derivation of (4.8). This follow since the energy

$$E[u] = \frac{1}{2}\int_\Omega u_x^2\,dx \tag{4.10}$$

is a Lyapunov functional for (4.1). At least formally, one has,

$$-\frac{d}{dt}E[u(t)] = -\int_\Omega u_x u_{xt}\,dx = -\int_\Omega u_{xx}(|u|^\beta u_{xxx})_x\,dx = \int_\Omega |u|^\beta u_{xxx}^2\,dx \ge 0.$$

Here we used the boundary conditions in (4.4). The time-integrated form of this relation,

$$\int_\Omega u_x(\tau)^2\,dx + 2\iint_{\Omega_\tau} |u|^\beta u_{xxx}^2\,dx\,dt = \int_\Omega u_{0,x}^2\,dx, \tag{4.11}$$

immediately yields (4.8). The energy estimate (4.11) holds rigorously for the regularized solutions u_ϵ to (4.9), and it carries over to the limit u as $\epsilon \to 0$. From here, the estimates in X_T follow by tedious but classical calculations.

2. Positivity of solutions

THEOREM 4.2. *Assume $\beta \ge 4$. If $u_0 > 0$, then $u > 0$ in $\overline{\Omega_T}$.*

As a consequence, the solution u is classical.

PROOF. The proof works by contradiction. Assume that u is not strictly positive in $\overline{\Omega_T}$. By continuity of u and positivity of u_0, there exists some smallest time $\tau > 0$ such that $\min_\Omega u(\tau; x) = 0$; let $\hat{x} \in \overline{\Omega}$ be a zero of $u(\tau)$. We recall that u is a classical solution on $(0,\tau) \times \Omega$, which justifies the manipulations below.
The key obervation is that, apart from the energy E given above, there exists another Lyapunov functional,

$$H[u] = \frac{1}{(\beta-1)(\beta-2)}\int_\Omega u^{2-\beta}\,dx. \tag{4.12}$$

Indeed, one finds for $0 < t < \tau$,

$$-\frac{d}{dt}H[u(t)] = -\frac{1}{1-\beta}\int_\Omega u^{1-\beta}u_t\,dx = -\int_\Omega u^{-\beta}u_x\,u^\beta u_{xxx}\,dx = \int_\Omega u_{xx}^2\,dx \geq 0.$$

Hence $H[u(t_1)] \geq H[u(t_2)]$ for $0 < t_1 \leq t_2 < \tau$. By continuity of u and positivity of u_0,

$$(\beta-1)(\beta-2)\lim_{t_1\searrow 0} H[u(t_1)] = \int_\Omega \lim_{t_1\searrow 0}\Big(u(t_1)^{2-\beta}\Big)\,dx = \int_\Omega u_0^{2-\beta}\,dx < \infty.$$

Moreover, by Fatou's Lemma,

$$(\beta-1)(\beta-2)\lim_{t_2\nearrow \tau} H[u(t_2)] \geq \int_\Omega \lim_{t_2\nearrow \tau}\Big(u(t_2)^{2-\beta}\Big)\,dx = \int_\Omega u(\tau)^{2-\beta}\,dx.$$

In combination, this implies

$$\int_\Omega u(\tau)^{2-\beta}\,dx < \infty. \tag{4.13}$$

On the other hand, since the energy $E[u(\tau)]$ is finite, the Sobolev embedding yields that $u(\tau)$ is Hölder-continuous with exponent $1/2$ on $\overline{\Omega}$. In particular, for all $x \in \Omega$,

$$0 \leq u(\tau; x) \leq K|x-\hat{x}|^{1/2}$$

with a finite constant $K > 0$. Consequently,

$$\int_\Omega u(\tau)^{2-\beta}\,dx \geq K^{2-\beta}\int_\Omega |x-\hat{x}|^{-(\beta/2-1)}\,dx.$$

But since $\beta \geq 4$, the last integral is infinite, in contradiction to (4.13). □

Theorem 4.2 is at the basis of proving that the (IBVP) possesses a non-negative weak solution for each H^1-regular non-negative initial datum. The argument for $0 < \beta < 4$ is a little intricate, see section 4 below. However, for $\beta \geq 4$, we may quite straightforwardely conclude

Corollary 4.1. *If $\beta \geq 4$ and $u_0 \geq 0$, then there exists a weak solution $u \geq 0$ to the (IBVP) in sense* (4.7).

Proof. The argument is only sketched here. One replaces the initial data by $u_{0\delta} = u_0 + \delta > 0$ and obtains positive solutions u_δ by Theorem 4.2 above. The necessary a priori estimates for the passage $\delta \to 0$ are the same as in the proof of Theorem 4.1, i.e. they follow from dissipation of E. Clearly, the uniform limit u of the positive functions u_δ is non-negative. □

3. Stationarity of the support

The entropy functional H in (4.12) has been proven extremely useful to show Ω-global positivity of solutions. In this section, we use a *localized version* of this entropy estimate to obtain a precise description of how positivity spreads in Ω if u_0 vanishes on some set. In fact, for $\beta \geq 4$, one obtains that the support of $u(t)$ is constant in time. We simply remark that in the regimes where $0 < \beta < 4$, the behavior of the support is much more complicated.

Theorem 4.3. *Assume $\beta \geq 4$. For given $u_0 \geq 0$, let $u \geq 0$ be the non-negative weak solution constructed in the proof of Corollary 4.1. Then* $\operatorname{supp}(u(t)) = \operatorname{supp}(u_0)$ *for all* $0 \leq t \leq T$.

Theorem 4.3 follows from Lemma 4.1, due to Bernis and Friedman [**9**], and 4.2, due to Beretta, Bertsch and dal Passo [**8**].

Lemma 4.1. *Under the above assumptions,* $\operatorname{supp}(u_0) \subset \operatorname{supp}(u(t))$.

Proof. For brevity, denote by $v = u_\delta > 0$ the positive, classical solution with initial data $v_0 = u_{0\delta} = u_0 + \delta$. For a non-negative, smooth function $\varphi : \Omega \to \mathbb{R}$, introduce the localized entropy

$$H_\varphi[u] = \frac{1}{(\beta-1)(\beta-2)}\int_\Omega u^{2-\beta}\,\varphi^4\,dx.$$

Fix φ such that $u_0(x) > 0$ for all $x \in \operatorname{supp}\varphi$; then

$$H_\varphi[v_0] \le H_\varphi[u_0] < \infty.$$

For any $0 < t < T$, we have

$$\begin{aligned}
\frac{d}{dt} H_\varphi[v(t)] &= \frac{1}{\beta-1} \int_\Omega v^{1-\beta} \, (v^\beta v_{xxx})_x \varphi^4 \, dx \\
&= \int_\Omega v_x v_{xxx} \varphi^4 \, dx + \frac{1}{\beta-1} \int_\Omega v v_{xxx} (\varphi^4)_x \, dx \\
&= -\int_\Omega v_{xx}^2 \varphi^4 \, dx - \frac{\beta}{\beta-1} \int_\Omega v_x v_{xx} (\varphi^4)_x \, dx - \frac{1}{\beta-1} \int_\Omega v v_{xx} (\varphi^4)_{xx} \, dx.
\end{aligned}$$

Integration by parts is justified since φ is smooth on Ω, and v is (sufficiently) smooth on the support of φ. Moreover, by smoothness of φ, one trivially has pointwise estimates on $\overline{\Omega}$,

$$|(\varphi^4)_x| \le A\varphi^3, \quad |(\varphi^4)_{xx}| \le B\varphi^2.$$

This gives on one hand

$$\int_\Omega |v_x v_{xx}| |(\varphi^4)_x| \, dx \le A \Big(\int_\Omega v_x^2 \varphi^2 \, dx \Big)^{1/2} \Big(\int_\Omega v_{xx}^2 \varphi^4 \, dx \Big)^{1/2}.$$

Note that the first integral on the right-hand side is controlled by $E[v(t)]^{1/2} \le E[u_0]^{1/2}$, hence bounded independently of t and δ. On the other hand,

$$\int_\Omega v |v_{xx}| |(\varphi^4)_{xx}| \, dx \le B \Big(\int_\Omega v^2 \, dx \Big)^{1/2} \Big(\int_\Omega v_{xx}^2 \varphi^4 \, dx \Big)^{1/2}.$$

The L^2-norm of v is bounded independently of δ and t. So, altogether,

$$\frac{d}{dt} H_\varphi[u(t)] \le C \Big(\int_\Omega u_{xx}^2 \varphi^4 \, dx \Big)^{1/2} - \int_\Omega u_{xx}^2 \varphi^4 \, dx \le \frac{1}{4} C^2,$$

where C does not depend on t or δ. By continuity of $v(t)$ at $t = 0$, it follows that

$$H_\varphi[v(t)] \le H_\varphi[v_0] + \frac{1}{4} C^2 T \le H_\varphi[u_0] + \frac{1}{4} C^2 T < \infty \quad \text{for all } t \le T. \tag{4.14}$$

Now, since the approximations $v = u_\delta$ converge uniformly in $\overline{\Omega_T}$ to the true solution u, it follows by Fatou's Lemma that also $H_\varphi[u(t)] < \infty$ for all $t \le T$.
At this point, one uses exactly the same argument as in the proof of Theorem 4.2: Assume that $u(t)$ vanishes at $\hat{x} \in \operatorname{supp}\varphi$. Due to the Hölder-regularity of $u(t)$ in x (induced by finiteness of $E[u(t)]$), the integral $H_\varphi[u(t)]$ diverges, contradicting (4.14) above. □

LEMMA 4.2. *Under the above assumptions,* $\operatorname{supp}(u(t)) \subset \operatorname{supp}(u_0)$.

PROOF. Also this argument is based on a contradiction. Assume that there is some non-empty open interval $I \subset \Omega$ on which $u_0 = 0$, but $u(t) \ge \eta > 0$ for some $t > 0$.
Denote again by v the approximating solution u_δ. Choose a non-negative, smooth $\psi : \Omega \to \mathbb{R}$, supported in I with positive integral. Then, with the usual justifications,

$$\begin{aligned}
\frac{d}{dt} \int_\Omega \psi \, \log v \, dx &= -\int_\Omega \psi \, \frac{(v^\beta v_{xxx})_x}{v} \, dx \\
&= \int_\Omega \psi_x \, v^{\beta-1} v_{xxx} \, dx - \int_\Omega \psi \, v^{\beta-2} v_x v_{xxx} \, dx.
\end{aligned}$$

Integration in time and an application of Hölder's inequality yield

$$
\begin{aligned}
-\int_\Omega \psi\,\log v_0\,dx &\le -\int_\Omega \psi\,\log v(t)\,dx \\
&\quad + \Big(\iint_{\Omega_T} \psi_x^2\, v^{\beta-2}\,dx\,dt\Big)^{1/2}\Big(\iint_{\Omega_T} v^\beta v_{xxx}^2\,dx\,dt\Big)^{1/2} \\
&\quad + \Big(\iint_{\Omega_T} \psi^2\, v^{\beta-4}\,dx\,dt\Big)^{1/2}\Big(\iint_{\Omega_T} v^\beta v_{xxx}^2\,dx\,dt\Big)^{1/2} \\
&\le -\int_\Omega \psi\,\log v(t)\,dx + C\,E[u_0]^{1/2}.
\end{aligned}
$$

In the last step we used smoothness of ψ, boundedness of v, $\beta \ge 4$, and the main energy dissipation estimate (4.11). The last expression is bounded, independently of $\delta > 0$. But this is in contradition to the fact that

$$
-\int_\Omega \psi\,\log v_0\,dx = -\log\delta \int_\Omega \psi\,dx \to +\infty
$$

as $\delta \to 0$. $\square$

4. Behavior for smaller parameters

From the physical point of view, the range $1 < \beta < 4$ is much more interesting than $\beta \ge 4$. Which of the results above survive in the lower range for β?
First of all, the condition $\beta \ge 4$ can be replaced by $\beta > 1$ in Corollary 4.1. The key idea of the ingenious proof from [**9**] is to regularize (4.1) in the following way:

$$
\partial_t u_\epsilon = -(f_\epsilon(u_\epsilon)u_{\epsilon,xxx})_x, \quad f_\epsilon(s) = \frac{|s|^{4+\beta}}{s^4 + \epsilon|s|^\beta}, \tag{4.15}
$$

and to supply it with the positive initial condition of the form

$$
u_{0\epsilon} = u_0 + \epsilon^{1/4}. \tag{4.16}
$$

The reason for the akward choice of f_ϵ is essentially that $f_\epsilon(s) \approx \epsilon^{-1}|s|^4$ for $s \approx 0$, which turns out to be sufficient to prove Theorem 4.2. Consequently, there exists a positive classical solution u_ϵ to (4.15)&(4.16). For $1 < \beta < 4$, the energy relation (4.11) leads to a priori estimates which are sufficient to conclude uniform convergence of u_{ϵ_n} along a suitable sequence ϵ_n to a weak solution u in the sense 4.7. As a limit of positive functions, u is non-negative.
Moreover, the following weakend form of Theorem 4.2 is available for $\beta > 2$.

Theorem 4.4. *If $\beta > 2$ and $\int_\Omega u_0^{2-\beta}\,dx < \infty$, then $u(t) > 0$ almost everywhere on Ω, at each time $t \le T$.*

Proof. Let $A > 0$ be much larger than the maximum of u on $\overline{\Omega_T}$, and introduce $h_\epsilon : \mathbb{R} \to \mathbb{R}$ by

$$
h_\epsilon(s) = \int_s^A \int_r^A \frac{dz\,dr}{f_\epsilon(z)},
$$

and in particular

$$
h_0(s) = \frac{1}{(\beta-1)(\beta-2)} s^{2-\beta} + \frac{A^{1-\beta}}{\beta-1}s - \frac{A^{2-\beta}}{\beta-2}. \tag{4.17}
$$

In substituion of H in (4.12),

$$
H_\epsilon[u] = \int_\Omega h_\epsilon(u)\,dx \tag{4.18}
$$

is a Lyapunov functional for (4.15). In fact, since $h_\epsilon''(s) = 1/f_\epsilon(s)$, one finds after integration by parts

$$
-\frac{d}{dt} H_\epsilon[u_\epsilon(t)] = \int_\Omega u_{\epsilon,xx}^2\,dx \ge 0.
$$

The goal is to pass to the limit $\epsilon \to 0$ in the resulting inequality

$$H_\epsilon[u_\epsilon(t)] \leq H_\epsilon[u_{0\epsilon}]. \tag{4.19}$$

The problem is that $h_\epsilon(0) \to +\infty$ as $\epsilon \to 0$. The left-hand side can be treated by Fatou's lemma (recall $h_\epsilon(s) \geq 0$ for $s \leq A$). For the right-hand side, one uses that by definition of h_ϵ,

$$h''_\epsilon(s) - h''_0(s) = \epsilon s^{-4} \quad \Longrightarrow \quad h_\epsilon(s) - h_0(s) = \epsilon(as^{-2} + bs + c),$$

and consequently, since $u_{0\epsilon} \geq \epsilon^{1/4}$,

$$0 \leq h_\epsilon(u_{0\epsilon}) - h_0(u_{0\epsilon}) \leq C\epsilon \cdot (\epsilon^{-1/4})^2 = C\epsilon^{1/2},$$

which converges to zero uniformly as $\epsilon \to 0$. Thus, estimate (4.19) holds woth $\epsilon = 0$. Thus, due to the particular form of H_0 given in (4.17),

$$\frac{1}{(\beta-1)(\beta-2)} \int_\Omega u(t)^{2-\beta}\,dx + \frac{A^{1-\beta}}{\beta-1} \int_\Omega u(t)\,dx \leq \frac{1}{(\beta-1)(\beta-2)} \int_\Omega u_0^{2-\beta}\,dx + \frac{A^{1-\beta}}{\beta-1} \int_\Omega u_0\,dx.$$

Finally, taking into account that the total mass is conserved by weak solutions to (4.1),

$$\int_\Omega u(t)^{2-\beta}\,dx \leq \int_\Omega u_0^{2-\beta}\,dx.$$

Thus, $u(t)$ cannot vanish on a set of positive measure. $\square$

With some technical effort, a similar procedure can be carried out to obtain localized entropy estimates [**9**]. These prove that the support of $u(t)$ cannot shrink as t increases.

5. Problems

This exercise sheet is entirely devoted to studying the large-time asymptotics of non-negative, strong solutions u to the *Hele-Shaw flow* (the thin film equation with $\beta = 1$),

$$\partial_t u(t;x) = -\partial_x\big(u(t;x)\,\partial_x^3 u(t;x)\big) \tag{4.20}$$

on $\mathbb{R}$. Our investigation is performed in several steps.

Problem 4.1. *Find a suitable rescaling of the form*

$$y = \lambda(t)x, \quad s = s(t), \quad u(t;x) = \lambda(t)v(s;y),$$

such that (4.20) *turns into*

$$\partial_s v(s;y) = -\partial_y\big(v(s;y)\,\partial_y^3 v(s;y)\big) + \partial_y\big(y\,v(s;y)\big). \tag{4.21}$$

Problem 4.2. *Find the explicit form of the non-negative, stationary solutions v_∞ to* (4.21)*. Also, determine the corresponding self-similar solutions for* (4.20)*. How regular are these?*
Hint: *Try with a suitable Barenblatt profile.*

Problem 4.3. *Rewrite* (4.21) *further in the Carrillo-Toscani-form*

$$\partial_s v = -a\,\partial_y^2\big(v^\alpha\,\partial_y^2 F\big) + b\,\partial_y\big(v^\beta\,\partial_y F\big), \quad \textit{with} \quad F(s;y) = c\,v(s;y)^\gamma + \frac{1}{2}y^2. \tag{4.22}$$

Calculate the values of the coefficients a, b and c, and the exponents α, β, γ.

Problem 4.4. *Prove* formally *(any integration by parts is allowed) that*

$$H[v] = \int_{\mathbb{R}} \Big(\sqrt{\frac{8}{3}}v(y)^{3/2} + \frac{1}{2}y^2\Big)\,dy \tag{4.23}$$

is an entropy for (4.21)*.*
Hint: *To prove the Lyapunov property, use equation* (4.22)*. To prove the equilibration property in $L^1(\mathbb{R})$, look up in your old notes.*

Problem 4.5. *Prove that the relative entropy H and its production D_H are related by*

$$H[v] - H[v_\infty] \le \frac{1}{2}D_H[v], \tag{4.24}$$

where v_∞ is the stationary solution with the same mass as the initial condition.
Hint: *Again, your old notes might be helpful.*

Problem 4.6. *Conclude the usual L^1-estimates for the asymptotics of v and u.*

CHAPTER 5

New Entropies for the Thin Film Equation

The goal of this lecture is to improve the positivity results for the thin film equation (4.1). Recall that the key idea for proving positivity of solutions and (non-)expansion properties of the support was a combinations of two Lyapunov functionals: the energy on one hand,

$$E[u] = \frac{1}{2}\int_\Omega u_x^2\,dx,$$

and the entropy on the other hand,

$$H[u] = \frac{1}{(\beta-1)(\beta-2)}\int_\Omega u^{-(\beta-2)}\,dx.$$

Neglecting technical details, the positivity argument for $\beta \geq 4$ in Theorem 4.2 went like this: Finiteness of $E[u(t)] \leq E[u_0]$ implies that each profile $u(t)$ is Hölder-continuous of degree $1/2$ in x. Assuming that $u(t;\hat{x}) = 0$, there is some finite constant $K > 0$ (depending only on $E[u_0]$) such that

$$0 \leq u(t;x) \leq K|x-\hat{x}|^{1/2}.$$

This, in turn, leads to (recalling $\beta \geq 4$)

$$H[u(t)] \geq K^{2-\beta}\int_\Omega |x-\hat{x}|^{-(\beta/2-1)}\,dx = +\infty,$$

which contradicts $H[u(t)] \leq H[u_0] < +\infty$. Non-shrinking of the support followed by localizing the entropy H, using a suitable cut-off function, see Lemma 4.1. Finally, the bound $H[u(t)] \leq H[u_0] < \infty$ was used to obtain positivity of each profile $u(t)$ almost everywhere in Ω in the range $2 < \beta < 4$, see Theorem 4.4.
The aim of this lecture is to extend this argument by using more general Lyapunov functionals. More precisely, we are looking for quantities of the form

$$E_p[u] = \frac{1}{2}\int_\Omega (u^{p/2})_x^2\,dx, \qquad H_\alpha[u] = \frac{1}{\alpha(\alpha-1)}\int_\Omega u^\alpha\,dx, \tag{5.1}$$

(other than just $p = 2$ and $\alpha = 2-\beta$), which are dissipated by (4.1). For $\alpha = 1$ or $\alpha = 0$, we replace the definition in (5.1) by

$$H_1[u] = \int_\Omega \big(u(\log u - 1) + 1\big)\,dx, \quad H_0[u] = \int_\Omega (u - \log u)\,dx,$$

respectively; notice that H_1 is the logarithmic entropy, up to an additive constant. The functionals H_α are strictly convex w.r.t. u for all $\alpha \in \mathbb{R}$, are non-negative for $\alpha \leq 0$ and for $\alpha \geq 1$, and non-positive for $0 < \alpha < 1$. Moreover, by Jensen's inequality, they are bounded below by the respective value of the homogeneous steady state,

$$H_\alpha[u] \geq H_\alpha[u_\infty], \quad u_\infty \equiv \frac{1}{|\Omega|}\int_\Omega u\,dx.$$

To simplify calculations in this lecture, the domain $\Omega = (-\pi, +\pi)$ is used instead of $(0,1)$; due to the homogeneity of the thin film equation (4.1), this corresponds merely to a rescaling of u and x. Moreover, we replace the boundary conditions (4.4) by periodic ones, so integration by parts does not produce boundary terms. It is a straight-forward (though annoying) exercise to verify that the results also hold under the condition (4.4).

1. Dissipated Entropies

THEOREM 5.1. *Assume $\alpha + \beta > 0$. Then the functional H_α is dissipated by (4.1) if and only if $3/2 \le \alpha + \beta \le 3$.*

In particular, $\alpha = 2 - \beta$ is a possible choice for each β. However, for the proof of positivity, the optimal value is $\alpha = 3/2 - \beta$. Using the same argument as before, one finds conservation of strict positivity for $\beta \ge 7/2$.
Before presenting the actual proof of Theorem 5.1, an outline of the argument is in place, since the idea generalizes to a variety of other situations. Differentiation of H_α along solutions to the thin film equation gives

$$\begin{aligned} D_\alpha[u(t)] &:= -\frac{d}{dt} H_\alpha[u(t)] = -\frac{1}{\alpha - 1} \int_\Omega u^{\alpha-1} \partial_t u \, dx = \frac{1}{\alpha-1} \int_\Omega u^{\alpha-1} (u^\beta u_{xxx})_x \, dx \\ &= - \int_\Omega u^{\alpha+\beta} \big(\frac{u_{xxx}}{u}\big)\big(\frac{u_x}{u}\big) \, dx. \end{aligned}$$

Integration by parts will be used to rewrite the last expression as in integral over something that is pointwise non-negative. In order to retain a range of α's as large as possible, the integration by parts should be carried out somewhat systematically. Here we follow the algebraic approach developed in [**27**]. Our paradigma is to conserve the apparent form of the integrand: it should be the product of u^γ (setting $\gamma = \alpha + \beta$), with a polynomial in the scaled derivatives u_x/u, u_{xx}/u etc. To keep calculations short, introduce the abbreviations

$$\xi_1 = u_x/u, \quad \xi_2 = u_{xx}/u, \quad \dots \tag{5.2}$$

With these notations,

$$D_\alpha[u] = - \int_\Omega u^\gamma \xi_3 \xi_1 \, dx.$$

Moreover, we adopt the following view on integration by parts:

To integrate by parts means
to add the x-derivative of some spatial periodic expression under the integral,
thus changing the form of the integrand, but not the value of the integral.

Suitable expressions to add, which do not alter the shape of the integrand, are necessarily linear combinations of

$$R_{\mathbf{m}} := u^\gamma \xi_1^{m_1} \xi_2^{m_2} \xi_3^{m_3}, \tag{5.3}$$

where $\mathbf{m}$ is a multi-index, and $m_1 + 2m_2 + 3m_3 = 3$. Their respective x-derivatives are

$$\begin{aligned} (R_{\mathbf{m}})_x = u^\gamma \Big(& m_1 \xi_1^{m_1-1} \xi_2^{m_2+1} \xi_3^{m_3} + m_2 \xi_1^{m_1} \xi_2^{m_2-1} \xi_3^{m_3+1} + m_3 \xi_1^{m_1} \xi_2^{m_2} \xi_3^{m_3-1} \xi_4 + \\ & + (\gamma - |\mathbf{m}|) \xi_1^{m_1+1} \xi_2^{m_2} \xi_3^{m_3} \Big), \end{aligned}$$

with $|\mathbf{m}| = m_1 + m_2 + m_3$. The ξ-dependent polynomial inside the brackets is referred to as *shift polynomial* in the following. In the situation at hand, there exist exactly 3 expression of type (5.3), namely

$$R_1 = u^\gamma \xi_1^3, \quad R_2 = u^\gamma \xi_2 \xi_1, \quad R_3 = u^\gamma \xi_3,$$

corresponding to the three shift polynomials

$$\begin{aligned} T_1 &= 3\xi_2 \xi_1^2 + (\gamma - 3)\xi_1^4, \\ T_2 &= \xi_3 \xi_1 + \xi_2^2 + (\gamma - 2)\xi_2 \xi_1^2, \\ T_3 &= \xi_4 + (\gamma - 1)\xi_3 \xi_1. \end{aligned}$$

The reason for introducing this formalism is a translation of analysis into algebra. The "analysis question"

Is H_α dissipated by the thin film equation?

is answered affirmatively, if the "algebra question"

Do there exist coefficients c_i such that
$S(\xi) = -\xi_1\xi_3 + c_1T_1 + c_2T_2 + c_3T_3$ *is non-negative for all ξ?*

has a positive answer.
The reader might suspect that the algebraic framework presented above provides an "overkill" to solve a relatively simple problem. However, we develop the general scheme in full generality for this easy example since it will be used in the more difficult calculations for E_p later.

LEMMA 5.1. *For α with $3/2 \leq \alpha+\beta \leq 3$, one has $D_\alpha[u] \geq 0$ for every smooth, positive $u : \Omega \to \mathbb{R}$ with periodic boundary conditions.*

PROOF. Let $\gamma = \alpha+\beta$ in the range $[3/2, 3]$ be fixed. By the preceeding considerations, it suffices to prove that there are real numbers c_1 to c_3 such that

$$\begin{aligned} 0 \leq S(\xi) &:= -\xi_1\xi_3 + c_1T_1 + c_2T_2 + c_3T_3 \\ &= c_3\xi_4 + \big(-1 + (\gamma-1)c_3 + c_2\big)\xi_3\xi_1 + c_2\xi_2^2 + \big((\gamma-2)c_2 + 3c_1\big)\xi_2\xi_1^2 + (\gamma-3)c_1\xi_1^4 \end{aligned}$$

for all $\xi = (\xi_1, \dots, \xi_4) \in \mathbb{R}^4$. The first observation is that $c_3 = 0$, since

$$\min_\xi S(\xi) \leq S(0,0,0,-c_3) = -c_3^2.$$

The second is that $c_2 = 1$, since

$$\min_\xi S(\xi) \leq \lim_{\epsilon\to 0} S\big(\epsilon, 0, \epsilon^{-1}(1-c_2), 0\big) = -(c_2-1)^2.$$

Thus only c_1 remains as a free parameter,

$$\begin{aligned} S(\xi) &= \xi_2^2 + (\gamma - 2 + 3c_1)\xi_2\xi_1^2 + (\gamma-3)c_1\xi_1^4 \\ &= \begin{pmatrix} \xi_2 \\ \xi_1^2 \end{pmatrix} \cdot \begin{pmatrix} 1 & \frac{1}{2}(\gamma-2+3c_1) \\ \frac{1}{2}(\gamma-2+3c_1) & (\gamma-3)c_1 \end{pmatrix} \cdot \begin{pmatrix} \xi_2 \\ \xi_1^2 \end{pmatrix}. \end{aligned}$$

The just defined quadratic form is non-negative iff the determinant of the matrix,

$$\Delta = (\gamma-3)c_1 - \frac{1}{4}(\gamma-2+3c_1)^2 = -\frac{1}{4}\big(9c_1^2 + 2\gamma c_1 + (\gamma-2)^2\big) \tag{5.4}$$

is non-negative. The quadratic polynomial in c_1 attains its maximal value at the point $c_1^* = -\gamma/9$, and the respective value is

$$\Delta^* = -\frac{1}{4}\big((\gamma-2)^2 - \gamma^2/9\big) = -\frac{4}{9}(2\gamma^2 - 9\gamma + 9) = -\frac{8}{9}(\gamma-3)(\gamma-3/2).$$

Hence, the maximum of Δ is non-negative if $3/2 \leq \gamma \leq 3$. □

LEMMA 5.2. *For α with either $0 < \alpha+\beta < 3/2$ or $\alpha+\beta > 3$, there exists a smooth, positive, periodic $\hat{u} : \Omega \to \mathbb{R}$ such that $D_\alpha[\hat{u}] < 0$.*

The proof presented below relies on an adaption of Laugensen's counterexample presented in [**29**]. The basic idea of the construction is suprisingly simple, while the technical details are suprisingly involved. Arguing on a purely formal level, one takes $\hat{u} = |x|^\sigma$ with $\sigma = 3/\gamma$ and puts it into a suitable representation of D_α (obtained after sufficiently many integrations by part). The result is

$$\begin{aligned} D_\alpha[\hat{u}] &= \int \hat{u}^\gamma \big(\frac{\hat{u}_{xx}}{\hat{u}}\big)\Big(\frac{\hat{u}_{xx}}{\hat{u}} + (\gamma-2)\big(\frac{\hat{u}_x}{\hat{u}}\big)^2\Big)\,dx \\ &= \int |x|^{\gamma\sigma}\sigma(\sigma-1)|x|^{-2}\Big(\sigma(\sigma-1)|x|^{-2} + (\gamma-2)\sigma^2|x|^{-2}\Big)\,dx \\ &= -2\sigma^2(1-\sigma)(2-\sigma)\int |x|^{-1}\,dx. \end{aligned} \tag{5.5}$$

By definition of σ, it is easily checked that the coefficient equals $-36\gamma^{-4}(\gamma-3)(\gamma-3/2)$, and hence is negative for $\gamma > 3$ and for $\gamma < 3/2$. On the other hand, the integral clearly diverges to $+\infty$. In summary, $D_\alpha[\hat{u}] = -\infty$ for $\alpha < 3/2 - \beta$ or $\alpha > 3 - \beta$. The aim of the following proof is to shows that there exists a smooth, positive and periodic version of $\hat{u}$.

The reader should not be mislead by the formal argument above that all power functions $|x|^\sigma$ with $\sigma \in \mathbb{R}$ constitute good *trial functions* for D_α; *the conclusion would be completely wrong!* The argument cannot be made rigorous in general for any other value than $\sigma = 3/\gamma$. In fact, it is an amusing exercise to verify that if $|x|^\sigma$ with just *a single value* $\sigma \neq 3/\gamma$ would be allowed as a trial function, then there is *always* a suitable representation of the integral in D_α such that $D_\alpha[|x|^\sigma] < 0$, thus excluding the existence of any entropy at all.

PROOF. Fix $\gamma = \alpha + \beta$ with $\gamma \notin [3/2, 3]$. Laugesen's choice for the trial function would be

$$\hat{u}(x) = (\epsilon + \sin^2 x)^{\tau/2}, \qquad \tau = \frac{3+\delta}{\gamma},$$

with positive parameters ϵ and δ to be chosen sufficiently small a posteriori. This function is smooth, positive in Ω, and satisfies periodic boundary conditions. Below, it is shown that

$$\lim_{\delta \to 0} \lim_{\epsilon \to 0} D_\alpha[\hat{u}] = -\infty,$$

with D_α as in (5.5). The assertion of the theorem follows by choosing first $\epsilon > 0$, then $\delta > 0$ small enough.
The first step is to prove

$$\lim_{\epsilon \to 0} D_\alpha[\hat{u}] = D_\alpha[|\sin x|^\tau]. \tag{5.6}$$

By elementary calculations,

$$\begin{aligned}
|\hat{u}_x| &= \tau(\epsilon + \sin^2 x)^{\tau/2-1} |\sin x \cos x| \le \tau(\epsilon + \sin^2 x)^{(\tau-1)/2}, \\
|\hat{u}_{xx}| &\le \tau(\tau-1)(\epsilon + \sin^2 x)^{\tau/2-2} \sin^2 x \cos^2 x + (\epsilon + \sin^2 x)^{\tau/2-1} |\cos^2 x - \sin^2 x| \\
&\le (1 + \tau(\tau-1))(\epsilon + \sin^2 x)^{(\tau-2)/2}.
\end{aligned}$$

This gives a pointwise estimate the expression under the integral in $D_\alpha[\hat{u}]$,

$$\hat{u}^\gamma \Big|\frac{\hat{u}_{xx}}{\hat{u}}\Big| \Big|\frac{\hat{u}_{xx}}{\hat{u}} + (\gamma-2)\Big(\frac{\hat{u}_x}{\hat{u}}\Big)^2\Big| \le C(\epsilon + \sin^2 x)^{(\gamma\tau-4)/2} \le C' |\sin x|^{\delta-1},$$

where C and C' are independent of $\epsilon > 0$. But $|\sin x|^{\delta-1}$ is integrable on Ω. By Lebesgue's dominated convergence theorem, (5.6) follows.
Concerning the subsequent limit of $\delta \to 0$, observe that

$$\begin{aligned}
D_\alpha[|\sin x|^\tau] &= \int_\Omega |\sin x|^{\gamma\tau} \Big(\tau(\tau-1)\Big(\frac{\cos x}{\sin x}\Big)^2 - \tau\Big)\Big(\tau(\tau-1)\Big(\frac{\cos x}{\sin x}\Big)^2 - \tau + (\gamma-2)\tau^2\Big(\frac{\cos x}{\sin x}\Big)^2\Big)\, dx \\
&= -\tau^2(1-\tau)(2+\delta-\tau) \int_\Omega |\sin x|^{\gamma\tau-4} \cos^4 x\, dx \\
&\quad + \tau^2(2-\gamma\tau) \int_\Omega |\sin x|^{\gamma\tau-2} \cos^2 x\, dx + \tau^2 \int_\Omega |\sin x|^{\gamma\tau}\, dx.
\end{aligned}$$

All appearing integrals are defined (as long as $\delta > 0$), since $\gamma\tau = 3 + \delta$. Moreover, since uniform convergence of the integrand implies convergence of the integral value, the last two integrals tend to finite values at $\delta \to 0$. On the other hand, for the first integral, there is some $c > 0$ such that

$$\int_\Omega |\sin x|^{\gamma\tau-4} \cos^4 x\, dx \ge c \int_{-\pi/4}^{\pi/4} |x|^{\delta-1}\, dx,$$

which converges to $+\infty$ for $\delta \to 0$. The elementary observation

$$-\tau^2(1-\tau)(2+\delta-\tau) \to -\sigma^2(1-\sigma)(2-\sigma) = -36\gamma^{-4}(\gamma-3/2)(\gamma-3) \tag{5.7}$$

as $\delta \to 0$ finishes the proof. □

2. Dissipated Energies

As the following formulas — including the final result — constitute complicated algebraic expressions, we choose to introduce a change of variables $(\alpha, p) \leftrightarrow (q, r)$ as follows:

$$p = 2 + q, \quad \beta = \frac{1}{4}(7 + q + 5r).$$

In terms of these variables, define the following elliptical regions in the q-r-plane:

$$\mathcal{E}_1 = \{P_1(q,r) < 0\}, \qquad P_1(q,r) = 7q^2 + 3r^2 - 3,$$
$$\mathcal{E}_2 = \{P_2(q,r) < 0\}, \qquad P_2(q,r) = 10q^2 + 15r^2 - 6.$$

Moreover, define the parallelogram between the four straight lines given by $\pm q \pm r = 1$, i.e.

$$\mathcal{P} = \{\ P_3(q,r) < 0\}, \qquad P_3(q,r) = (1 + q + r)(1 + q - r)(1 - q + r)(1 - q - r).$$

The arrangement of these three objects in the plane is rather special: The boundary of $\mathcal{P}$ is tangent to $\mathcal{E}_2$; and the four points of tangency coincide with the four points of intersection with $\mathcal{E}_1$. In terms of these geometric quantities, the main result reads as follows.

Theorem 5.2. *Suppose that p and β are such that $(q,r) \in \mathcal{E}_1 \cap \mathcal{P}$, or $(q,r) \in \mathcal{E}_2$. Then E_p is dissipated. If $p \neq 2$ and $P_3(q,r) > 0$, then E_p is not dissipated in general.*

This result has been obtained independently in [**29**] and [**27**]. The proof below is very close to the one from [**27**].

In the original variables, the new energies are all situated in the region $1/2 \le p \le 3$. Thus, additonal regularity estimates on the solution can be derived from them. Concerning the proof of positivity properties, however, no application of these energies is known so far. Notice that the Bernis-Friedman strategy always pairs energies with entropies; but the available entropies for $\beta \le 3$ remain bounded on film rupture, no matter how smoothly the solution touches the zero line.

The energy E_2 coincides with (4.10) and is always dissipated. Theorem 5.2 makes no statement about the behavior of E_p on the set $P_3(q,r) = 0$. The interested reader might want to extend the arguments below one step further to resolve also this.

Theorem 5.2 leaves several questions open, which are not answered until now. For instance, it is unknown what happens at points (q,r) that lie neither in the region of dissipation, nor in the region of "definite non-dissipation". Also, Theorem 5.2 does not answer the question which energies E_p remain bounded for all times if there are finite initially (even if they are not dissipated).

The strategy to prove Theorem 5.2 is exactly the same as for Theorem 5.1; only that now six spatial derivatives are involved, and the calculations become more complicated. Let us start by evaluating the energy dissipation,

$$\begin{aligned} D_p[u(t)] &:= -\frac{d}{dt}E_p[u(t)] = -\int_\Omega (u^{p/2})_x \partial_t (u^{p/2})_x \, dx \\ &= \frac{p}{2}\int_\Omega (u^{p/2})_{xx} u^{p/2-1} \partial_t u \, dx \\ &= \frac{p}{2}\int_\Omega (u^{p/2-1}(u^{p/2})_{xx})_x u^\beta u_{xxx} \, dx \\ &= \frac{p^2}{4}\int_\Omega u^{p+\beta}\Big(\xi_3 + 2q\xi_1\xi_2 + \frac{q(q-1)}{2}\xi_1^3\Big)\xi_3 \, dx, \end{aligned}$$

using the abbreviations from (5.2). This time, we set $\gamma = p + \beta$. Altogether, there are 7 possible shift polynomials, which we shall not list here. Arguing as in the proof of Lemma 5.1, most of them are irrelevant, since they contain products involving ξ_4, ξ_5 and ξ_6 in first power, while there are no higher powers to dominate these quantities. (One has to be careful, however, with such arguments, since the undesired terms might cancel when several shift polynomials are linearly combined in a clever way; in fact, this frequently happens in the multi-dimensional context.) The remaining three shift polynomials originate from

$$R_1 = u^\gamma \xi_1^5, \quad R_2 = u^\gamma \xi_2 \xi_1^3, \quad R_3 = u^\gamma \xi_2^2 \xi_1,$$

and read

$$\begin{aligned} T_1 &= 5\xi_2\xi_1^4 + (\gamma-5)\xi_1^6, \\ T_2 &= \xi_3\xi_1^3 + 3\xi_2^2\xi_1^2 + (\gamma-4)\xi_2\xi_1^4, \\ T_3 &= 2\xi_3\xi_2\xi_1 + \xi_2^3 + (\gamma-3)\xi_2^2\xi_1^2. \end{aligned}$$

Another argument shows that T_3 is not useful, since it introduces a ξ_2^3 in the game — and there is no ξ_2^4 to control it. Thus we end up with the algebraic question if there exist real numbers c_1 and c_2 such that

$$\begin{aligned} 0 \le S(\xi) &:= \Big(\xi_3 + 2q\xi_1\xi_2 + \frac{q(q-1)}{2}\Big)\xi_3 + c_1T_1 + c_2T_2 \\ &= \xi_3^2 + 2q\xi_1\xi_2\xi_3 + (\frac{q(q-1)}{2} + c_2)\xi_1^3\xi_3 + 3c_2\xi_2^2\xi_1^3 + (5c_1 + (\gamma-4)c_2)\xi_2\xi_1^4 + (\gamma-5)c_1\xi_1^6 \\ &= \begin{pmatrix} \xi_3 \\ \xi_2\xi_1 \\ \xi_1^3 \end{pmatrix} \cdot A \cdot \begin{pmatrix} \xi_3 \\ \xi_2\xi_1 \\ \xi_1^3 \end{pmatrix}. \end{aligned}$$

Here the matrix A is a linear combination, $A = A_0 + \frac{1}{2}c_1A_1 + \frac{1}{2}c_2A_2$, with

$$A_0 = \begin{pmatrix} 1 & z & \frac{z(z-1)}{4} \\ z & 0 & 0 \\ \frac{z(z-1}{2}) & 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 5 & \frac{5(q+r-1)}{2} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 6 & \frac{5q+5r-1}{4} \\ 1 & \frac{5q+5r-1}{4} & 0 \end{pmatrix}.$$

One needs to study when A defines a non-negative quadratic form. Denote by Δ the determinant of A, and by

$$\delta = 3c_2 - z^2$$

the determinant of the top-left 2×2-submatrix $\tilde{A}$. We distinguish two cases for non-negativity of A:

(1) either $\delta > 0$ and $\Delta \ge 0$,
(2) or $\delta = 0$ and A has zero as its smallest eigenvalue.

Before discussing these cases, we perform some simplifications. The complete expression for Δ is extremely large; however, it is easy to check from the form of the A_i that

$$\Delta = -\frac{25}{4}c_1^2 - \frac{3}{4}c_2^3 + \star c_2^2 + \star c_1c_2 + \star c_1 + \star c_2 + \star.$$

In particular, for each fixed c_2, this is an upside-down parabola in c_1. Moreover, the position c_1^* of the critical point of this parabola depends in an affine manner on c_2. In our investigation of the non-negativity of Δ, it suffices to restrict attention to $c_1 = c_1^*$. Define accordingly $A^* = A_0 + c_1^*A_1 + c_2A_2$, which has determinant

$$\Delta^*(c_2) = \Delta(c_1^*, c_2) = -\frac{3}{4}c_2^3 + \star c_2^2 + \star c_2 + \star.$$

We claim that Δ^* factors into δ and a quadratic polynomial $Q(c_2)$. For this, we need to show that $\Delta^*(\hat{c}_2) = 0$ for $\hat{c}_2 = z^2/3$.

On one hand, $\Delta^*(\hat{c}_2) \le 0$. Assume on the contrary that $\Delta^*(\hat{c}_2) > 0$. By continuity, $\Delta^*(\hat{c}_2+0) > 0$ and $\delta(\hat{c}_2+0) > 0$, so $A^*(c_2)$ has three genuinely positive eigenvalues for $c_2 = \hat{c}_2+0$. As $\Delta^*(\hat{c}_2) > 0$, all of them remain strictly positive on a small neighborhood of $\hat{c}_2$. But this contradicts $\delta(\hat{c}_2) = 0$.

In order to see that $A^*(\hat{c}_2) \ge 0$, it suffices to verify that $A = A_0 + c_1A_1 + \hat{c}_2A_2$ has a non-trivial kernel for a suitable choice of c_1. But the particular form of A_1 allows to extend the non-trivial kernel of the submatrix $\tilde{A}$ to a non-trivial kernel of A.

Knowing the approximate form of Δ^*, one can calculate the polynomial Q with some effort,

$$Q(c_2) = -\frac{1}{8}(2c_2^2 + (q^2 + r^2 - 1)c_2 + \frac{1}{2}q^2r^2).$$

We specialize to case one now, looking for a $c_2 > \hat{c}_2 = z^2/3$ with $\Delta^*(c_2) \geq 0$. One possibility is obviously that

$$0 < Q(\hat{c}_2) = -\frac{q^2}{144}P_2(q,r), \tag{5.8}$$

in which case c_2 simply needs to be chosen sufficiently close to $\hat{c}_2$. Another possibility is that Q increases at $c_2 = \hat{c}_2$,

$$0 < Q'(\hat{c}_2) = -\frac{4}{3}P_1(q,r),$$

towards its maximum value, which is non-negative,

$$0 \leq Q_{max} = Q\Big(-\frac{1}{4}(q^2 + r^2 - 1)\Big) = -\frac{1}{64}P_3(q,r).$$

The choice $c_2 = -(q^2 + r^2 - 1)/4$ yields $\Delta(c_2) \geq 0$. There are no further possibilities.
We turn to the second case in order to see if it provides genuinely new information. So assume that $A^*(\hat{c}_2)$ has zero as its lowest eigenvalue.
First, we argue that if Δ^* changes sign at $\hat{c}_2$ (i.e. $\hat{c}_2$ is not one of the — at most two — critical points of Δ^*), then we are back in the case (5.8) discussed above. Indeed, since there are three eigenvalues of A^*, and one is always strictly positive, it follows that exactly one eigenvalue changes sign when c_2 passes $\hat{c}_2$. The other two eigenvalues remain on the same side — the non-negative one, by our assumption on $A^*(\hat{c}_2)$. As it is impossible to have three positive eigenvalues of $A^*(\hat{c}_2 - 0)$, the sign transition of Δ^* must be from negative to positive. But this implies (5.8).
The remaining case is that Δ^* has a critical point at $\hat{c}_2$, i.e.

$$0 = Q(\hat{c}_2) = -\frac{q^2}{144}P_2(q,r).$$

This situation shall not discussed further here; it only concerns the *boundary* of the ellipse $\mathcal{E}_2$, and the (trivial) line $q = 0$.
The proof that E_p is not dissipated for $P_3 > 0$ is another application of Laugesen's construction. More a priori estimates are now necessary, but since no new ideas enter the proof, it is omitted here.

A concluding remark. There is one huge advantage of the algebraic entropy construction method, which has not been visible in this lecture. Namely, the proof of non-negativity of polynomials is a well-known task in *computational algebraic geometry.* A variety of numerical tools has been developed exactly to answer questions of the type "Do there exist parameters c such that the polynomial is non-negative for all ξ?" An algorithm which in principle can answer *any* such question (provided time and memory suffice) has been implemented e.g. in the program `Mathematica`. For instance, Theorem 5.1 can be proven using `Mathematica 5.1` in less than five minutes. Moreover, `Matlab`-users can download packages which give a numerical answer to the more restrictive question "Can parameters c be chosen such that the resulting polynomial in ξ is the *sum of squares* of other polynomials?"

3. Problems

PROBLEM 5.1. *Given a smooth initial condition* $u_0 : [-\pi, +\pi] \to \mathbb{R}$, *denote by* $u(t;x)$ *for* $t > 0$ *the corresponding — classical and unique — solution to the linear fourth order problem*

$$\partial_t u(t;x) = -\partial_x^4 u(t;x), \qquad u(0;x) = u_0(x) \tag{5.9}$$

with periodic boundary conditions. Find an u_0 *such that*

- u_0 *is positive, of unit mass, and extends to a smooth* 2π*-periodic function on* $\mathbb{R}$.
- *The solution* $u(t;x)$ *to* (5.9) *is negative at some point* $(\hat{t}, \hat{x})$ *with* $\hat{t} > 0$, $\hat{x} \in \mathbb{R}$.

Determine an upper bound $T > 0$ *on* $\hat{t}$, *solely in terms of* $E = \frac{1}{2}\int_{-\pi}^{+\pi} |\partial_x u_0(x)|^2\,dx$.
Hint: *Any smooth function* $f : [-\pi, +\pi] \to \mathbb{R}$ *of zero average with* $f(0) = f(1)$ *satisfies* $\|f\|_\infty \leq \sqrt{\pi/6}\,\|f'\|_2$. *Moreover, Poincaré's inequality for intervals might be useful.*

PROBLEM 5.2. *Consider the* Logarithmic Fourth Order *(alias* DLSS, *alias* Quantum Diffusion*...) equation*

$$\partial_t u(t;x) = -\partial_x^2\big(u(t;x)\partial_x^2 \log u(t;x)\big), \tag{5.10}$$

with periodic boundary conditions on $[-\pi, +\pi]$. *Determine all values of* $\alpha \in \mathbb{R}$ *such that the functional*

$$H_\alpha[u] := \frac{1}{\alpha(\alpha-1)}\int_{-\pi}^{+\pi} u(x)^\alpha\,dx \tag{5.11}$$

is dissipated along arbitrary positive and smooth solutions $u(t;x)$. *Moreover, for those* α, *determine the optimal value* $\mu_\alpha \geq 0$ *such that*

$$-\frac{d}{dt}H_\alpha[u(t)] \geq \mu_\alpha \int_{-\pi}^{+\pi} \big|\partial_x u(t;x)\big|^4\,dx. \tag{5.12}$$

Hint: *To prove optimality, apply Laugesen's trick.*

PROBLEM 5.3. *The formal part of Laugesen's trick consists of using* $\hat{u}(x) = |x|^\sigma$ *as a* trial function *in a suitable representation of the the entropy dissipation* $D[u]$. *Why is the respective rigorous argument* necessarily *restricted to one particular value of* σ*?*

CHAPTER 6

Decay Rates for the Thin Film Equation

This last lecture is more in the original spirit of the course. Entropy methods are used to establish various proofs for convergence of solutions to the thin film equation to the steady state. Numerous results on this topic exist, which are, however, scattered throughout the literature. Below, a selection from the results given in [**11, 19, 31, 32, 33**] is presented.

1. Relaxation for small β on bounded domains

Recall the initial boundary value problem (IBVP) from lecture 4,

$$\partial_t u = -(u^\beta u_{xxx})_x, \quad u_x(0) = u_x(1) = u_{xxx}(0) = u_{xxx}(1) = 0, \quad u(0;x) = u_0(x) > 0, \tag{6.1}$$

with slip-parameter $\beta > 0$. Once again, the spatial domain is $\Omega = (0,1)$, and for convenience, the (preserved) mass of the solution should equal one. By simple scaling arguments, all estimates on u immediately translate into respective estimates for solutions on an arbitrary finite interval and with arbitrary mass.
Theorems 6.1 and 6.2 are concerned with the convergence of the solution $u(t)$ to the homogeneous steady state,

$$u_\infty \equiv \int_\Omega u_0 \, dx = 1.$$

To be on safe grounds, positive (and hence smooth and classical) solutions u are considered only. The relaxation behavior in the general free-boundary situation is somewhat delicate. In fact, Theorem 4.3 states that for $\beta \geq 4$, the support of the solution u does not change in time. Hence, an initially compactly (inside Ω) supported solution is going to approach some *non-trivial* stationary profile in the long-time limit instead of the homogeneous state u_∞.

1.1. L^1-estimates.

Recall the definition of the entropies H_α in (5.1).

THEOREM 6.1. *Let $0 < \beta < 2$, and u be a positive solution to* (6.1). *Then u relaxes to homogeneity at an exponential rate,*

$$\|u(t) - u_\infty\|_{L^1} \leq C\big(H_{2-\beta}[u_0] - H_{2-\beta}[u_\infty]\big)^{1/2} \exp\big(-\pi^4(1-\beta/2)t \cdot\big). \tag{6.2}$$

Here C is a constant, only depending on β.

With some effort, using the results from Lecture 5, it is possible to extend the β-range in Theorem 6.1 to $0 < \beta < 3$, see [**11**], and — with even more effort — to all $\beta > 0$, see [**31**]. Also, the rates defined in (6.2) are far from optimal in general. Concerning the optimization of exponential rates, refer to [**15**], for instance. However, the crucial observation is that relaxation in $L^1(\Omega)$ indeed happens on a global exponential time scale. This is different from the $H^1(\Omega)$-relaxation treated in Theorem 6.2.

PROOF. Recall from Lecture 4 that the entropy

$$H[u] = H_{2-\beta} = \int_\Omega \frac{u^{2-\beta}}{(1-\beta)(2-\beta)} \, dx$$

satisfies for classical solutions u to (6.1)

$$-\frac{d}{dt} H[u(t)] = \int_\Omega u_{xx}^2 \, dx \geq C_P \int_\Omega u_x^2 \geq C_P^2 \Big(\int_\Omega u^2 \, dx - 1 \Big), \tag{6.3}$$

where $C_P = \pi^2$ is the Poincaré constant for Ω. From the estimate on u_x it is immediate that $u(t)$ converges to the homogeneous profile $u_\infty \equiv 1$ in $H^1(\Omega)$ as $t \to \infty$, implying that $H[u(t)] \to H[u_\infty]$. Moreover, by elementary considerations,

$$\frac{1}{1-\beta} s^{2-\beta} - \frac{2-\beta}{1-\beta} s + 1 \le (s-1)^2 \tag{6.4}$$

holds for all $s \ge 0$, provided $0 < \beta < 2$ with $\beta \neq 1$. Integrating (6.4) with $s = u(t)$ on Ω, taking into account that the mass of $u(t)$ is one, yields

$$(2-\beta)\big(H[u(t)] - H[u_\infty]\big) \le \int_\Omega u(t)^2\, dx - 1.$$

A respective estimate is readily verified also for $\beta = 1$. In combination with (6.3), we obtain

$$H[u(t)] - H[u_\infty] \le \exp\big(-\pi^2(2-\beta)t\big)\big(H[u_0] - H[u_\infty]\big)$$

by the usual Gronwall argument. Finally, an application of the Csiszar-Kullback inequality from Proposition 1.1 gives the desired decay estimate (6.2). $\square$

1.2. H^1-estimates. The setup is the same as before.

THEOREM 6.2. *Let $0 < \beta \le 2$, and u a positive solution to* (6.1). *Then u relaxes to homogeneity in $H^1(\Omega)$ at algebraic rate,*

$$\|u(t) - u_\infty\|_{H^1} \le (A + Bt)^{-1/4}. \tag{6.5}$$

Here both A and B depend on u_0 and β.

PROOF. Instead of the entropy H, the energy

$$E[u] = \frac{1}{2}\int_\Omega u_x^2\, dx$$

is now used to obtain the relaxation estimate. Recall that, for positive solutions u,

$$D_E[u(t)] = -\frac{d}{dt}E[u(t)] = \int_\Omega u^\beta u_{xxx}^2\, dx.$$

Following [**33**], the energy $E[u]$ is estimated in terms of a power of its own dissipation $D_E[u]$. Using the homogeneous Neumann boundary conditions and unity of mass,

$$\begin{aligned}\int_\Omega u_x^2\, dx = -\int_\Omega u u_{xx}\, dx &\le \Big(\int_\Omega u u_{xx}^2\, dx\Big)^{1/2} \\ &= \Big(\int_\Omega u u_x u_{xxx}\, dx\Big)^{1/2} \\ &= (\sup_\Omega u^{2-\beta})^{1/4}\Big(\int_\Omega u_x^2\, dx\Big)^{1/4}\Big(\int_\Omega u^\beta u_{xxx}^2\, dx\Big)^{1/4}.\end{aligned}$$

As Ω is one-dimensional, the supremum of u can be estimated in terms of $E[u]$,

$$\sup_\Omega u \le 1 + \int_\Omega |u_x|\, dx \le C(u_0) := 1 + \big(2E[u_0]\big)^{1/2}.$$

Thus,

$$D_E[u(t)] \ge 8C(u_0)^{2-\beta}E[u(t)]^3.$$

By comparison with the ODE $\dot{y} = -Cy^3$, one concludes

$$E[u(t)] \le (E[u_0]^{-2} + 16C(u_0)^{2-\beta}t)^{-1/2}.$$

Next, the Poincaré inequality yields at once that

$$\int_\Omega \big(u(t) - 1\big)^2\, dx \le 2C_P E[u(t)],$$

and thus the $H^1(\Omega)$-estimate in (6.5). $\square$

Two comments are in place here. The first is that Theorem 6.1 can be generalized to multiple dimensions, whereas Theorem 6.2 (seemingly) cannot. The bottleneck in the proof above is the continuous embedding of $H^1(\Omega)$ into $L^\infty(\Omega)$, which does no longer exist in two (or more) dimensions. In fact, there is no guarantee that in multiple dimensions, arbitrary solutions remain bounded in L^∞.
The second comment is that Theorem 6.2 is interesting only in the initial phase of the relaxation. Once the solution u is L^∞-close to the steady state u_∞, the convergence is dictated by the linearization of the thin film equation, and thus is exponentially fast. More precisely, by the H^1-estimate in (6.5), there exists a time $T > 0$ such that $\|u(t) - 1\|_\infty \leq 1/2$ for all $t \geq T$. And then,

$$-\frac{d}{dt}E[u(t)] = \int_\Omega u^\beta u_{xxx}^2 \, dx \geq (1/2)^\beta \int u_{xxx}^2 \, dx \geq 2^{1-\beta} C_P^2 E[u(t)].$$

This is sufficient to conclude *exponential* relaxation of $u(t)$ towards u_∞ in H^1 and L^1. However, in contrast to Theorem 6.1, one does not obtain a global and universal exponential rate in H^1, since the time T depends on the initial condition u_0 and could be arbitrarily large. An argument of this type has seemingly first been given in [**13**], and is at the basis of the considerations in [**15**].

2. Relaxation of the rescaled Hele-Shaw flow

The thin film equation with parameter $\beta = 1$,

$$\partial_t u = -(u u_{xxx})_x \tag{6.6}$$

plays a distinct role. First of all, it is the only thin film equation that constitutes a Wasserstein gradient flow. The corresppnding potential coincides with the energy,

$$E[u] = \frac{1}{2} \int u_x^2 \, dx. \tag{6.7}$$

Moreover, it further is the only thin-film equation for which the self-similar solution on $\mathbb{R}$,

$$u_s(t; x) = t^{-1/5} V(t^{-1/5} x),$$

is explicitly known: it is the *Smyth-Hill-profile*, given by

$$V(y) = \frac{1}{48}(\mu^2 - y^2)_+^2, \tag{6.8}$$

where $\mu > 0$ is a mass parameter. In the following, the intermediate asymptotics of equation (6.6) are investigated, i.e. the convergence of its solutions to to self-similarity. In analogy to the linear and non-linear Fokker Planck equations in Lectures 2 and 3, a scaling is performed such that the self-similar profile V becomes a stationary solution,

$$\partial_s v = -(v v_{yyy})_y + (yv)_y.$$

A little more general, we shall be concerned with

$$\partial_s v = -2(v v_{yyy})_y + \lambda(yv)_y, \tag{6.9}$$

where $\lambda > 0$ determines the strength of the confinement potential, and the factor two is introduced for notational convenience later. The respective steady state becomes

$$v_\infty(y) = \frac{\lambda}{48}(\mu^2 - y^2)_+^2.$$

Notice that the profile v_∞ is of Barenblatt type (3.3), where $m = 3/2$. However, the dependence on the strength λ of the quadratic confinement potential is different.

2.1. L^1-estimates. The following result, taken from [**19**], establishes an intimate connection between the theory for porous medium equations and the one for thin film equations, and is probably one of the most surprising results in this field.

THEOREM 6.3. *The functional*

$$H[v] = \int \left(\frac{4}{\sqrt{3}} v^{3/2} + \frac{\lambda}{2} y^2 v\right) dy \tag{6.10}$$

is an entropy for the rescaled Hele-Shaw flow (6.9). *In fact, H converges exponentially at rate 2λ, and consequently*

$$\|v(t) - v_\infty\|_{L^1} \le K e^{-\lambda t}, \tag{6.11}$$

where K only depends on $H[v_0]$.

PROOF. There is a somewhat ingenious way to prove decay of (6.10). Namely, one rewrites (6.9) in the following form:

$$\partial_s v = -\frac{2}{\sqrt{3}} \big(v^{3/2} F_{yy}\big)_{yy} + \big(v F_y\big)_y, \quad F = (2\sqrt{3} v^{1/2} + \frac{\lambda}{2} y^2).$$

From here, the following easy calculation reveals the dissipation property:

$$\begin{aligned} D_H[v(s)] &= -\frac{d}{ds} H[v(s)] \\ &= -\int_{\mathbb{R}} (2\sqrt{3} v^{1/2} + \frac{1}{2} y^2)\, \partial_s v \, dy \\ &= +\frac{2}{\sqrt{3}} \int_{\mathbb{R}} F \big(v^{3/2} F_{yy}\big)_{yy} \, dy - \int_{\mathbb{R}} F \big(v F_y\big)_y \, dy \\ &= +\frac{2}{\sqrt{3}} \int_{\mathbb{R}} v^{3/2} F_{yy}^2 \, dy + \int_{\mathbb{R}} v F_y^2 \, dy. \end{aligned}$$

The term with the second order derivatives is now neglected. This seems a very rough estimate, but due to the particular form of the entropy production, one is still able to conclude exponential convergence of H. To this end, we recall the functional inequality (3.6) relating entropy and entropy production for the porous medium equation. Choosing exponent $m = 3/2$, dimension $d = 1$, and $\lambda = 1$ *there* (the λ in Lecture 3 and the λ in (6.9) are a priori independent of each other), (3.6) reads

$$\int_{\mathbb{R}} \big(2u^{3/2} + \frac{1}{2} x^2 u\big)\, dx - \int_{\mathbb{R}} \big(2u_\infty^{3/2} + \frac{1}{2} x^2 u_\infty\big)\, dx \le \frac{1}{2} \int_{\mathbb{R}} u \big|3(u^{1/2})_x + x\big|^2 \, dx,$$

where u_∞ denotes the respective Barenblatt profile (3.3),

$$u_\infty(x) = \frac{1}{36} (\sigma - x^2)^2,$$

with some $\sigma > 0$ determining the mass. By substituting

$$x = \sqrt{\lambda} y, \quad u(x) = \frac{4}{3} v(y), \tag{6.12}$$

and trivial manipulations, this estimate turns exactly into

$$H[v] - H[v_\infty] \le \frac{1}{2\lambda} D_H[v]. \tag{6.13}$$

This obviously implies exponential convergence of H at rate 2λ. The L^1-estimate in (6.11) follows by the Csiszar-Kullback inequality proven in Lecture 3. □

COROLLARY 6.1. *Let some non-negative initial condition $u_0 \in L^1(\mathbb{R}) \cap H^1(\mathbb{R})$ with vanishing first moment, finite second moment, and finite entropy be given. Then there exists a strong solution to the Hele-Shaw flow* (6.6)*, which satisfies*

$$\|u(x,t) - t^{-1/5} V(t^{-1/5}\|_{L^1} \le C \sqrt{H[u_0] - H[V]} (5t+1)^{-1/5}. \tag{6.14}$$

Here V is the Smyth-Hill-profile (6.8) *with the same mass as u_0.*

It is most remarkable that the proof neglects the seemingly best piece of the entropy dissipation, namely the term containing the second order derivatives, and still obtains an exponential decay estimate in L^1. Naturally, the rate in (6.14) is (presumably) rather suboptimal. A linearization of (6.6) around v_∞ on a bounded domain yields [**10**] that the dominant eigenvalue should be $\lambda = -15$, corresponding to an approximation in the intermediate asymptotics of $\approx t^{-3/2}$.

2.2. H^1-estimates. The rescaled equation (6.9) is still a gradient flow in Wasserstein metrics, with potential

$$E[v] = \int (v_y^2 + \frac{\lambda}{2} y^2 v)\,dy.$$

It is natural to investigate the rate of equilibration of $E[v(t)]$. In fact, convergence of $v(t)$ "in energy" implies equilibration of $v(t)$ in $H^1(\mathbb{R})$.

LEMMA 6.1. *Provided $v \geq 0$ has finite energy and the same mass as v_∞,*

$$\big\|(v - v_\infty)_y\big\|_{L^2}^2 \leq E[v] - E[v_\infty].$$

PROOF. The proof heavily relies on the special shape of v_∞. For reference below, note that

$$v_{\infty,y}(y) = -\frac{\lambda}{12}(\mu^2 - y^2)_+ y, \quad v_{\infty,yy}(y) = \frac{\lambda}{4}\big(y^2 - \frac{1}{3}\mu^2\big)\mathbf{1}_{|y|<m}.$$

Moreover, by straightforward integation,

$$\int_{\mathbb{R}} v\,dy = \int_{\mathbb{R}} v_\infty\,dy = \frac{\lambda}{48}\int_{-\mu}^{+\mu} \big(\mu^2 - y^2\big)^2\,dy = \frac{1}{45}\lambda\mu^5,$$

and similarly,

$$\int_{\mathbb{R}} y^2 v_\infty\,dy = \frac{\lambda}{48}\int_{-\mu}^{+\mu} y^2\big(\mu^2 - y^2\big)^2\,dy = \frac{1}{315}\lambda\mu^7, \quad \int_{\mathbb{R}} v_{\infty,y}^2\,dy = \frac{\lambda}{3}\int_{\mathbb{R}} y^2 v_\infty\,dy = \frac{1}{945}\lambda^2\mu^7.$$

Putting this together, one obtains

$$\begin{aligned}
\int_{\mathbb{R}} (v - v_\infty)_y^2\,dy &= \int_{\mathbb{R}} v_y^2\,dy + \int_{\mathbb{R}} v_{\infty,y}^2\,dy - 2\int_{\mathbb{R}} v_y v_{\infty,y}\,dy \\
&= E[v] - \frac{\lambda}{2}\int_{\mathbb{R}} y^2 v\,dy + \int_{\mathbb{R}} v_{\infty,y}^2\,dy + 2\int_{\mathbb{R}} v v_{\infty,yy}\,dy \\
&= E[v] - \frac{\lambda}{2}\int_{|y|>\mu} y^2 v\,dy - E[v_\infty] + \underbrace{2\int_{\mathbb{R}} v_{\infty,y}^2\,dy + \frac{\lambda}{2}\int_{\mathbb{R}} y^2 v_\infty\,dy}_{=\lambda^2\mu^7/270} + \frac{\lambda\mu^2}{6}\int_{|y|<\mu} v\,dy \\
&= E[v] - E[v_\infty] - \frac{\lambda}{2}\int_{|y|>\mu} y^2 v\,dy + \frac{\lambda}{2}\int_{|y|>\mu} \frac{1}{3}\mu^2 v\,dy \\
&\leq E[v] - E[v_\infty],
\end{aligned}$$

since obviously $\mu^2/3 < y^2$ on the set $\{|y| > \mu\}$. $\square$

There is an interesting approach [**14**] to proving $E[v(t)] \to E[v_\infty]$, which is based on the *equipartition* property of E. Partition the energy into its *kinetic* and its *potential* contribution,

$$E[v] = E_k[v] + E_p[v], \quad \text{with} \quad E_k[v] = \int_{\mathbb{R}} v_y^2\,dy, \quad E_p[v] = \frac{\lambda}{2}\int_{\mathbb{R}} y^2 v\,dy.$$

A technical, but direct argument (using the famous *Nash trick*) provides exponential convergence of $E_k[v(t)]$ to $E_k[v_\infty]$. The hard part is to estimate the distance of $E_p[v(t)]$ to $E_p[v_\infty]$. Here, the idea of equilibration enters as follows. In the steady state, $E_k[v_\infty] : E_p[v_\infty] = 2:3$, independently of $\lambda > 0$ and $\mu > 0$. Introduce the deviation from the perfect partition,

$$\Delta[v] = 3E_k[v] - 2E_p[v].$$

A variety of calculations allows to estimate $|\Delta[v]|$ in terms of the square root of the energy dissipation $D_E = -dE/dt$. Knowing that

$$\int_{t_1}^{t_2} \Delta[v(t)]^2\, dt \le C(E[v(t_1)] - E[v(t_2)]),$$

and monotonicity of $E[v(t)]$, one concludes

$$|\Delta[v(t)]| \le Ct^{-1/2}.$$

In view of the exponential convergence of $E_k[v(t)]$, one thus finds equilibration of $E[v(t)]$ at the algebraic rate $t^{-1/2}$.

This result, however, is suboptimal and can be improved as follows [**32**].

THEOREM 6.4. *For solutions v to* (6.9)*, the energy $E[v]$ converges exponentially at rate 2λ. Moreover, v itself equilibrates exponentially fast in $H^1(\mathbb{R})$,*

$$\|v(t) - v_\infty\|_{H^1} \le Ce^{-\lambda t}, \tag{6.15}$$

where C only depends on $E[v_0]$.

PROOF. We shall actually only provide a pretty formal argument that yields an exponential decay rate of $10\lambda/9$ instead of 2λ. Deeper investigations of the Wasserstein nature of (6.9) are necessary to conclude the (presumably optimal) rate.

Also here, the essential idea is to rewrite equation (6.9) in a clever way. But first, let us introduce the following functional,

$$G[v] = \frac{4}{3}\int_{\mathbb{R}} v^{3/2}\, dy + \frac{\Lambda}{2}\int_{\mathbb{R}} y^2 v\, dy.$$

and an associated equation of porous medium type,

$$\partial_\tau v = (v(2v^{1/2} + \frac{\Lambda}{2}y^2)_y)_y = \frac{2}{3}(v^{3/2})_{yy} + \Lambda(yv)_y. \tag{6.16}$$

Here $\Lambda > 0$ is such that $3\Lambda^2 = \lambda$. The dissipation of $G[v]$ along the flow of (6.16) amounts to

$$\begin{aligned} D_G[v] &= \int_{\mathbb{R}} v(2v^{1/2} + \frac{\Lambda}{2}y^2)_y^2\, dy \\ &= \int_{\mathbb{R}} v_y^2\, dy + \Lambda^2 \int_{\mathbb{R}} y^2 v\, dy + 2\Lambda \int_{\mathbb{R}} yv^{1/2}v_y\, dy \\ &= \int_{\mathbb{R}} v_y^2\, dy + \Lambda^2 \int_{\mathbb{R}} y^2 v\, dy - \frac{4\Lambda}{3}\int_{\mathbb{R}} v^{3/2}\, dy. \end{aligned}$$

Finally, introduce the following equation related to D_G,

$$\begin{aligned} \partial_\sigma v &= (v(-2v_{yy} + \Lambda^2 y^2 - 2\Lambda v^{1/2})_y)_y \\ &= -2(vv_{yyy})_y + 2\Lambda^2(yv)_y - \frac{2}{3}(v^{3/2})_{yy}. \end{aligned}$$

The key observation is that (6.9) can be restated as follows:

$$\partial_s v = \partial_\sigma v + \Lambda\partial_\tau v,$$

while the energy takes the form

$$E[v] = D_G[v] + \Lambda G[v].$$

A combination of these items provides an estimate the dissipation of E. Indeed,

$$D_E[v] = -\partial_\sigma D_G[v] - \Lambda\partial_\tau D_G[v] - \Lambda\partial_\sigma G[v] - \Lambda^2\partial_\tau G[v].$$

The four terms on the right hand side are treated separately. For the first, one has,

$$\begin{aligned} -\partial_\sigma D_G[v] &= \int_{\mathbb{R}} (2v_{yy} - \Lambda^2 y^2 + 2\Lambda v^{1/2})\partial_\sigma v\, dy \\ &= \int_{\mathbb{R}} v(2v_{yy} - \Lambda^2 y^2 + 2\Lambda v^{1/2})_y^2\, dy \ge 0. \end{aligned}$$

The second term constitutes the second τ-derivative of G. Using similar considerations that produced inequality (6.13) from (3.6), a variation of the estimate (3.8) gives

$$D_G[v] \leq \frac{1}{2\Lambda}(-\partial_\tau D_G[v]). \tag{6.17}$$

Omitting straightforward calculations, one finds that the third term verifies

$$\partial_\sigma G[v] = \partial_\tau D_G[v],$$

so estimate (6.17) applies again. Finally, by definition, $-\partial_\tau G = D_G$. Altogether, this yields

$$D_E[v] \geq 0 + 2\Lambda^2 D_G[v] + 2\Lambda^2 D_G[v] + \Lambda^2 D_G[v] = \frac{5}{3}\lambda D_G[v]. \tag{6.18}$$

To finish the argument, notice that inequality (3.6) can be restated as

$$G[v] - G[v_\infty] \leq \frac{1}{2\Lambda} D_G[v].$$

Thus,

$$E[v] - E[v_\infty] = D_G[v] + \Lambda(G[v] - G[v_\infty]) \leq \frac{3}{2} D_G[v].$$

Substituting this into (6.18) yields

$$D_E[v] \geq \frac{10\lambda}{3}(E[v] - E[v_\infty]),$$

which implies exponential convergence of the energy $E[v(t)]$. The H^1-estimate in (6.15) is a consequence of Lemma 6.1 in combination with the exponential convergence in L^1 from (6.11). $\square$

3. Problems

PROBLEM 6.1. *Prove the* logarithmic Sobolev inequality

$$\int_0^1 f(x)\log f(x)\,dx - \Big(\int_0^1 f(x)\,dx\Big)\log\Big(\int_0^1 f(x)\,dx\Big) \le C_1 \int_0^1 \big|\partial_x\sqrt{f(x)}\big|^2\,dx \tag{6.19}$$

for positive and smooth functions f on the interval $[0,1]$, which satisfy homogeneous Neumann boundary conditions. Calculate the optimal value of the constant C_1.
Hint: *There are dozens of ways to prove* (6.19)*. My favourite one is to consider the entropy dissipation along solutions $f(t)$ of the heat equation on $[0,1]$ with $f(0)=f$.*

PROBLEM 6.2. *By deep results from measure-capacity theory* [**22**]*, the logarithmic Sobolev inequality* (6.19) *implies the L^q-*logarithmic Sobolev inequality

$$\int_0^1 f(x)\log f(x)\,dx - \Big(\int_0^1 f(x)\,dx\Big)\log\Big(\int_0^1 f(x)\,dx\Big) \le C_q\Big(\int_0^1 \big|\partial_x\sqrt{f(x)^q}\big|^2\,dx\Big)^{1/q} \tag{6.20}$$

for all $q>1$. Prove that (6.20) *implies further the* Beckner interpolation inequalities,

$$\int_0^1 f(x)\,dx - \Big(\int_0^1 f(x)^{1/p}\Big)^p \le K_{p,q}\Big(\int_0^1 \big|\partial_x\sqrt{f(x)^q}\big|^2\,dx\Big)^{1/q} \tag{6.21}$$

for all $p>1$. Express $K_{p,q}$ in terms of C_q.
Hint: *First, prove convexity of*

$$F(p) := \Big(\int_0^1 f(x)^{1/p}\Big)^p$$

for $p>0$, with fixed f. Then perform a Taylor expansion of $F(p)$ around $p=1$.

PROBLEM 6.3. *Use* (6.21) *to describe the convergence behavior of the functionals*

$$E_\alpha[u] = \frac{1}{\alpha(\alpha-1)}\int_0^1 u(x)^\alpha\,dx$$

for $\alpha>1$ along solutions $u(t;x)$ to the porous medium equation,

$$\partial_t u(t;x) = \partial_x^2\big(u(t;x)^m\big),\quad u_x(t;0)=u_x(t;1)=0,\quad u(0;x)=u_0(x)>0, \tag{6.22}$$

where $m>1$. Calculate the algebraic convergence rates in dependence of m and α.
Remark: *Existence, uniqueness, smoothness and positivity of solutions to* (6.22) *are granted.*

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