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A UNIFIED THEORY FOR SOME NON-NEWTONIAN FLUIDS UNDER SINGULAR FORCING*

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Abstract. We consider a model of steady, incompressible non-Newtonian flow with neglected convective term under external forcing. Our structural assumptions allow for certain nondegenerate power-law or Carreau-type fluids. Within our setting, we provide the full-range theory, namely, existence, optimal regularity, and uniqueness of solutions, not only with respect to forcing belonging to Lebesgue spaces, but also with respect to their refinements, namely, the weighted Lebesgue spaces, with weights in a respective Muckenhoupt class. The analytical highlight is derivation of existence and uniqueness theory for forcing with its regularity well below the natural duality exponent, via estimates in weighted spaces. It is a generalization of [M. Bulíček, L. Diening, and S. Schwarzacher, Anal. PDE, 9 (2016), pp. 1115–1151] to incompressible fluids. Moreover, two technical results, needed for our analysis, may be useful for further studies. They are the solenoidal, weighted, biting div-curl lemma and the solenoidal Lipschitz approximations on domains.

Key words. non-Newtonian fluids, existence of very weak solutions, existence theory beyond duality pairing, optimal regularity, uniqueness of solutions, solenoidal, weighted, biting div-curl lemma

AMS subject classifications. 35Q35, 35D99, 35J57, 35J60

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1. Introduction.

In a bounded domain $\Omega \subset \mathbb{R}^n$ with a C^1 boundary, we consider the following stationary nonlinear Stokes system:

$$\begin{aligned} -\operatorname{div} \mathcal{S}(x, \varepsilon(v)) + \nabla\pi &= -\operatorname{div} f && \text{in } \Omega,\\ \operatorname{div} v &= 0 && \text{in } \Omega,\\ v &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $v : \Omega \to \mathbb{R}^n$ describes the unknown velocity of the fluid, $\pi : \Omega \to \mathbb{R}$ describes the unknown pressure, and $f : \Omega \to \mathbb{R}^{n\times n}$ is the given forcing. The nonlinear stress tensor is a prescribed, matrix-valued mapping $\mathcal{S} : \Omega \times \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$. We use the notation $\varepsilon(v) = \frac{1}{2}(\nabla v + \nabla^T v)$.

We will introduce a setting which allows us for $f \in L^q(\Omega)$ with *any* $q \in (1,\infty)$ to provide the full-range theory related to (1.1), namely, existence, regularity, and uniqueness of its solutions (hence the eponymous ‘unified theory’) in an arbitrary space dimension.

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Succinctly, it will suffice that $\mathcal{S}$ is monotone and linear-at-infinity (i.e., uniformly in x, $\mathcal{S}(x,\eta) \to \mu\eta$ as $\eta \to \infty$ for some $\mu > 0$) and $\mathcal{S}(x,\eta)\cdot\eta$ has quadratic growth. Typically, the precise restrictions related to uniqueness will be actually slightly stronger. For detailed assumptions, we refer to section 1.2.

Observe that for $\mathcal{S}(x,\eta)\cdot\eta$ growing quadratically, in case $f \notin L^2(\Omega)$, the operator $f \mapsto \varepsilon(v)$ related to (1.1) is no longer coupled via duality. In simpler words, v cannot be expected to remain an admissible test function. Therefore, the standard monotone operator theory fails. This is the analytic reason for calling such f a rough forcing and the related solutions a very weak solution. Providing the "unified theory" for rough forcings is the focal point of our article.

Within our assumptions, system (1.1) models a steady flow of such incompressible non-Newtonian fluids with neglected inertial forces (no convective term) that behave asymptotically Newtonian for large shear rates. This includes famous models of incompressible non-Newtonian fluids, such as (nondegenerate) power-law fluids as well as Carreau-type fluids. For instance, we allow for $\mathcal{S}(x,\eta) = s(x,|\eta|)\eta$ with

$$\begin{aligned} &s(x,|\eta|) = \mu + (\nu_0 + \nu_1|\eta|^2)^{\frac{p-2}{2}} && \text{for } p \in (1,2] \text{ and } \mu > 0,\ \nu_0,\nu_1 \geq 0, \\ &s(x,|\eta|) = \min\{\mu, (\nu_0 + \nu_1|\eta|^2)^{\frac{p-2}{2}}\} && \text{for } p \in (2,\infty] \text{ and } \mu > 0,\ \nu_0,\nu_1 \geq 0. \end{aligned} \tag{1.2}$$

An important example among the substances described via stresses as above is blood, paint, or ketchup. For a discussion of the physical model see Málek, Rajagopal, and Růžička [28] and Málek and Rajagopal [29].

The analysis for such fluids was initiated by Ladyzhenskaya [25, 26] and Lions [27]. In case of partial differential systems inspired by non-Newtonian flows, as our (1.1), there is no general local $C^{1,\alpha}$ smoothness result of the homogeneous problem, since the system depends merely on the symmetric part of the gradient.[1] This distinguishes the non-Newtonian models from, unless similar, nonlinear partial differential systems with a p-Laplace structure. As far as we know, the best available regularity results for steady non-Newtonian models are higher regularity related to testing, roughly speaking, with Δu and the related partial regularity; see, for instance, Bildhauer and Fuchs [3] and Breit and Fuchs [10]. In the case of quadratic growths, Fuchs and Seregin were able to prove boundedness of gradients; see [23, 24]. However, nonlinear Calderón–Zygmund theory for non-Newtonian flows is generally not provided for $f \in L^q(\Omega)$ with large q's, compare Diening and Kaplický [16], even in the case of quadratic growths. Therefore, the regularity theory for (1.1) with high-integrable forcings is also interesting for us.

1.1. Context and main novelties.

First, let us recall the case of the classical steady Stokes system with a rough forcing

$$\begin{aligned} -\Delta v + \nabla\pi &= \operatorname{div} f && \text{in } \Omega, \\ \operatorname{div} u &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{1.3}$$

[1]In case of dependence on the full gradient, i.e., for p-Laplace type equations, a special structure of the system was revealed that allows providing local $C^{1,\alpha}$ smoothness; see, for instance, the seminal work by Uhlenbeck [45]. This structure is sometimes referred to as *the Uhlenbeck structure*. Interestingly, for $p > 2$ this result was announced earlier by Uraltseva, compare Remark 2, p. 221 in [46].

The existence of a solution (v, p) to (1.3), as well as its uniqueness and optimal regularity

$$f \in L^q(\Omega) \implies \nabla v \in L^q(\Omega),\ \pi \in \mathring{L}^q(\Omega), \tag{1.4}$$

for $q \in (1, \infty)$ is classical. (The circle above L^q denotes null mean values and disambiguates the pressure.) The first such result is due to Cattabriga [14], where the case of three space dimensions and smooth, bounded domain is considered. For further results (all space dimensions and more general domains), we refer to Borchers and Miyakawa [6, section 3] and [7] as well as Solonnikov [38] with their references.

Equations and systems with a more complex structure do not allow us to build such a unified theory as (1.4) with $q \in (1, \infty)$. Recall that even a linear, elliptic, homogeneous equation can have a nonsmooth solution v such that $\nabla v \notin L^2$, as long as its bounded coefficients are nonsmooth; see Serrin [37]. This, compared with the fact that a linear, homogeneous equation with bounded coefficients admits a smooth solution v as long as $\nabla v \in L^2$, indicates that the case of $\nabla v \in L^q$, $q < 2$, is peculiarly interesting.

If the studied problem becomes nonlinear and vectorial, even smooth coefficients and smooth forcing do not ensure existence of smooth solutions; recall Šverák and Yan [39] with its references. In fact, the existence or regularity theory is available only for special cases, where the nonlinearity has an appropriate structure. Its canonical examples are monotonicity for the existence theory and the Uhlenbeck structure for regularity. It is important to observe that, up to now, both of them are insufficient to obtain existence (all the more — optimal regularity, even if the notion of optimality is clear) of solutions to problems with rough forcing, i.e., of the type $\operatorname{div} f$ with integrability of f substantially below the duality exponent dictated by the energy estimate.

In this paper we develop, under suitable assumptions on the nonlinear shear stress $\mathcal{S}$, the *unified theory* for (1.1), as follows:

(i) Existence of its solutions, for forcing within the entire integrability range $q \in (1, \infty)$, including the difficult case of q's below the duality exponent (equal 2 within our structure).

(ii) Optimal regularity estimates and uniqueness of solutions.

The existence part and its methodology is the main novelty here. Our results generalize the ones of Bulíček, Diening, and Schwarzacher [11] to incompressible steady flows with no inertial forces.

We find at least two of our technical results, needed to accomplish the main goal, to be interesting by themselves. These are the solenoidal, weighted, biting div-curl lemma, potentially useful for identification of limits of nonlinearities appearing in mathematical fluid dynamics, as well as our version of the solenoidal Lipschitz approximation lemma.

1.2. Main result. For a tensor $Q \in \mathbb{R}^{n \times n}$, its symmetrization is denoted by $Q^s = \frac{Q+Q^T}{2}$. We provide existence of a solution to (1.1), with $f \in L^q(\Omega)$ for all $q \in (1, \infty)$, with the related optimal regularity estimate, under the following assumptions.

Assumption 1.1. Let $\mathcal{S}(\cdot, \cdot) : \Omega \times \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$ be a Carathéodory mapping such that for positive numbers c_0, c_1, c_2, μ it holds that

$$c_0 |Q^s|^2 - c_2 \le \mathcal{S}(x, Q^s) \cdot Q, \qquad |\mathcal{S}(x, Q^s)| \le c_1 |Q| + c_2,$$

$$0 \le (\mathcal{S}(x, Q^s) - \mathcal{S}(x, P^s)) \cdot (Q - P),$$

as well as it is *linear-at-infinity*, i.e.,

$$\lim_{|Q^s|\to\infty} \frac{|\mathcal{S}(x,Q^s) - \mu Q^s|}{|Q^s|} = 0 \tag{1.5}$$

for all $Q, P \in \mathbb{R}^{n\times n}$ and uniformly in x.

The obtained solution is unique among distributional solutions, in case one additionally has the following.

Assumption 1.2. Tensor $\mathcal{S}$ verifies

$$0 < (\mathcal{S}(x,Q^s) - \mathcal{S}(x,P^s)) \cdot (Q-P)$$

and

$$\lim_{|Q^s|\to\infty} \left| \frac{\partial \mathcal{S}(x,Q^s)}{\partial Q^s} - \mu \mathrm{Id} \right| = 0 \tag{1.6}$$

for all $Q^s \neq P^s \in \mathbb{R}^{n\times n}$ and uniformly in x.

Remark 1.3 (admissible stress tensors). The canonical stress tensors admissible by Assumption 1.1 are

$$\mathcal{S}(x,\eta) = s(x,|\eta|)\eta, \quad \text{with } \ 0 \le s(x,\lambda) \le C, \quad \lim_{\lambda\to\infty} s(x,\lambda) = \mu, \tag{1.7}$$

as long as they are monotonous. Both Assumptions 1.1 and 1.2 are satisfied by the introductory example (1.2).

We are ready to state our main results. The definitions of notions used in their formulations (most of them standard) can be found in section 2.2.

THEOREM 1.4. *Let $\mathcal{S}$ satisfy Assumption* 1.1 *and $\partial\Omega \in C^1$. If $f \in L^q(\Omega)$ with $1 < q < \infty$, then* (1.1) *admits a weak solution $(v,\pi) \in W^{1,q}_0(\Omega) \times \mathring{L}^q(\Omega)$.*

Moreover, for any $(v,\pi) \in W^{1,s}_0(\Omega) \times \mathring{L}^s(\Omega)$ with an $s > 1$, solving (1.1)*, the following estimate holds:*

$$\|\nabla v\|_{L^q(\Omega)} + \|\pi\|_{L^q(\Omega)} \le C\left(1 + \|f\|_{L^q(\Omega)}\right). \tag{1.8}$$

The constant depends on q, the C^1-property of Ω, and the quantities in Assumption 1.1.

If Assumption 1.2 *is additionally fulfilled, then (v,π) solving* (1.1) *is unique in $W^{1,q}_0(\Omega) \times \mathring{L}^q(\Omega)$.*

Theorem 1.4 as stated is complete, since it gives at once existence, optimal integrability, and uniqueness. However, for $q < 2$, the L^q-a-priori information is not enough to develop an existence theory. To this end, we need to derive more accurate estimates, namely in weighted Lebesgue spaces. An additional benefit of this technique is that one immediately obtains the following generalization of Theorem 1.4 over the weighted Lebesgue spaces with the Muckenhoupt weight A_q. Let us notice that due to nonlinearity of our problem, there necessarily appears an additive constant in (1.8).

THEOREM 1.5. *Let $\mathcal{S}$ satisfy Assumption* 1.1 *and $\partial\Omega \in C^1$. If $f \in L^q_\omega(\Omega)$ with $1 < q < \infty$ and $\omega \in A_q$, then* (1.1) *admits a solution $(v,\pi) \in W^{1,q}_{0,\omega}(\Omega) \times \mathring{L}^q_\omega(\Omega)$.*

Moreover, for any $(v,\pi) \in W^{1,s}_{0,\tilde\omega}(\Omega) \times \mathring{L}^s_{\tilde\omega}(\Omega)$ solving (1.1)*, with an $s > 1$ and $\tilde\omega \in A_s$, the following estimate holds:*

$$\|\nabla v\|_{L^q_\omega(\Omega)} + \|\pi\|_{L^q_\omega(\Omega)} \le C\left(1 + \|f\|_{L^q_\omega(\Omega)}\right). \tag{1.9}$$

The constant depends on q, A_q, the C^1-property of Ω, and the quantities in Assumption 1.1.

If Assumption 1.2 *is additionally fulfilled, then* (v,π) *solving* (1.1) *is unique in* $W^{1,q}_{0,\omega}(\Omega) \times \mathring{L}^q_\omega(\Omega)$.

Let us remark that Theorem 1.5 is optimal with respect to weighted spaces, since the Laplace operator is continuous in weighted Lebesgue spaces L^q_ω, as long as $\omega \in A_q$, $q \in (1,\infty)$; see, for instance, Sawyer [35, Theorem A]. Observe that our Theorem 1.5 covers the entire range $q \in (1,\infty)$.

Let us present a short heuristics, explaining why weighted estimates are essential for an existence theory in the case of rough data. Namely, by the choice of a proper weight, the estimate (1.9) (utilized for a regularised problem) implies that both $\varepsilon(v)$ and $\mathcal{S}(\cdot,\varepsilon(v))$ are in a weighted L^2_ω space (uniformly in a regularization). This fact establishes a duality relation between $\varepsilon(v)$ and $\mathcal{S}(\cdot,\varepsilon(v))$ which is unavailable in case of rough data within the standard Lebesgue spaces. Exploited correctly, this duality will eventually allow us to adapt a very weak version of the Minty trick.

Remark 1.6 (measure-valued forcings are included). Since forcing of (1.1) is in a divergence form, we indeed cover cases of a very general forcing, for instance, bounded Radon measures. Indeed, for a vector-valued bounded Radon measure μ, let us solve $-\operatorname{div}\nabla h = \mu$. Hence $\nabla h \in L^r$ with any $r \in [1, \frac{n}{n-1})$, so $\nabla h = f$ is within the scope of Theorems 1.4 and 1.5.

For the sake of completeness and to demonstrate the generality of our approach, let us finally present the respective result for systems with inhomogeneous boundary conditions and prescribed compressibility d. Namely, let us consider

$$\begin{aligned} -\operatorname{div}\mathcal{S}(x,\varepsilon(v)) + \nabla\pi &= -\operatorname{div} f && \text{in } \Omega,\\ \operatorname{div} v &= d && \text{in } \Omega,\\ \gamma(v) &= g && \text{on } \partial\Omega, \end{aligned} \tag{1.10}$$

where γ is the trace operator. In the result below, $T^q_\omega(\Omega)$ denotes the weighted trace space; see section 2.2. The following holds.

COROLLARY 1.7. *Let $f, d \in L^q_\omega(\Omega)$, $g \in T^q_\omega(\Omega)$ with $1 < q < \infty$, $\omega \in A_q$, and let $\mathcal{S}$ satisfy Assumption* 1.1. *Then* (1.1) *admits a solution $(v,\pi) \in W^{1,q}_\omega(\Omega) \times \mathring{L}^q_\omega(\Omega)$. Moreover, if any solution of* (1.10) *for an $s > 1$ enjoys $(v,\pi) \in W^{1,s}(\Omega) \times \mathring{L}^s(\Omega)$, $\gamma(v) = g$, then it satisfies*

$$\|\nabla v\|_{L^q_\omega(\Omega)} + \|\pi\|_{L^q_\omega(\Omega)} \le C(1 + \|f\|_{L^q_\omega(\Omega)} + \|d\|_{L^q_\omega(\Omega)} + \|g\|_{\hat{T}^q_\omega(\Omega)}). \tag{1.11}$$

The constant C depends on q, A_q, the C^1-property of Ω, and the quantities in Assumption 1.1.

If Assumption 1.2 *additionally holds, then* (v,π) *solving* (1.1) *is unique in* $(\gamma^{-1}(g) + W^{1,q}_0(\Omega)) \times \mathring{L}^q_\omega(\Omega)$.

Finally, let us state the following remark.

Remark 1.8 (a slight relaxation of assumptions). In Assumptions 1.1 and 1.2 the linearity-at-infinity can be relaxed. Indeed, in place of (1.5) it suffices to require that for an $\varepsilon_0(c_0,c_1,c_2)$ and a $m_0 > 0$ it holds that $\frac{|\mathcal{S}(x,Q^s) - \mu Q^s|}{|Q^s|} \le \varepsilon_0$ for all $|Q| \ge m_0$. Analogously (1.6) can be replaced by $\left|\frac{\partial \mathcal{S}(x,Q^s)}{\partial Q^s} - \mu \mathrm{Id}\right| \le \varepsilon_0$ for all $|Q| \ge m_0$.

1.3. Main technical results. Let us gather in this section two technical results that we would like to highlight as potentially useful in mathematical fluid dynamics. First is the solenoidal, weighted, biting div-curl lemma, which is a solenoidal version of Theorem 2.6 of [11], itself being a far generalization of the original Murat–Tartar result; see [31, 32, 41, 42].

THEOREM 1.9 (solenoidal, weighted, biting div–curl lemma). *Let $\Omega \subset \mathbb{R}^n$ denote an open, bounded set. Assume that for a given $q \in (1,\infty)$ and $\omega \in A_q$, there is a sequence of measurable, tensor-valued functions $a^k, s^k : \Omega \to \mathbb{R}^{n\times n}$, $k \in \mathbb{N}$, such that k-uniformly*

$$\|a^k\|_{L^q_\omega(\Omega)} + \|s^k\|_{L^{q'}_\omega(\Omega)} \le C. \tag{1.12}$$

Furthermore, assume that for every bounded sequence $\{c^k\}_{k=1}^\infty$ in $W^{1,\infty}_0(\Omega)$ and for every bounded solenoidal sequence $\{d^k\}_{k=1}^\infty$ in $W^{1,\infty}_{0,\mathrm{div}}(\Omega)$ such that

$$\nabla c^k \rightharpoonup^* 0 \qquad \text{weakly}^* \text{ in } L^\infty(\Omega), \qquad \nabla d^k \rightharpoonup^* 0 \qquad \text{weakly}^* \text{ in } L^\infty(\Omega)$$

one has

$$\lim_{k\to\infty} \int_\Omega s^k \cdot \nabla d^k \,\mathrm{d}x = 0, \tag{1.13}$$

$$\lim_{k\to\infty} \int_\Omega a^k_i \partial_{x_j} c^k - a^k_j \partial_{x_i} c^k \,\mathrm{d}x = 0 \qquad\qquad \text{for all } i,j = 1,\dots,n \tag{1.14}$$

and that

$$tr(a^k) \quad \text{converges pointwisely almost everywhere in } \Omega. \tag{1.15}$$

Then, there exists a (nonrelabeled) subsequence (a^k, b^k) and a nondecreasing sequence of measurable subsets $\Omega_j \subset \Omega$, with $|\Omega \setminus \Omega_j| \to 0$ as $j \to \infty$, such that

$$a^k \rightharpoonup a \qquad\qquad \text{weakly in } L^1(\Omega), \tag{1.16}$$

$$s^k \rightharpoonup s \qquad\qquad \text{weakly in } L^1(\Omega), \tag{1.17}$$

$$a^k \cdot s^k \omega \rightharpoonup a \cdot s\,\omega \qquad\qquad \text{weakly in } L^1(\Omega_j) \quad \text{for all } j \in \mathbb{N}. \tag{1.18}$$

The proof of Theorem 1.9, presented in section 4, relies among others on the following fine-tuning of the solenoidal Lipschitz truncations.

THEOREM 1.10 (solenoidal Lipschitz approximations on domains). *Let $\Omega \subset \mathbb{R}^n$ and $s > 1$. Let $g \in W^{1,s}_{0,\mathrm{div}}(\Omega)$. Then for any $\lambda > 1$ there exists a* solenoidal Lipschitz truncation $g^\lambda \in W^{1,\infty}_{\mathrm{div}}(\Omega)$ *such that*

$$g^\lambda = g \quad \text{and} \quad \nabla g^\lambda = \nabla g \qquad\qquad \text{in } \{M(\nabla g) \le \lambda\} \cap \Omega, \tag{1.19}$$

$$|\nabla g^\lambda| \le |\nabla g| \chi_{\{M(\nabla g)\le\lambda\}} + C\,\lambda \chi_{\{M(\nabla g)>\lambda\}} \qquad \text{almost everywhere.} \tag{1.20}$$

Further, if $\nabla g \in L^p_\omega(\Omega)$ for some $1 \le p < \infty$ and $\omega \in A_p$, then

$$\begin{aligned} \int_\Omega |\nabla g^\lambda|^p \omega \,\mathrm{d}x &\le C \int_\Omega |\nabla g|^p \omega \,\mathrm{d}x, \\ \int_\Omega |\nabla (g - g^\lambda)|^p \omega \,\mathrm{d}x &\le C \int_{\Omega\cap\{M(\nabla g)>\lambda\}} |\nabla g|^p \omega \,\mathrm{d}x, \end{aligned} \tag{1.21}$$

where the constant C depends on $(A_p(\Omega), \Omega, N, p)$.

The proof of Theorem 1.10 can be found in section 4.

1.4. Further research. Let us point out the significance of our results for future research, particularly the flexibility of the developed existence scheme.

First, consider the full Navier–Stokes analogue of (1.1). It involves an additional convective term. However, in three dimensions, it is possible to treat it as a right-hand side with respect to a priori estimates and as a compact perturbation with respect to the existence analysis. This will be presented in our future work. For results on existence of solutions to steady non-Newtonian Navier–Stokes flows with nonrough forcing, see Diening, Málek, and Steinhauer [18] and Bulíček et al. [12].

Another generalization is related to considering degeneracies, for instance, the degenerate power-law model $\mathcal{S}(x,Q) = \nu|Q|^{p-2}Q$. Recently, it has become possible to establish an existence theory for the related p-Laplace system; see Bulíček and Schwarzacher [13]. Even though it holds only for exponents q being close to the natural exponent p, it is the first existence proof for degenerate systems below the duality exponent. A generalization to degenerate fluids seems achievable. It would also match the regularity theory available for the degenerate Stokes systems; compare Diening and Kaplický [16] and Diening, Kaplický, and Schwarzacher [17].

Finally, we wish to emphasize that the very weak weighed duality relation discovered here has a considerable potential for numerical schemes and their analysis.

2. Preliminaries.

2.1. Structure of the paper.

This section gathers certain auxiliary tools for the proofs. Section 3 presents an a priori type estimate: in Theorem 3.1, we provide a quantitative regularity estimate (1.8), under an additional assumption that the solution of (1.1) belongs to a certain $L^s(\Omega)$ regularity class, $s > 1$. This result relies on a regularity theory for weighted linear Stokes that we partially needed to provide in this paper as well. Section 4 contains proofs of the main technical results, namely, of Theorems 1.9 and 1.10. Finally, section 5 provides proofs of our main theorems, presented in section 1.2.

2.2. Basic notation and definitions.

2.2.1. Function spaces. For $p \in [1,\infty)$ and ω being a weight, i.e., a measurable function that is almost everywhere finite and positive, let us define the weighted Lebesgue space $L^p_\omega(\Omega)$ and its norm $\|\cdot\|_{L^p_\omega}$ as

$$L^p_\omega(\Omega) := \left\{ f : \Omega \to \mathbb{R}^n;\ \text{measurable},\ \|f\|_{L^p_\omega} := \left(\int_\Omega |u(x)|^p \omega(x)\,\mathrm{d}x \right)^{\frac{1}{p}} < \infty \right\}.$$

The space $\mathring{L}^p_\omega(\Omega)$ contains all functions $f \in L^p_\omega(\Omega)$ with $\int_\Omega f\,dx = 0$.

The weighted Sobolev space $W^{1,p}_\omega(\Omega)$ consists of all functions where both the distributional derivative ∇f and f are in $L^p_\omega(\Omega)$.

The homogeneous Sobolev space $\hat{W}^{1,p}_\omega(\Omega)$ is the space of all functions such that $\nabla f \in L^p_\omega(\Omega)$ (and f belongs to the natural embedded space; $\hat{W}^{1,p}_\omega(\Omega) \neq W^{1,p}_\omega(\Omega)$ only in unbounded domains).

Since weights may have a certain impact on the exact shape of trace space, one typically defines it only semiexplicitly as $\gamma(W^{1,p}_\omega(\Omega) \cap W^{1,1}(\Omega))$, where $\gamma : W^{1,1}(\Omega) \to L^1(\partial\Omega)$ is the canonical trace operator. In case of an unbounded domain, one additionally localizes the domain by an intersection with a ball. For more details, compare Fröhlich [21, section 3.3] and [22] with their references. The zero trace subspaces of $W^{1,p}_\omega(\Omega)$ and $\hat{W}^{1,p}_\omega(\Omega)$ are denoted by $W^{1,p}_{0,\omega}(\Omega)$ and $\hat{W}^{1,p}_{0,\omega}(\Omega)$, respectively. For brevity, we will write $T^q_\omega(U)$ for $\gamma(W^{1,p}_\omega(\Omega) \cap W^{1,1}(\Omega))$ and $\hat{T}^q_\omega(U)$ for $\gamma(\hat{W}^{1,p}_\omega(\Omega) \cap W^{1,1}_{\mathrm{loc}}(\overline{\Omega}))$.

All the mentioned spaces are Banach spaces. In the case considered here, namely, the case of Muckenhoupt weights $\omega \in A_p$ and $p \in (1,\infty)$, the above defined spaces are additionally reflexive and separable. These and more properties are discussed in Stein [40, Chapter 3], for instance. Moreover, by (2.8) below, we find in the case of Muckenhoupt weights $\omega \in A_p$ that $W^{1,p}_\omega(\Omega) \subset W^{1,1}(\Omega)$ and $\hat{W}^{1,p}_\omega(\Omega) \subset W^{1,1}_{\mathrm{loc}}(\overline{\Omega})$; hence functions that are bounded in $W^{1,p}_\omega(\Omega)$, $\hat{W}^{1,p}_\omega(\Omega)$ possess weak derivatives and well-defined traces.

Finally, $W^{1,q}_{0,\mathrm{div}\,,\omega}(\Omega)$ is defined as the closure of $C^\infty_{0,\mathrm{div}}(\Omega)$ (the smooth, compactly supported, and solenoidal functions) with respect to the $W^{1,q}_\omega$-norm.

For any vector- or tensor-valued $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ we define its Hardy–Littlewood maximal function Mf in a standard manner as follows:

$$Mf(x) := \sup_{R>0} \fint_{B_R(x)} |f(y)|\,\mathrm{d}y,$$

where $B_R(x)$ denotes a ball with radius R centered at $x \in \mathbb{R}^n$.

2.2.2. A notion of solution. Let us introduce the standard definition.

DEFINITION 2.1 (distributional solution). *A couple* $(v,\pi) \in W^{1,1}_{0,\mathrm{div}}(\Omega) \times L^1(\Omega)$ *is a distributional solution to* (1.1) *iff for any* $\varphi \in C^\infty_0(\Omega)$ *it holds that*

$$\int_\Omega \mathcal{S}(x,\varepsilon(v))\nabla\varphi - \pi \operatorname{div}\varphi = \int_\Omega f\nabla\varphi.$$

An analogous definition, with natural modifications, will be used for the inhomogeneous problem.

In the following, we will sometimes call (v,π) a weak solution, provided it belongs to the optimal regularity class (with respect to regularity of f).

2.3. An algebraic lemma. Let us begin with an algebraic lemma, which can be found as Lemma 4.1 in Bulíček, Diening, and Schwarzacher [11].

LEMMA 2.2. *Let* $\mathcal{S}$ *fulfill Assumptions* 1.1 and 1.2. *Then for every* $\delta > 0$ *there exists* C *such that for all* $x \in \Omega$ *and all* $Q, P \in \mathbb{R}^{n\times n}$ *there holds*

$$|\mathcal{S}(x,Q) - \mathcal{S}(x,P) - \mu(Q-P)| \le \delta|Q-P| + C(\delta). \tag{2.1}$$

2.4. Muckenhoupt weights. To provide optimal regularity and to mimic the L^2 duality, we resort to L^2_ω with a weight ω from the Muckenhoupt class.

DEFINITION 2.3. *For* $p \in [1,\infty)$, *we say that a weight* ω *belongs to the* Muckenhoupt class A_p *iff there exists a positive constant* A *such that for every ball* $B \subset \mathbb{R}^k$ *it holds that*

$$\left(\fint_B \omega\,\mathrm{d}x\right)\left(\fint_B \omega^{-(p'-1)}\,\mathrm{d}x\right)^{\frac{1}{p'-1}} \le A \qquad \text{if } p \in (1,\infty), \tag{2.2}$$

$$M\omega(x) \le A\,\omega(x) \qquad \text{if } p = 1. \tag{2.3}$$

We denote by $A_p(\omega)$ *the smallest constant* A *for which the inequality* (2.2), *respectively,* (2.3), *holds.*

2.4.1. Basic properties. For $1 \leq p \leq q < \infty$ it holds that $A_p \subset A_q$. The maximum $\omega_1 \vee \omega_2$ and minimum $\omega_1 \wedge \omega_2$ of two A_p-weights is again an A_p-weight. For $p = 2$, since $\frac{1}{\omega_1 \wedge \omega_2} \leq \frac{1}{\omega_1} + \frac{1}{\omega_2}$ almost everywhere, we have straightforwardly

$$\fint_B (\omega_1 \wedge \omega_2)\,\mathrm{d}x \fint_B \frac{1}{\omega_1 \wedge \omega_2}\,\mathrm{d}x \leq A_2(\omega_1) + A_2(\omega_2). \tag{2.4}$$

For $\omega \in A_q$, $q \in (1, \infty)$ we will write $\omega' = \omega^{-\frac{1}{q-1}}$. Hölder inequality gives $\omega \in A_q \iff \omega' \in A_{q'}$.

2.4.2. Relation to the maximal function. Due to the celebrated result of Muckenhoupt [30], we know that $\omega \in A_p$ for $1 < p < \infty$ is equivalent to the existence of a constant A', such that for all $f \in L^p_\omega(\mathbb{R}^n)$

$$\int |Mf|^p \omega\,\mathrm{d}x \leq A' \int |f|^p \omega\,\mathrm{d}x. \tag{2.5}$$

Another link between the maximal function and A_p-weights is given by the next lemma.

Lemma 2.4. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ be such that $Mf < \infty$ almost everywhere in $\mathbb{R}^n$. Then for all $\alpha \in (0, 1)$ we have $(Mf)^\alpha \in A_1$. Furthermore, for all $p \in (1, \infty)$ and all $\alpha \in (0, 1)$ there holds $(Mf)^{-\alpha(p-1)} \in A_p$.*

For a proof, see pp. 229–230 in Torchinsky [43] and p. 5 in Turesson [44]. Lemma 2.4 implies that

$$g \in L^s(\Omega) \text{ for an } s \in (1, 2) \implies g \in L^2_{\omega_1}(\Omega) \text{ with } \omega_1 = (Mg)^{s-2} \in A_2, \tag{2.6}$$

because[2]

$$\int g^2 (Mg)^{s-2}\,\mathrm{d}x \leq \int g^s\,\mathrm{d}x \leq \int (Mg)^2 (Mg)^{s-2}\,\mathrm{d}x \leq A' \int g^2 (Mg)^{s-2}\,\mathrm{d}x. \tag{2.7}$$

Finally, we will also need that for every $p \in (1, \infty)$ and $\omega \in A_p$, there exists an $s \in (1, \infty)$ depending only on $A_p(\omega)$, such that $L^p_\omega(\Omega) \hookrightarrow L^s_{\mathrm{loc}}(\Omega)$. Moreover, the related inequality

$$\left(\fint_B |f|^s\,\mathrm{d}x\right)^{\frac{1}{s}} \leq C(A_p(\omega)) \left(\fint_B \omega\,\mathrm{d}x\right)^{\frac{1}{p}} \left(\int_B |f|^p \omega\,\mathrm{d}x\right)^{\frac{1}{p}} \tag{2.8}$$

holds. See formula (3.5) from [11].

2.4.3. A miracle of extrapolation. The seminal work by Rubío de Francia [34] implies that if a linear operator is bounded between $L^{p_0}_\omega$ for a $p_0 \in (1, \infty)$ and every $\omega \in A_{p_0}$, then it is also bounded for L^p_ω for every $p \in (1, \infty)$ and every $\omega \in A_p$. We will refer to this fact as a "miracle of extrapolation"; compare Theorem 1.4 of the monograph [15] by Cruz-Uribe, Martell, and Pérez and its references.

2.5. Very weak compactness. Since we were unable to locate an exact reference, we provide proof of the following very weak compactness result.

Lemma 2.5. *For $\omega \in A_q$ it holds that $L^q_\omega \hookrightarrow (W^{1,q'}_{\omega',0})^*$, with the embedding being (sequentially) compact.*

[2] Here and in what follows, when we deal with maximal function and a function defined on a domain, we extend the function over the full space by 0.

Proof. Let us pick a uniformly bounded sequence $g_j \in L^q_\omega(\Omega)$, $\|g_j\|_{L^q_\omega(\Omega)} \le c$. Since $L^q_\omega(\Omega)$ is reflexive, the weak compactness implies that on a subsequence $g_j \rightharpoonup g$. By subtracting the limit, we may assume with no loss of generality that $g \equiv 0$. By the dual norm definition, we find $\psi_i \in W^{1,q'}_{\omega',0}(\Omega)$, such that $\|\psi_j\|_{W^{1,q'}_{\omega',0}(\Omega)} = 1$ and $\|g_j\|_{(W^{1,q'}_{\omega',0}(\Omega))^*} \le 2\langle g_j, \psi_j\rangle$. Moreover, we find by Theorem 2.3 in [22] a convergent (nonrelabeled) subsequence $\psi_j \to \psi$, in $L^{q'}_{\omega'}(\Omega)$. This implies, by the weak-strong coupling, that $\|g_j\|_{(W^{1,q'}_{\omega',0}(\Omega))^*} \le 2\langle g_j, \psi_j\rangle \to 0$ on a subsequence, which is the (sequential) compactness of our embedding. $\square$

2.6. Convergence tools. In order to identify the limit correctly, we will use the following biting lemma.

LEMMA 2.6. *Let Ω be a bounded domain in $\mathbb{R}^n$ and let $\{v^k\}_{n=1}^\infty$ be a bounded sequence in $L^1(\Omega)$. Then there exists a nondecreasing sequence of measurable subsets $\Omega_j \subset \Omega$ with $|\Omega \setminus \Omega_j| \to 0$ as $j \to \infty$ such that for every $j \in \mathbb{N}$ and every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $A \subset \Omega_j$ with $|A| \le \delta$ and all $n \in \mathbb{N}$ the following holds:*

$$\int_A |v^k|\,\mathrm{d}x \le \varepsilon. \tag{2.9}$$

The Chacon's biting lemma from Ball and Murat [2] has as its the thesis weak-L^1 precompactness, which implies the thesis of Lemma 2.6 in view of the Dunford–Pettis theorem.

3. Regularity estimate. The main result of this section is Theorem 3.1 below. It shows that any distributional solution (v, π) to (1.1) enjoys an optimal regularity estimate, provided additionally $\nabla v, \pi \in L^s(\Omega)$, for an $s > 1$. The relation between q, ω, the right-hand side f, and s will become clear in the next section.

THEOREM 3.1. *Let Ω be a bounded domain with $\partial\Omega \in \mathcal{C}^1$ and let $\mathcal{S}$ satisfy Assumption* 1.1. *Let $f \in L^q_\omega(\Omega)$ with $1 < q < \infty$, $\omega \in A_q$, $s \in (1, \infty)$, and $\mathcal{S}$ satisfy Assumption* 1.1. *Then any distributional solution of* (1.1) *that enjoys additionally $(v, \pi) \in W^{1,s}_0(\Omega) \times L^s(\Omega)$ satisfies*

$$\|\nabla v\|_{L^q_\omega(\Omega)} + \|\pi - \langle\pi\rangle\|_{L^q_\omega(\Omega)} \le C\left(1 + \|f\|_{L^q_\omega(\Omega)}\right). \tag{3.1}$$

Analogously for the inhomogeneous case, if $d \in L^q_\omega(\Omega)$ and $g \in T^q_\omega(\Omega)$, then any distributional solution of (1.10) *that enjoys additionally $(v, \pi) \in W^{1,s}(\Omega) \times L^s(\Omega), \gamma(u) = g$, satisfies*

$$\|\nabla v\|_{L^q_\omega(\Omega)} + \|\pi - \langle\pi\rangle\|_{L^q_\omega(\Omega)} \le C\left(1 + \|f\|_{L^q_\omega(\Omega)} + \|d\|_{L^q_\omega(\Omega)} + \|g\|_{\hat{T}^q_\omega(\Omega)}\right), \tag{3.2}$$

where all constants depend only on Assumption 1.1, *$A_q(\omega), q$, and on the modulus of continuity of $\partial\Omega$.*

The proof of Theorem 3.1 occupies the end of this section. As the main ingredient of its proof, we need the following.

3.1. L^q_ω-theory for linear Stokes.

LEMMA 3.2. *Let Ω be a bounded domain with $\partial\Omega \in \mathcal{C}^1$ and let $(w, p) \in W^{1,q}_\omega(\Omega) \times L^q_\omega(\Omega)$ be a distributional solution to*

$$\begin{aligned} -\operatorname{div}(\varepsilon(w)) + \nabla p &= -\operatorname{div} F && \text{in } \Omega,\\ \operatorname{div} w &= d && \text{in } \Omega,\\ \gamma(w) &= g && \text{on } \partial\Omega. \end{aligned} \tag{3.3}$$

Then for any $F, d \in L^q_\omega(\Omega)$ *and* $g \in T^q_\omega(\Omega)$ *with* $q \in (1,\infty)$ *and* $\omega \in A_q$

$$\|w\|_{W^{1,q}_\omega(\Omega)} + \|p - \langle p\rangle\|_{L^q_\omega(\Omega)} \le C\left(\|F\|_{L^q_\omega(\Omega)} + \|d\|_{L^q_\omega(\Omega)} + \|g\|_{\hat{T}^q_\omega(\Omega)}\right), \tag{3.4}$$

where $C = C(q, A_q(\omega), \partial\Omega)$.

We could not find the exact reference concerning the case of a bounded domain, so we provide the proof.

Proof. Let U be either the full space $\mathbb{R}^n$ or the half-space $\mathbb{R}^n_+$. Recall that by $\hat{W}^{1,q}_\omega(U)$ we denote the homogeneous Sobolev space. In view of Theorems 5.1 and 5.2 by Fröhlich [21] (cases of $\mathbb{R}^n$ and $\mathbb{R}^n_+$, respectively), for every $f \in (\hat{W}^{1,q'}_{0,\omega'}(U))^*$, $d \in L^q_\omega(U)$, and $g \in \hat{T}^q_\omega(U)$, the problem

$$\begin{aligned} -\mathrm{div}\,(\varepsilon(w)) + \nabla p &= f && \text{in } U,\\ -\mathrm{div}\, w &= d && \text{in } U\\ (\gamma(w) &= g && \text{on } \mathbb{R}^{n-1} \quad \text{in case of U } = \mathbb{R}^n_+) \end{aligned} \tag{3.5}$$

admits a unique weak solution $(w,p) \in \hat{W}^{1,q}_\omega(U) \times L^q_\omega(U)$ that enjoys the estimate

$$\|\nabla w\|_{L^q_\omega(U)} + \|p\|_{L^q_\omega(U)} \le C\left(\|f\|_{(\hat{W}^{1,q'}_{0,\omega'}(U))^*} + \|d\|_{L^q_\omega(U)} + \|g\|_{\hat{T}^q_\omega(U)}\right), \tag{3.6}$$

where the term involving g naturally appears only for the half-space, $C = C(q, A_q, \Omega)$, $q \in (1,\infty)$, and $\omega \in A_q$. In particular, for $f = -\mathrm{div}\, F$, where $F \in L^q_\omega(U)$

$$\|\nabla w\|_{L^q_\omega(U)} + \|p\|_{L^q_\omega(U)} \le C\left(\|F\|_{L^q_\omega(U)} + \|d\|_{L^q_\omega(U)} + \|g\|_{\hat{T}^q_\omega(U)}\right), \tag{3.7}$$

so our thesis follows.[3]

Hence to finish our proof, we are left with performing the last step: from full space and half space to a bounded domain. Unluckily, the available results (see Fröhlich [22] or Schumacher [36]) do not cover the needed case of weak forcing div F, $F \in L^q_\omega$, so let us provide some details of this last step.

Recall that our goal here is merely the optimal regularity and not existence. Hence in what follows, we assume to have a distributional solution of a considered problem and we aim at showing (3.7) for that w.

First, let us consider a distributional solution to problem (3.5) on $\Omega = E$ being a bent half-space with a small bend and with $g = 0$. By a small bend we mean that there exists smooth $\Sigma : \mathbb{R}^n_+ \ni \tilde{x} \to E \ni x$ having the form $\Sigma(\tilde{x}_1, \dots \tilde{x}_{n-1}, \tilde{x}_n) = (\tilde{x}_1, \dots \tilde{x}_{n-1}, \tilde{x}_n + \sigma(\tilde{x}_1, \dots \tilde{x}_{n-1}))$ with small derivatives of σ (Σ being a small perturbation of identity). The distributional formulation of (3.5)

$$\begin{aligned} \int_E w^i_{x_j}\varphi^i_{x_j} - p\varphi^i_{x_i} &= \int_E f^i\varphi^i,\\ \int_E w^i\psi_{x_i} &= \int_E d\psi \end{aligned}$$

translates for new functions $w \circ \Sigma = \tilde{w}$, etc., via a straightforward computation, with an observation that a change of variables is volume-preserving, into

[3] In fact, the cited results from [21] consider ∇w in place of $\varepsilon(w)$ in the problem formulation. The same result, which we need here, holds for (3.5) by a redefinition of the pressure to $p - \mathrm{div}\, w$.

$$\int_{\mathbb{R}^n_+}(\tilde{w}^i_{\tilde{x}_j} - \sigma_{\tilde{x}_j}\tilde{w}^i_{\tilde{x}_n})(\tilde{\varphi}^i_{\tilde{x}_j} - \sigma_{\tilde{x}_j}\tilde{\varphi}^i_{\tilde{x}_n})\mathbf{1}_{\{j<n\}} + \tilde{w}^i_{\tilde{x}_n}\tilde{\varphi}^i_{\tilde{x}_n} - \tilde{p}\big((\tilde{\varphi}^i_{\tilde{x}_i} - \sigma_{\tilde{x}_i}\tilde{\varphi}^i_{\tilde{x}_n})\mathbf{1}_{\{i<n\}} + \tilde{\varphi}^n_{\tilde{x}_n}\big)$$
$$= \int_{\mathbb{R}^n_+} \tilde{f}^i\tilde{\varphi}^i,$$
$$\int_{\mathbb{R}^n_+} \tilde{w}^i(\tilde{\psi}_{\tilde{x}_i} - \sigma_{\tilde{x}_i}\tilde{\psi}_{\tilde{x}_n})\mathbf{1}_{\{i<n\}} + \tilde{w}^n\tilde{\psi}_{\tilde{x}_n} = \int_{\mathbb{R}^n_+} \tilde{d}\tilde{\psi}$$

(no summation over repeated n's), i.e., after reordering

$$\begin{aligned}\int_{\mathbb{R}^n_+} \tilde{w}^i_{\tilde{x}_j}\tilde{\varphi}^i_{\tilde{x}_j} - \tilde{p}\tilde{\varphi}^i_{\tilde{x}_i} &= \int_{\mathbb{R}^n_+} \sigma_{\tilde{x}_j}\tilde{w}^i_{\tilde{x}_n}\tilde{\varphi}^i_{\tilde{x}_j}\mathbf{1}_{\{j<n\}} \\ &\quad + (\tilde{w}^i_{\tilde{x}_j}\sigma_{\tilde{x}_j} - \tilde{w}^i_{\tilde{x}_n}\sigma^2_{\tilde{x}_j} - \tilde{p}\sigma_{\tilde{x}_i})\mathbf{1}_{\{j<n\}}\tilde{\varphi}^i_{\tilde{x}_n} + \tilde{f}^i\tilde{\varphi}^i, \\ \int_{\mathbb{R}^n_+} \tilde{w}^i\tilde{\psi}_{\tilde{x}_i} &= \int_{\mathbb{R}^n_+} \big(\tilde{d} - (\tilde{w}^i\sigma_{\tilde{x}_i}\mathbf{1}_{\{i<n\}})_{\tilde{x}_n}\big)\tilde{\psi}.\end{aligned}$$

This shows that $(\tilde{w}, \tilde{p})$ solves distributionally

$$\begin{aligned}-\mathrm{div}\,(\nabla\tilde{w}) + \nabla\tilde{p} &= -\mathrm{div}\,B + \tilde{f} && \text{in } \mathbb{R}^n_+, \\ -\mathrm{div}\,\tilde{w} &= D && \text{in } \mathbb{R}^n_+, \\ \gamma(\tilde{w}) &= 0 && \text{on } \mathbb{R}^{n-1}\end{aligned}$$

with

$$B^{ij} = \begin{cases}\sigma_{\tilde{x}_j}\tilde{w}^i_{\tilde{x}_n} & \text{for } j<n, \\ (\tilde{w}^i_{\tilde{x}_j}\sigma_{\tilde{x}_j} - \tilde{w}^i_{\tilde{x}_n}\sigma^2_{\tilde{x}_j} - \tilde{p}\sigma_{\tilde{x}_i})\mathbf{1}_{\{j<n\}} & \text{for } j=n,\end{cases}$$

and

$$D = \tilde{d} - (\tilde{w}^i\sigma_{\tilde{x}_i}\mathbf{1}_{\{i<n\}})_{\tilde{x}_n}.$$

Hence we can use (3.7) on $\Omega = \mathbb{R}^n_+$ for $(\tilde{w}, \tilde{p})$ and data $-\mathrm{div}\,B + \tilde{f}, D$. It gives, after taking into account the form of B, D,

$$\begin{aligned}&\|\nabla\tilde{w}\|_{L^q_\omega(\mathbb{R}^n_+)} + \|\tilde{p}\|_{L^q_\omega(\mathbb{R}^n_+)} \le c\|\tilde{f}\|_{(\hat{W}^{1,q'}_{0,\omega'}(\mathbb{R}^n_+))^*} \\ &+ c(|\nabla\sigma|_\infty + |\nabla\sigma|^2_\infty)(\|\nabla\tilde{w}\|_{L^q_\omega(\mathbb{R}^n_+)} + \|\tilde{p}\|_{L^q_\omega(\mathbb{R}^n_+)}) + c\|\tilde{d}\|_{L^q_\omega(\mathbb{R}^n_+)} + |\nabla^2\sigma|^2_\infty\|\tilde{w}\|_{L^q_\omega(\mathbb{R}^n_+)}.\end{aligned}$$

Smallness of the bend, i.e., of derivatives of σ in relation to $c = c(q, A_q, \mathbb{R}^n_+)$, implies then

$$\|\nabla\tilde{w}\|_{L^q_\omega(\mathbb{R}^n_+)} + \|\tilde{p}\|_{L^q_\omega(\mathbb{R}^n_+)} \le c\|\tilde{f}\|_{(\hat{W}^{1,q'}_{0,\omega'}(\mathbb{R}^n_+))^*} + c\|\tilde{d}\|_{L^q_\omega(\mathbb{R}^n_+)} + \delta\|\tilde{w}\|_{L^q_\omega(\mathbb{R}^n_+)}. \tag{3.8}$$

Since Σ is a small, volume-preserving perturbation of identity, (3.8) gives for (w, p) solving problem (3.5) with data $f, d, g = 0$ on $\Omega = E$ being a bended half-space with a small bend

$$\|\nabla w\|_{L^q_\omega(E)} + \|p\|_{L^q_\omega(E)} \le c(\|f\|_{(\hat{W}^{1,q'}_{0,\omega'}(E))^*} + \|d\|_{L^q_\omega(E)} + \|w\|_{L^q_\omega(E)}). \tag{3.9}$$

Next, let us consider a distributional solution (w, p) to our target problem (3.3), still with $g = 0$. For a cutoff function η with a small support and an arbitrary test function, vector valued φ and scalar valued ψ, we have that

$$\int_V w^i_{x_j}(\varphi^i\eta)_{x_j} - p(\varphi^i\eta)_{x_i} = \int_V F^{ij}(\varphi^i\eta)_{x_j},$$
$$\int_V w^i(\psi\eta)_{x_i} = \int_V d\psi\eta$$

for V being either the little-banded half-space E, when we localize near the boundary $\partial\Omega$, or $\mathbb{R}^n$, when we localize away from the boundary. Observe that due to our assumption that $\partial\Omega \in C^1$ and $\partial\Omega$ is compact, we can always find a small absolute number δ such that the intersection $B_\delta \cap \partial\Omega$ can be described with local coordinates σ, such that $\|\nabla\sigma\|_\infty$ is conveniently small for any $B_\delta \subset \mathbb{R}^n$. We introduce a partition of unity η^k on Ω, where η^k have support on a B_δ and a number c. The localized functions $\bar{w} = w\eta^k$, $\bar{p} = p\eta^k - c$ satisfy distributionally

$$\begin{aligned} -\mathrm{div}\,(\nabla\bar{w}) + \nabla\bar{p} &= h - \mathrm{div}\,H \qquad \text{in } V,\\ -\mathrm{div}\,\bar{w} &= w^i\eta^k_{x_i} + \bar{d} \qquad \text{in } V\\ (\bar{w} = 0 \qquad &\text{on } \partial E \quad \text{in case of } V = E),\end{aligned}$$

where

$$h^i = F^{ij}\eta_{x_j} + w^i_{x_j}\eta^k_{x_j} + p\eta^k_{x_i}, \qquad H^{ij} = w^i\eta^k_{x_j} + F^{ij}\eta^k.$$

Estimates (3.6), (3.9) give

$$\begin{aligned}&\|\nabla\bar{w}\|_{L^q_\omega(V)} + \|\bar{p}\|_{L^q_\omega(V)}\\ &\le C(\eta^k, q, A_q, \Omega)\Big(\|F\|_{L^q_\omega(\Omega)} + \|d\|_{L^q_\omega(\Omega)} + \|w\|_{L^q_\omega(\Omega)} + \|\nabla w\|_{(\hat{W}^{1,q'}_{0,\omega'}(\Omega))^*} + \|p\|_{(\hat{W}^{1,q'}_{0,\omega'}(\Omega))^*}\Big).\end{aligned}$$

Hence, summing over k and using the weighted Poincaré inequality (see Theorem 2.3 of [22]) and choosing $c = \langle p\rangle_\Omega$, we arrive at

$$\begin{aligned}&\|w\|_{W^{1,q}_\omega(\Omega)} + \|p - \langle p\rangle\|_{L^q_\omega(\Omega)}\\ &\le C(q, A_q, \Omega)\left(\|F\|_{L^q_\omega(\Omega)} + \|w\|_{L^q_\omega(\Omega)} + \|d\|_{L^q_\omega(\Omega)} + \|p\|_{(\hat{W}^{1,q'}_{0,\omega'}(\Omega))^*}\right).\end{aligned} \tag{3.10}$$

To conclude, we need to show that (3.10) implies the thesis (3.4). To this end we will use the classical Agmon–Douglis–Nirenberg reasoning by contradiction. Recall that we work, by assumption, with $(w,p) \in W^{1,q}_\omega(\Omega) \times L^q_\omega(\Omega)$. Assume that (3.4) is false, i.e., that there is a sequence $(w_j, p_j) \in W^{1,q}_\omega(\Omega) \times L^q_\omega(\Omega)$, $F_j, d_j \in L^q_\omega(\Omega)$ solving (3.3) such that

$$C_j := \|w_j\|_{W^{1,q}_\omega(\Omega)} + \|p_j - \langle p_j\rangle\|_{L^q_\omega(\Omega)} \ge j(\|F_j\|_{L^q_\omega(\Omega)} + \|d_j\|_{L^q_\omega(\Omega)}).$$

Due to linearity of (3.3), $W_j := \frac{w_j}{C_j}$ and $P_j := \frac{p_j - \langle p_j\rangle}{C_j}$ solve (3.3) with force $R_j := \frac{F_j}{C_j}$ and compressibility $D_j := \frac{d_j}{C_j}$. Observe we have $\langle P_j\rangle = 0$. Hence we have by our above assumption

$$1 = \|W_j\|_{W^{1,q}_\omega(\Omega)} + \|P_j\|_{L^q_\omega(\Omega)} \ge j\left(\|R_j\|_{L^q_\omega(\Omega)} + \|D_j\|_{L^q_\omega(\Omega)}\right).$$

It means that we can find a (nonrelabeled) subsequence and respective limits:

$$
\begin{aligned}
\nabla W_j &\to \nabla W_\infty && \text{weakly in } L^q_\omega(\Omega), \quad \text{strongly in } (W^{1,q'}_{0,\omega'}(\Omega))^*,\\
W_j &\to W_\infty && \text{weakly in } W^{1,q}_\omega(\Omega), \quad \text{strongly in } L^q_\omega(\Omega),\\
P_j &\to P_\infty && \text{weakly in } L^q_\omega(\Omega), \quad \text{strongly in } (W^{1,q'}_{0,\omega'}(\Omega))^*,\\
R_j &\to 0, D_j \to 0 && \text{strongly in } L^q_\omega(\Omega),
\end{aligned}
$$

where the first two strong limits follow from compact embeddings $W^{1,q}_\omega \hookrightarrow L^q_\omega \hookrightarrow (W^{1,q'}_{0,\omega'})^*$; see Theorem 2.3 of [22] for the former and Lemma 2.5 for the latter.

Moreover, taking limit $j \to \infty$ in (3.3) solved by (W_j, P_j), with data $R_j, D_j, 0$, we see that $(W_\infty, P_\infty) = (0, 0)$ in view of uniqueness of the zero solution (with zero mean pressure) to (3.3). The uniqueness of the zero solution follows, for instance, from the fact that within Muckenhoupt weights we have for a bounded Ω that $L^p_\omega(\Omega) \hookrightarrow L^s(\Omega)$ for a certain $s > 1$. Consequently, we can use a classical uniqueness theorem in L^s, which can be found, for instance, in section 3 of Borchers and Miyakawa [6]. Hence

$$
\begin{aligned}
1 &= \|W_j\|_{W^{1,q}_\omega(\Omega)} + \|P_j\|_{L^q_\omega(\Omega)}\\
&\le C(q, A_q, \Omega)\left(\|R_j\|_{L^q_\omega(\Omega)} + \|W_j\|_{L^q_\omega(\Omega)} + \|D_j\|_{L^q_\omega(\Omega)} + \|P_j\|_{(W^{1,q'}_{0,\omega,}(\Omega))^*}\right) \overset{j\to 0}{\to} 0,
\end{aligned}
$$

which contradicts (3.10).

We have reached the thesis (3.4) for $g = 0$. In order to include the nonhomogeneous case $g \neq 0$, recall that the trace space $T^q_\omega(\Omega)$ (or its homogeneous version $\mathring{T}^q_\omega(\Omega)$) is defined via the existence of an extension $\gamma^{-1} : T^q_\omega(\Omega) \to W^{1,q}_\omega(\Omega)$, which is linear and bounded). Therefore (3.3) can be transferred into $(\tilde{w}, p) := (w - \gamma^{-1}g, p)$, which is a solution to the following system:

$$
\begin{aligned}
-\mathrm{div}\,(\varepsilon\tilde{w}) + \nabla p &= -\mathrm{div}\,(F - \varepsilon\gamma^{-1}g) && \text{in } \Omega,\\
\mathrm{div}\,\tilde{w} &= d - \mathrm{div}\,(\gamma^{-1}g) && \text{in } \Omega,\\
\gamma(\tilde{w}) &= 0 && \text{on } \partial\Omega,
\end{aligned}
$$

and the result can be achieved using the estimate for homogeneous boundary data. ∎

3.2. Proof of Theorem 3.1. Recall for section 2.4 that due to the *miracle of extrapolation*, it is sufficient to prove the desired estimates in the case $L^2_\omega(\Omega)$, with $\omega \in A_2$. By our assumption, (v, π) solves (1.1) and $\nabla v, \pi \in L^s(\Omega)$ for some $s \in (1, \infty)$. Due to boundedness of Ω, we can assume without loss of generality that $s \in (1, 2]$. The first idea behind our estimate is to approximate ω by ω_j such that $\nabla v, \pi \in L^2_{\omega_j}(\Omega)$. By (2.7), we have for $\tilde{\omega}_1 = (M\nabla v)^{s-2} \in A_2$ and $\nabla u \in L^2_{\omega_1}(\Omega)$ as well as for $\tilde{\omega}_2 = (Mp)^{s-2} \in A_2$ and $\pi \in L^2_{\omega_2}(\Omega)$. Let us take $\tilde{\omega}_3 = \min\{\tilde{\omega}_1, \tilde{\omega}_2\}$ and $\omega_j = \min\{j\tilde{\omega}_3, \omega\}$. Obviously, $\nabla u \in L^2_{\omega_j}(\Omega)$ and $f \in L^2_{\omega_j}(\Omega)$. But moreover, by (2.4), we find that $A_2(\omega_j) \le A_2(\omega) + A_2(\omega_3)$, since $A_q(\omega_1) = A_q(j\omega_3)$ directly by definition. For this ω_j we perform now the following a priori estimate.

Let us rewrite (1.1) as a distributional formulation of the linear Stokes problem

$$
\int_\Omega \mu\, \varepsilon(v) \cdot \nabla\varphi + \pi \mathrm{div}\,\varphi \,\mathrm{d}x = \int_\Omega (f - \mathcal{S}(x, \varepsilon(v)) + \mu\, \varepsilon(v)) \cdot \nabla\varphi. \tag{3.11}
$$

Since $\nabla v \in L^2_{\omega_j}$, we can use an estimate of Lemma 3.2 and Assumption 1.1 to provide the following absorption with $C = C(A_2(\omega) + A_2(\omega_3), \Omega)$:

$$\|\nabla v\|^2_{L^2_{\omega_j}(\Omega)} + \|\pi - \langle\pi\rangle\|^2_{L^2_{\omega_j}(\Omega)} \le C\int_\Omega |f|^2\omega_j + |\mathcal{S}(x,\varepsilon(v)) - \mu\,\varepsilon(v)|^2\omega_j$$
$$\le C\int_\Omega (|f|^2 + 2c_1^2m^2 + 2c_2^2 + 2\mu^2m^2)\,\omega_j + C\int_{\{|\varepsilon(v)|\ge m\}} \frac{|\mathcal{S}(x,\varepsilon(v)) - \mu\varepsilon(v)|^2}{|\varepsilon(v)|^2}|\varepsilon(v)|^2\omega_j.$$

Due to the assumed linearity-at-infinity we can find such $m = m_0$ that the last summand on the right-hand side. above does not exceed half of the first of the left-hand side. Consequently

$$\|\nabla v\|_{L^2_{\omega_j}(\Omega)} + \|\pi - \langle\pi\rangle\|_{L^2_{\omega_j}(\Omega)} \le C(A_2(\omega) + A_2(\omega_3), \mu, \Omega)\left(1 + \|f\|_{L^2_{\omega_j}(\Omega)}\right). \tag{3.12}$$

Observe that the above constant is j-uniform. Next, we let $j \to \infty$ in (3.12). For the right-hand side, we use the fact that $\omega_j \le \omega$, and for the left-hand side we use the monotone convergence theorem (notice here that $\omega_j \nearrow \omega$ since $\omega_3 < \infty$ almost everywhere). Consequently

$$\|\nabla v\|_{L^2_\omega(\Omega)} + \|\pi - \langle\pi\rangle\|_{L^2_\omega(\Omega)} \le C(A_2(\omega) + A_2(\omega_3), \Omega)\left(1 + \|f\|_{L^2_\omega(\Omega)}\right). \tag{3.13}$$

This implies the quantitative estimate, but with C still depending on $A_2(\omega_3)$. Therefore we use from (3.13) only the qualitative information $\nabla v, \pi \in L^2_\omega$ and redo the absorption for ω alone. Consequently one gets the desired estimate with dependence on $A_2(\omega)$ alone. Therefore the extrapolation [15, Theorem 1.4] can be applied and the theorem is proved.

4. Proofs of the technical results. This section contains proofs of Theorem 1.9 (solenoidal, biting, weighted div-curl lemma) and Theorem 1.10 (solenoidal Lipschitz truncations). Let us begin with the latter, since it is needed in the proof of the former.

4.1. Lipschitz truncations. Since even the optimal regularity of (1.1) for $q < 2$ is insufficient for u to be a test function, we resort to Lipschitz truncations. It is a standard tool by now, originally developed in Acerbi and Fusco [1] and Frehse, Málek, and Steinhauer [20] (see also Diening, Málek, and Steinhauer [18]). Recently a further advance was provided that is important for the fluid dynamics considerations, namely, a solenoidal Lipschitz truncation; compare Breit, Diening, and Fuchs [8] and Breit, Diening, and Schwarzacher [9]. Let us present weighted estimates for the solenoidal Lipschitz truncations developed in [9] and fine-tune them for our purposes.

LEMMA 4.1 (solenoidal Lipschitz approximation on balls). *Let $B \subset \mathbb{R}^n$ be a ball and $s > 1$. Let $g \in W^{1,s}_{0,\mathrm{div}}(B)$. Then, for all $\lambda > \lambda_0$, there exists a* Lipschitz truncation *$g^\lambda \in W^{1,\infty}_{0,\mathrm{div}}(2B)$ such that*

$$g^\lambda = g \quad \text{and} \quad \nabla g^\lambda = \nabla g \qquad \text{in } \{M(\nabla g) \le \lambda\} \subset 2B, \tag{4.1}$$
$$|\nabla g^\lambda| \le |\nabla g|\chi_{\{M(\nabla g)\le\lambda\}} + C\,\lambda\chi_{\{M(\nabla g)>\lambda\}} \qquad \text{almost everywhere.} \tag{4.2}$$

Further, if $\nabla g \in L^p_\omega(\Omega)$ for some $1 \le p < \infty$ and $\omega \in A_p$, then

$$\begin{aligned} \int_{2B} |\nabla g^\lambda|^p\omega\,\mathrm{d}x &\le C\int_B |\nabla g|^p\omega\,\mathrm{d}x, \\ \int_{2B} |\nabla(g - g^\lambda)|^p\omega\,\mathrm{d}x &\le C\int_{B\cap\{M(\nabla g)>\lambda\}} |\nabla g|^p\omega\,\mathrm{d}x, \end{aligned} \tag{4.3}$$

where the constant C depends on $(A_p(\Omega), \Omega, N, p)$ and $\lambda_0 = c(s,n)\left(⨍_B |\nabla g|^s\,\mathrm{d}x\right)^{\frac{1}{s}}$.

Proof. All statements except for (4.3) are already contained in [9, Lemma 4.3 and Theorem 4.4]. Please observe, that although the construction there is done in the three-dimensional case, the arguments are in fact valid in all dimensions by replacing the inverse-curl operator with its n-dimensional analogue as defined in Remark 2.18 in [9].

The first inequality of (4.3) follows directly from the second, so it is enough to prove the latter.

Let us extend both g and g^λ by 0 outside B and $2B$, respectively. It follows from (4.1) and (4.2) that

$$\begin{aligned}\|\nabla(g-g^\lambda)\|_{L^p_\omega(\mathbb{R}^n)} &= \|\nabla(g-g^\lambda)\chi_{\{M(\nabla g)>\lambda\}}\|_{L^p_\omega(\mathbb{R}^n)}\\ &\le \|\nabla g\,\chi_{\{M(\nabla g)>\lambda\}}\|_{L^p_\omega(B)} + C\,\|\lambda\,\chi_{\{M(\nabla g)>\lambda\}}\|_{L^p_\omega(\mathbb{R}^n)}.\end{aligned} \tag{4.4}$$

The second term will be handled by a Calderón–Zygmund-type covering argument. As $\{M(\nabla g)>\lambda\}\subset 2B$ is open, for every $x\in\{M(\nabla g)>\lambda\}$ there is a ball $B_{r(x)}(x)\subset\{M(\nabla g)>\lambda\}$ such that

$$\lambda < \fint_{B_r(x)} |\nabla g|dx \le 2\lambda. \tag{4.5}$$

These balls cover $\{M(\nabla g)>\lambda\}$. Next, using the Besicovich covering theorem, we extract from this cover a countable subset B_i which is locally finite, i.e.,

$$\#\{j\in\mathbb{N};\, B_i\cap B_j\neq\emptyset\}\le C(n). \tag{4.6}$$

In the following, for a measurable set A we write $|A|_\omega=\int_A \omega dx$. Using (4.5), (2.2), and (4.6), we have the following estimate:

$$\begin{aligned}&\|\lambda\,\chi_{\{M(\nabla g)>\lambda\}}\|^p_{L^p_\omega(\mathbb{R}^n)} = \lambda^p|\{M(\nabla g)>\lambda\}|_\omega \le \sum_i \lambda^p|B_i|_\omega \le \sum_i \left(\fint_{B_i}|\nabla g|\,dx\right)^p|B_i|_\omega\\ &\le \sum_i \fint_{B_i}|\nabla g|^p\omega\,dx\left(\fint_{B_i}\omega^{-(p'-1)}\,dx\right)^{\frac{1}{p'-1}}|B_i|_\omega \le A_p(\omega)\sum_i\int_{B_i}|\nabla g|^p\omega\,dx\\ &\le C(n)\,A_p(\omega)\int_{\{M(\nabla g)>\lambda\}}|\nabla g|^p\omega\,dx = C(n)\,A_p(\omega)\int_B|\nabla g|^p\chi_{\{M(\nabla g)>\lambda\}}\omega\,dx.\end{aligned}$$

This directly leads to the inequality

$$\|\lambda\,\chi_{\{M(\nabla g)>\lambda\}}\|_{L^p_\omega(\mathbb{R}^n)} \le C(n)\,A_p(\omega)^{\frac{1}{p}}\|\nabla g\,\chi_{\{M(\nabla g)>\lambda\}}\|_{L^p_\omega(B)},$$

which used in (4.4) finishes the proof of the desired estimate (4.3). ∎

Next, we provide a proof of Theorem 1.10. We lose the zero trace of its counterpart on balls from the preceding Lemma 4.1 but deal with Lipschitz truncation on general domains Ω.

Proof of Theorem 1.10. We use the construction of [9, section 4]. The fact that g has zero trace in a ball is used only in Lemmas 4.2 and 4.3 there, so all other results can be directly applied to our situation. The construction of g^λ and Lemma 4.1 of [9] are valid in all dimensions and for a general domain Ω with no changes, except for the replacing of the inverse-curl operator with the n-dimensional analogue, as defined in Remark 2.18 of [9]. Moreover, by using for general Ω the local estimates intended for balls in [9], one loses only information of the zero trace, but all estimates hold and the solenoidality is preserved. For instance, the argument in the proof of Lemma 4.3 implies in our case that $g^\lambda \in W^{1,1}_{\mathrm{div}}(\Omega)$ (no zero trace). Moreover, the proof of Theorem 4.4 of [9] implies all the above assertions needed by us, with the exception of (1.21). These weighted estimates follow from redoing the argument in Lemma 4.1, by replacing B there with Ω. ∎

4.2. Solenoidal, generalized div-curl lemma. Let us focus now on the proof of Theorem 1.9. It is divided into several steps for clarity.

4.2.1. Preliminary Step 0. First, by the reflexivity and separability of $L^q_\omega, L^{q'}_\omega$ together with the assumption (1.12) and due to the embedding $L^{q'}_\omega(\Omega) \hookrightarrow L^{1+\delta}(\Omega)$ (compare with (2.8)), we find a subsequence

$$s^k \rightharpoonup s \quad \text{weakly in } L^{q'}_\omega(\Omega) \cap L^1(\Omega), \qquad a^k \rightharpoonup a \quad \text{weakly in } L^q_\omega(\Omega) \cap L^1(\Omega). \tag{4.7}$$

In the following, it remains to show (1.18). Since we aim to show convergence on a (large) subset of Ω we may assume without loss of generality that $\partial\Omega$ is C^∞-smooth.

4.2.2. Step 1. Reduction to the nonsolenoidal case. Let us consider the linear Stokes problem

$$\begin{aligned} -\mathrm{div}\,(\varepsilon(w^k)) + \nabla p^k &= -\mathrm{div}\, s^k \quad && \text{in } \Omega, \\ \mathrm{div}\, w &= 0 && \text{in } \Omega \end{aligned} \tag{4.8}$$

with homogeneous boundary values. Lemma 3.2 and assumption (1.12) imply that

$$\|\nabla w^k\|_{L^{q'}_\omega(\Omega)} + \|p^k\|_{\mathring{L}^{q'}_\omega(\Omega)} \le C\left(1 + \|s^k\|_{L^{q'}_\omega(\Omega)}\right) \le C. \tag{4.9}$$

Hence assumption (1.12) and the embedding (2.8) implies that we may pass to a subsequence, such that

$$p^k \rightharpoonup p \qquad \text{weakly in } L^{q'}_\omega(\Omega). \tag{4.10}$$

Let us consider $b^k =: s^k + p^k \mathrm{Id}$. Assume for a moment that for every bounded sequence $\{c^k\}_{k=1}^\infty$ in $W^{1,\infty}_0(\Omega)$ such that

$$\nabla c^k \rightharpoonup^* 0 \qquad \text{weakly}^* \text{ in } L^\infty(\Omega)$$

one has

$$\lim_{k\to\infty} \int_\Omega b^k \cdot \nabla c^k \,\mathrm{d}x = 0. \tag{4.11}$$

Then, making in the nonsolenoidal, weighted, biting div-curl lemma, i.e., Theorem 2.6 of [11], the choices

$$a^k =: a^k, \quad b^k =: b^k,$$

we see via our assumptions and (4.9) that the assumptions of the nonsolenoidal lemma are satisfied. Its thesis implies existence of a subsequence such that

$$a^k \rightharpoonup a \qquad \text{weakly in } L^1(\Omega), \tag{4.12}$$

$$b^k \rightharpoonup b \qquad \text{weakly in } L^1(\Omega), \tag{4.13}$$

$$a^k \cdot b^k \omega \rightharpoonup a \cdot b\,\omega \qquad \text{weakly in } L^1(\Omega_j) \quad \text{for all } j \in \mathbb{N}. \tag{4.14}$$

Due to (4.7), we identify $b = s + p\mathrm{Id}$. Finally, assumption (1.15) gives, after decreasing Ω_j slightly, via Egoroff's theorem

$$p^k \mathrm{Id} \cdot a^k \omega = p^k \ \mathrm{tr}(a^k)\omega \rightharpoonup p \ \mathrm{tr}(a)\omega = p\mathrm{Id} \cdot a\, \omega \qquad \text{weakly in } L^1(\Omega_j), \tag{4.15}$$

thanks to (4.10), uniqueness of the limiting a, and the strong-weak coupling.

Subtracting from (4.14) with $b^k =: s^k + p^k\mathrm{Id}$ and $b = s + p\mathrm{Id}$ the formula (4.15) we arrive at (1.18). The limits (1.16), (1.17) are given as (4.7) and (4.12).

Consequently, we are left with justifying the compactness condition (4.11). Since the first equation of (4.8) can be rewritten as

$$\operatorname{div} b^k = \operatorname{div} \nabla w^k, \tag{4.16}$$

the condition (4.11) is equivalent to the strong-L^1 precompactness of ∇w^k. We will accomplish this in the following three steps.

4.2.3. Step 2. Solenoidal truncations. Let us use Theorem 1.10 to truncate solenoidally w^k at height λ, producing $w^{k,\lambda}$. For the following *dual forcing* given by

$$Q(\eta) := |\eta|^{q'-2}\eta,$$

let us consider the following auxiliary linear Stokes problem:

$$\begin{aligned} -\operatorname{div}(\varepsilon(z^{k,\lambda})) + \nabla t^{k,\lambda} &= -\operatorname{div} Q(\nabla w^{k,\lambda}) && \text{in } \Omega, \\ \operatorname{div} z^{k,\lambda} &= 0 && \text{in } \Omega \end{aligned} \tag{4.17}$$

with null boundary values. Boundedness of $Q(\nabla w^{k,\lambda})$ for a fixed λ and Lemma 3.2 imply that for any finite p one has

$$\|z^{k,\lambda}\|_{W^{1,p}_{0,div}} + \|t^{k,\lambda}\|_{L^p} \le C(\lambda)$$

and the regularity is inherited by the limiting equation with respect to $k \to \infty$, which reads

$$\begin{aligned} -\operatorname{div}(\varepsilon(z^{\lambda})) + \nabla t^{\lambda} &= -\operatorname{div}\, Q_\lambda && \text{in } \Omega, \\ \operatorname{div} z^{\lambda} &= 0 && \text{in } \Omega \end{aligned} \tag{4.18}$$

with null boundary values. The above Q_λ denotes the L^q_ω weak limit of $Q(\nabla w^{k,\lambda})$ (since $Q(\nabla w^k)$, hence $Q(\nabla w^{k,\lambda})$ is k-uniformly bounded in L^q_ω).

For a non relabeled subsequence of Q_λ, let us immediately denote its L^q_ω weak limit by Q_0.

4.2.4. Step 3. A nonweighted weak-L^1 limit for truncations. Our aim in this step is to show, for a fixed λ (possibly, again on a nonrelabeled subsequence), that for $k \to \infty$

$$Q(\nabla w^{k,\lambda}) \cdot \nabla w^k \rightharpoonup Q_\lambda \cdot \nabla w \qquad \text{weakly in } L^1(\Omega). \tag{4.19}$$

Due to (4.9) and boundedness of $\nabla w^{k,\lambda}$, we see that $Q(\nabla w^{k,\lambda}) \cdot \nabla w^k$ is k-uniformly $L^{q'}_\omega \subset L^{1+\delta}$ integrable, hence equi-integrable. Consequently, it possesses a weakly-L^1 converging subsequence. Now, to identify it with $Q_\lambda \cdot \nabla w$, it suffices to show that for all $\eta \in \mathcal{D}(\Omega)$ we have

$$\lim_{k\to\infty} \int_\Omega Q(\nabla w^{k,\lambda}) \cdot \nabla w^k\, \eta = \int_\Omega Q_\lambda \cdot \nabla w\, \eta. \tag{4.20}$$

Let us write

$$\int_\Omega Q(\nabla w^{k,\lambda})\cdot\nabla w^k\eta=\int_\Omega(Q(\nabla w^{k,\lambda})-\varepsilon(z^{k,\lambda}))\cdot\nabla w^k\eta+\int_\Omega\varepsilon(z^{k,\lambda})\cdot\nabla w^k\eta=:I^{k,\lambda}+II^{k,\lambda}.$$

One has

$$\begin{aligned}I^{k,\lambda}&=\int_\Omega\big(Q(\nabla w^{k,\lambda})-\varepsilon(z^{k,\lambda})\big)\cdot\nabla(w^k\eta)\,\mathrm{d}x-\int_\Omega\big(Q(\nabla w^{k,\lambda})-\varepsilon(z^{k,\lambda})\big)\cdot\big(w^k\otimes\nabla\eta\big)\,\mathrm{d}x\\&=\int_\Omega t^{k,\lambda}\mathrm{div}\,(w^k\eta)\,\mathrm{d}x-\int_\Omega\big(Q(\nabla w^{k,\lambda})-\varepsilon(z^{k,\lambda})\big)\cdot\big(w^k\otimes\nabla\eta\big)\,\mathrm{d}x\\&=\int_\Omega t^{k,\lambda}w^k\nabla\eta\,\mathrm{d}x-\int_\Omega\big(Q(\nabla w^{k,\lambda})-\varepsilon(z^{k,\lambda})\big)\cdot\big(w^k\otimes\nabla\eta\big)\,\mathrm{d}x,\end{aligned}$$

where for the second equality above we used (4.17) and for the last one solenoidality of w^k. We have obtained formulas with a coupling of w^k, strong converging in $L^{q'}_\omega\subset L^{1+\delta}$, and the remainders weak converging in L^p with any finite p. Hence we can pass to the limit and recover it by reverse equalities as follows:

$$\begin{aligned}\lim_{k\to\infty}I^{k,\lambda}&=\int_\Omega t^\lambda w\nabla\eta\,\mathrm{d}x-\int_\Omega(Q_\lambda-\varepsilon(z))\cdot(w\otimes\nabla\eta)\,\mathrm{d}x\\&=\int_\Omega t^\lambda\mathrm{div}\,(w^k\eta)\,\mathrm{d}x-\int_\Omega(Q_\lambda-\varepsilon(z))\cdot(w\otimes\nabla\eta)\,\mathrm{d}x\\&=\int_\Omega(Q_\lambda-\varepsilon(z^\lambda))\cdot\nabla w\;\eta.\end{aligned}$$

Function $w^k\eta$ with the Bogovskii correction[4] is admissible in (4.17). Therefore we can write for $II^{k,\lambda}$

$$\begin{aligned}II^{k,\lambda}&=\int_\Omega\varepsilon(w^k)\cdot\nabla(z^{k,\lambda}\eta)dx-\int_\Omega\varepsilon(w^k)\cdot(z^{k,\lambda}\otimes\nabla\eta)dx\\&=\int_\Omega\varepsilon(w^k)\cdot\nabla(z^{k,\lambda}\eta-\mathrm{Bog}(z^{k,\lambda}\otimes\nabla\eta))dx+\int_\Omega\varepsilon(w^k)\cdot\nabla\left(\mathrm{Bog}(z^{k,\lambda}\otimes\nabla\eta)\right)dx\\&\quad-\int_\Omega\nabla w^k\cdot(z^{k,\lambda}\otimes\nabla\eta)dx\\&=\int_\Omega s^k\cdot\nabla(z^{k,\lambda}\eta-\mathrm{Bog}(z^{k,\lambda}\otimes\nabla\eta))dx+\int_\Omega\nabla w^k\cdot\nabla\left(\mathrm{Bog}(z^{k,\lambda}\otimes\nabla\eta)\right)dx\\&\quad-\int_\Omega\nabla w^k\cdot(z^{k,\lambda}\otimes\nabla\eta)dx,\end{aligned}$$

where for the second equality above we used, this time, (4.8). We use our assumption (1.14) to pass to the limit in the first term above. For the last two terms, we invoke continuity of the Bogovskii operator in L^p spaces to pass to the respective limits, thanks to the strong-weak coupling. Using for the limit (4.8) to reverse, we see that

$$\lim_{k\to\infty}II^{k,\lambda}=\int_\Omega\varepsilon(z^\lambda)\cdot\nabla w\;\eta.$$

Putting together limits for $I^{k,\lambda}$ and $II^{k,\lambda}$, we obtain (4.20), thus (4.19).

[4]Compare Bogovskii [4, 5] and Diening, Růžička, and Schumacher [19].

4.2.5. Step 4. A weighted weak-L^1 biting limit. Our goal here is to show that $Q(\nabla w^k)\cdot\nabla w^k\omega$ tends to $Q_0\cdot\nabla w\ \omega$ weakly in $L^1(\Omega)$; in fact, we will have to decrease Ω slightly. Recall that (4.19) does not involve a weight ω. Therefore, we decompose an arbitrary $\omega\in A_{q'}$ as follows:

$$\omega=\frac{\omega}{1+\delta\omega}+\frac{\delta\omega^2}{1+\delta\omega}$$

with the former summand bounded for any $\delta>0$. Let us write

$$\begin{aligned}
&Q(\nabla w^k)\cdot\nabla w^k\omega-Q_\lambda\cdot\nabla w\ \omega\\
&=\left(Q(\nabla w^{k,\lambda})\cdot\nabla w^k-Q_\lambda\cdot\nabla w\right)\omega+\left(Q(\nabla w^k)-Q(\nabla w^{k,\lambda})\right)\cdot\nabla w^k\ \omega\\
&=\left(Q(\nabla w^{k,\lambda})\cdot\nabla w^k-Q_\lambda\cdot\nabla w\right)\frac{\omega}{1+\delta\omega}+\left(Q(\nabla w^{k,\lambda})\cdot\nabla w^k-Q_\lambda\cdot\nabla w\right)\frac{\delta\omega^2\mathbf{1}_{\{\omega\le\lambda\}}}{1+\delta\omega}\\
&+\left(Q(\nabla w^{k,\lambda})\cdot\nabla w^k-Q_\lambda\cdot\nabla w\right)\frac{\delta\omega^2\mathbf{1}_{\{\omega>\lambda\}}}{1+\delta\omega}+\left(Q(\nabla w^k)-Q(\nabla w^{k,\lambda})\right)\cdot\nabla w^k\ \omega\\
&=:III_\delta^{k,\lambda}+IV_\delta^{k,\lambda}+V_\delta^{k,\lambda}+VI^{k,\lambda}.
\end{aligned}\tag{4.21}$$

We will deal with $III_\delta^{k,\lambda}$ and $IV_\delta^{k,\lambda}$ directly via (4.19). Indeed, (4.19) extends automatically to its weighted version, as long as the involved weight is bounded. Therefore, as for fixed λ,δ the respective weights are bounded, we have for an arbitrary $\psi\in L^\infty(\Omega)$

$$\lim_{k\to\infty}\int III_\delta^{k,\lambda}\psi=0,\qquad \lim_{k\to\infty}\int IV_\delta^{k,\lambda}\psi=0.\tag{4.22}$$

In relation to $V_\delta^{k,\lambda}$ we write, using the Hölder inequality,

$$\begin{aligned}
&\int V_\delta^{k,\lambda}\psi\le\|\psi\|_\infty\int\left(|Q(\nabla w^{k,\lambda})||\nabla w^k|+|Q_\lambda||\nabla w|\right)\frac{\delta\omega^2\mathbf{1}_{\{\omega>\lambda\}}}{1+\delta\omega}\\
&\le\|\psi\|_\infty\|Q(\nabla w^{k,\lambda})\|_{L^q_{\frac{\delta\omega^2\mathbf{1}_{\{\omega>\lambda\}}}{1+\delta\omega}}}\|\nabla w^k\|_{L^{q'}_{\frac{\delta\omega^2\mathbf{1}_{\{\omega>\lambda\}}}{1+\delta\omega}}}+\|\psi\|_\infty\int|Q_\lambda||\nabla w|\frac{\delta\omega^2}{1+\delta\omega}\\
&\le\|\psi\|_\infty\|\nabla w^k\|^2_{L^{q'}_{\omega\mathbf{1}_{\{\omega>\lambda\}}}}+\|\psi\|_\infty\int|Q_\lambda||\nabla w|\frac{\delta\omega^2}{1+\delta\omega},
\end{aligned}\tag{4.23}$$

where for the second inequality we used growth of Q, (1.21) and $\frac{\delta\omega^2}{1+\delta\omega}\le\omega$ almost everywhere.

Let us apply the biting lemma, Lemma 2.6, on the sequence $|\nabla w^k|^{q'}\omega$; compare (4.9). Consequently, there is a sequence Ω_j such that $|\Omega\setminus\Omega_j|\to 0$ and for any $K\subset\Omega_j$ it holds that

$$\int_K|\nabla w^k|^{q'}\omega\le\varepsilon\tag{4.24}$$

k-uniformly, as long as $|K|\le\delta_{\varepsilon,j}$. The Chebyshev inequality for ω, integrable by definition, indicates that the role of K may play $\{\omega>\lambda\}$ for sufficiently large λ, as long as we restrict ourselves to Ω_j in (4.22) and (4.23). Indeed, in tandem with the above application of the biting lemma, for every j and ε there exists λ_j^ε, such that

$$\int_{\{w>\lambda\}\cap\Omega_j} |\nabla v^k|^{q'} \omega \leq \varepsilon \quad \text{ for every } \quad \lambda \geq \lambda_j^\varepsilon. \tag{4.25}$$

Consequently, a restriction to Ω_j does not change (4.22) and allows us to write via use of (4.25) in (4.23) that for any ε and each $\lambda \geq \lambda_j^\varepsilon$

$$\int_{\Omega_j} V_\delta^{k,\lambda}\psi \leq C\varepsilon + C(\|\psi\|_\infty)\int_{\Omega_j} |Q_\lambda||\nabla w|\frac{\delta\omega^2}{1+\delta\omega}. \tag{4.26}$$

Since $|Q_\lambda||\nabla w|\frac{\delta\omega^2}{1+\delta\omega} \leq |Q_\lambda||\nabla w|\omega$ with the latter integrable via the Hölder inequality, the Lebesgue dominated convergence used for the last summand of (4.26) implies altogether

$$\limsup_{\delta\to\infty}\limsup_{k\to\infty}\int_{\Omega_j} V_\delta^{k,\lambda}\psi \leq C\varepsilon + 0. \tag{4.27}$$

Finally, let us focus on $VI^{k,\lambda}$ of (4.21). We deal with it using again the biting lemma, together with the weak-L^1 estimate for the maximal function

$$|\{M(\nabla w^k) > \lambda\}| \leq \frac{c\|\nabla w^k\|_{L^1(\Omega)}}{\lambda} \leq \frac{C}{\lambda},$$

which indicates that here the role of the biting set K may play $\{M(\nabla w^k) > \lambda\}$ for sufficiently large λ. Indeed, in tandem with the above application of the biting lemma, for every j and ε there exists λ_j^ε, such that

$$\int_{\{M(\nabla w^k)>\lambda\}\cap\Omega_j} |\nabla w^k|^{q'} \omega \leq \varepsilon \quad \text{ for every } \quad \lambda \geq \lambda_j^\varepsilon. \tag{4.28}$$

Let us use Theorem 1.10 to write

(4.29)
$$\begin{aligned}
&\left|\int_{\Omega_j} (Q(\nabla w^k) - Q(\nabla w^{k,\lambda}))\cdot \nabla w^k\ \omega\ \psi\right| \\
&= \left|\int_{\{M(\nabla(w^k))>\lambda\}\cap\Omega_j} (Q(\nabla w^k) - Q(\nabla w^{k,\lambda}))\cdot \nabla w^k\ \omega\ \psi\right| \\
&\leq C\|\psi\|_\infty\left(\int_\Omega |\nabla w^{k,\lambda}|^{q'}\omega + |\nabla w^k|^{q'}\omega\right)^{\frac{1}{q}}\left(\int_{\{M(\nabla(w^k))>\lambda\}\cap\Omega_j} |\nabla(w^k)|^{q'}\omega\right)^{\frac{1}{q'}},
\end{aligned}$$

where, for the inequality, we used growth of Q.

Putting together (4.29) and (4.28) we see that for every j and ε there exists λ_j^ε such that

$$\left|\int_{\Omega_j} VI^{k,\lambda}\psi\right| \leq C\|\varphi\|_\infty\varepsilon^{\frac{1}{q'}} \qquad \text{ for every } \quad \lambda \geq \lambda_j^\varepsilon. \tag{4.30}$$

Altogether, integrating (4.21) over Ω_j, taking in its right-hand side

$$\limsup_{\lambda\to\infty}\ \limsup_{\delta\to\infty}\ \limsup_{k\to\infty},$$

and using (4.22), (4.27), and (4.30), we see that for any j it holds that

$$Q(\nabla w^k)\cdot\nabla w^k\omega \rightharpoonup Q_0\cdot\nabla w\ \omega \qquad \text{weakly in } L^1(\Omega_j). \tag{4.31}$$

4.2.6. Step 5. Justifying the compactness condition (4.11) via weighted monotonicity. Finally, (4.31) together with radial unboundedness (coercivity) and strict monotonicity of Q imply

$$\nabla w^k \to \nabla w \qquad \text{a.e. in } \Omega_j.$$

For more details on this step, compare, for instance, pp. 52–53 of the book by Roubíček [33]. The diagonal argument gives us a subsequence such that

$$\nabla w^k \to \nabla w \qquad \text{a.e. in } \Omega.$$

This together with (4.9) implies uniform integrability and hence via the Vitali's theorem L^1 strong sequential precompactness of ∇w^k.

The proof of Theorem 1.9 is complete.

5. Proofs of main results. This section is dedicated to the proofs of Theorem 1.4, Theorem 1.5, and Corollary 1.7. Theorem 1.4 is a special case of Theorem 1.5, so let us focus on the latter. The main ingredients of its proof are a priori estimates provided by Theorem 3.1, limit identification by Theorem 1.9, and weighed considerations that allow us to provide optimal regularity.

5.1. Existence. Step 1. Approximate problems. Recall that an arbitrary $f \in L^q_\omega(\Omega)$ with $\omega \in A_q$, $1 < q < \infty$, is a force of the considered problem (1.1). We have by (2.8) that $f \in L^{s_0}(\Omega)$ for an $s_0 \in (1,2)$. Formula (2.6) with $\alpha = 2 - s_0$ implies that $(Mf)^{s_0-2} \in A_2$, hence also $\omega_0 := (1+Mf)^{s_0-2}$ belongs to A_2. Consequently we have $f \in L^2_{\omega_0}(\Omega)$; compare (2.7).

Let us define $f^k := f\chi_{\{|f|<k\}}$. Then

$$f^k \to f \quad \text{strongly in } L^2_{\omega_0}(\Omega) \cap L^{s_0}(\Omega) \cap L^q_\omega(\Omega). \tag{5.1}$$

For our $f^k \in L^2(\Omega)$ we can use the standard monotone operator theory to find $v^k \in W^{1,2}_0(\Omega)$ satisfying

$$\int_\Omega \mathcal{S}(x, \varepsilon(v^k)) \cdot \nabla\varphi = \int_\Omega f^k \cdot \nabla\varphi \qquad \text{for all } \varphi \in W^{1,2}_{0,\mathrm{div}}(\Omega). \tag{5.2}$$

It is equivalent to finding $(v^k, \pi^k) \in W^{1,2}_0(\Omega) \times \mathring{L}^2(\Omega)$ solving weakly (1.1).

By Theorem 3.1 (used three times, for $L^q_\omega(\Omega), L^{s_0}(\Omega)$ and for $L^2_{\omega_0}(\Omega)$), we find that uniformly in k

$$\begin{aligned} \|\nabla v^k\|_{L^q_\omega(\Omega)} + \|\pi^k\|_{L^q_\omega(\Omega)} &\le C(1 + \|f^k\|_{L^q_\omega(\Omega)}) \le C(1 + \|f\|_{L^q_\omega(\Omega)}), \\ \|\nabla v^k\|_{L^2_{\omega_0}(\Omega)} + \|\pi^k\|_{L^2_{\omega_0}(\Omega)} + \|\nabla v^k\|_{L^{s_0}(\Omega)} + \|\pi^k\|_{L^{s_0}(\Omega)} &\le C(1 + \|f^k\|_{L^2_{\omega_0}(\Omega)}) \le C_f. \end{aligned} \tag{5.3}$$

5.2. Existence. Step 2. Limit passage. Using the estimate (5.3), the reflexivity of the corresponding spaces, the unique identification of the limit v in $W^{1,1}(\Omega)$, and the growth of Assumption 1.1, we obtain for a (nonrelabeled) subsequence

$$v^k \rightharpoonup v \qquad \text{weakly in } W^{1,s_0}_0(\Omega), \tag{5.4}$$

$$(\nabla v^k, \pi^k) \rightharpoonup (\nabla v, \pi) \qquad \text{weakly in } L^2_{\omega_0}(\Omega) \cap L^{s_0}(\Omega) \cap L^q_\omega(\Omega), \tag{5.5}$$

$$\mathcal{S}(x, \varepsilon(v^k)) \rightharpoonup \mathcal{S}_0 \qquad \text{weakly in } L^2_{\omega_0}(\Omega) \cap L^{s_0}(\Omega) \cap L^q_\omega(\Omega). \tag{5.6}$$

Hence the lower weak semicontinuity implies via (5.3)

$$\begin{aligned} \|\nabla v\|_{L^q_\omega(\Omega)} + \|\pi\|_{L^q_\omega(\Omega)} &\le C(1+\|f\|_{L^q_\omega(\Omega)}), \\ \|\nabla v\|_{L^{s_0}(\Omega)} + \|\nabla v\|_{L^2_{\omega_0}(\Omega)} &\le C_f. \end{aligned} \tag{5.7}$$

Convergences (5.6) and (5.1) used in (5.2) imply

$$\int_\Omega \mathcal{S}_0 \cdot \nabla\varphi = \int_\Omega f \cdot \nabla\varphi \qquad \text{for all } \varphi \in W^{1,\infty}_{0,\mathrm{div}}(\Omega). \tag{5.8}$$

Hence, to complete the proof of Theorem 1.5, it remains to identify the limit properly, i.e., to show

$$\mathcal{S}_0(x) = \mathcal{S}(x, \nabla v(x)) \qquad \text{in } \Omega, \tag{5.9}$$

because then the optimal regularity will be given by the first line of (5.7).

5.3. Existence. Step 3. Limit identification. This is the central part of our proof. Its crucial part will follow from the solenoidal, weighted, biting div-curl lemma, i.e., Theorem 1.9.

Recall that the classical way of identifying the limit in nonlinear problems, namely, use of monotonicity and dealing with the most nonlinear part via the equation, is impossible in our very weak setting, since one cannot use u as a test function in (5.8).

Observe also that taking the weighed limits is crucial to end up with optimal regularity related to f (recall our weight ω_0 is related to Mf).

Let us use Theorem 1.9 with the following choices:

$$q = q' = 2, \quad \omega = \omega_0, \quad a^k = \nabla v^k, \quad s^k = \mathcal{S}(\cdot, \varepsilon(v^k)).$$

The uniform boundedness assumption (1.12) is satisfied thanks to (5.3). The compactness assumption (1.13) holds thanks to the weak formulation (5.2) with d^k as the test function. Finally, the compensation assumptions (1.14), (1.15) hold automatically, since our a^k is a gradient of a solenoidal function.

Thesis of Theorem 1.9 provides thence, for a nonrelabeled subsequence and a nondecreasing sequence of measurable subsets $\Omega_j \subset \Omega$ with $|\Omega \setminus \Omega_j| \to 0$ as $j \to \infty$, that

$$\mathcal{S}(\cdot, \varepsilon(v^k)) \cdot \nabla v^k \omega_0 \rightharpoonup \mathcal{S}_0 \cdot \nabla v\, \omega_0 \qquad \text{weakly in } L^1(\Omega_j). \tag{5.10}$$

The last needed step, from (5.10) to (5.9), will be performed via monotonicity. Let us take any $B \in L^2_{\omega_0}(\Omega)$. Using (5.10), (5.5), and (5.6), we get

$$(\mathcal{S}(x,\varepsilon(v^k)) - \mathcal{S}(x,B)) \cdot (\nabla v^k - B)\, \omega_0 \rightharpoonup (\mathcal{S}_0 - \mathcal{S}(x,B)) \cdot (\nabla u - B)\, \omega_0 \quad \text{weakly in } L^1(\Omega_j). \tag{5.11}$$

Monotonicity of $\mathcal{S}$ implies that the limit is signed as well, thus

$$\int_{\Omega_j} (\mathcal{S}_0 - \mathcal{S}(x,B)) \cdot (\nabla v - B)\, \omega_0 \,\mathrm{d}x \ge 0 \tag{5.12}$$

for any $j \in \mathbb{N}$. Consequently

$$\infty > \int_\Omega (\mathcal{S}_0 - \mathcal{S}(x,B)) \cdot (\nabla v - B)\, \omega_0 \ge \int_{\Omega \setminus \Omega_j} (\mathcal{S}_0 - \mathcal{S}(x,B)) \cdot (\nabla v - B)\, \omega_0.$$

Observe that the integrals above are well defined due to (5.5), (5.6), and the assumed growth of $\mathcal{S}$. Therefore, recalling that $|\Omega \setminus \Omega_j| \to 0$ as $j \to \infty$, we let $j \to \infty$ and obtain

$$\infty > \int_\Omega (\mathcal{S}_0 - \mathcal{S}(x, B)) \cdot (\nabla v - B)\, \omega_0 \,\mathrm{d}x \geq 0 \qquad \text{for all } B \in L^2_{\omega_0}(\Omega).$$

Choosing $B := \nabla u - \varepsilon G$ with an arbitrary $G \in L^\infty(\Omega)$, we get

$$\infty > \int_\Omega (\mathcal{S}_0 - \mathcal{S}(x, \nabla v - \varepsilon G)) \cdot G\, \omega_0 \,\mathrm{d}x \geq 0.$$

Finally, using the Lebesgue dominated convergence theorem, Assumption 1.1 (growth and continuity), we let $\varepsilon \to 0_+$ to deduce

$$\int_\Omega (\mathcal{S}_0 - \mathcal{S}(x, \nabla v)) \cdot G\, \omega_0 \,\mathrm{d}x \geq 0.$$

Choosing

$$G := -\frac{\mathcal{S}_0 - \mathcal{S}(x, \nabla v)}{1 + |\mathcal{S}_0 - \mathcal{S}(x, \nabla v)|}$$

and utilizing that ω_0 is strictly positive almost everywhere in Ω, we arrive at validity of (5.9) a.e. in Ω. Consequently

$$\int_\Omega \mathcal{S}(x, \nabla v) \cdot \nabla \varphi = \int_\Omega f \cdot \nabla \varphi \qquad \text{for all } \varphi \in W^{1,\infty}_{0,\mathrm{div}}(\Omega) \tag{5.13}$$

with estimate (5.7).

We have ended the proof of the existence part of Theorem 1.5. The estimate (1.9) is given by Theorem 3.1. Hence, to conclude the proof of Theorem 1.5, we are left with showing its uniqueness statement.

5.4. Uniqueness. Recall that now the tensor $\mathcal{S}$ satisfies additionally Assumption 1.2. A difference between two solutions u_1 and u_2 to (1.1) with the same force $f \in L^q_\omega(\Omega)$ satisfies

$$\int_\Omega \big(\mathcal{S}(x, \varepsilon(v_1)) - \mathcal{S}(x, \varepsilon(v_2))\big) \cdot \nabla \varphi \,\mathrm{d}x = 0 \tag{5.14}$$

with the admissible class of φ dictated by the optimal L^q_ω-regularity of v_1, v_2; see (1.9). Hence, if we could have chosen $\varphi = v_1 - v_2$, the assumed strict monotonicity would imply $v_1 = v_2$. Therefore in the case $L^q_\omega(\Omega) \subset L^2(\Omega)$ the proof is finished. But generally, we find that $f \in L^{s_0}(\Omega)$, merely for some $s_0 \in (1, 2]$; compare section 5.1. Such L^{s_0}-regularity seems insufficient, since possibly $s_0 < 2$. Nevertheless, we will be able to show that $\nabla(v_1 - v_2) \in L^2(\Omega)$ via the weighted estimates and conclude the uniqueness using this extra regularity for the difference.

To begin with, let us recall that $f \in L^2_{\omega_0}(\Omega)$ for $\omega_0 = (1 + Mf)^{s_0 - 2}$ and therefore also $\nabla v_1, \nabla v_2 \in L^2_{\omega_0}(\Omega)$. Let us rewrite the identity (5.14) into the form

$$\int_\Omega (\varepsilon(v_1 - v_2)) \nabla \varphi = \mu^{-1} \int_\Omega \big(\mu\, \varepsilon(v_1) - \mathcal{S}\big(x, \varepsilon(v_1)\big) - \big(\mu\, \varepsilon v_2 - \mathcal{S}(x, \varepsilon(v_2))\big)\nabla \varphi, \tag{5.15}$$

which is valid for all $\varphi \in W^{1,\infty}_{0,\mathrm{div}}(\Omega)$.

Let $w^j := \min\{1, (j\omega_0)\}$ and observe that $\varepsilon(v_1 - v_2) \in L^2_{\omega_j}(\Omega)$ for a fixed j, since $\varepsilon(v_1 - v_2) \in L^2_{\omega_0}(\Omega)$ in view of the previous subsection. Moreover, $A_p(\omega_j) \leq \max(1, A_p(\omega_0))$ in view of definition 2.3 and basic properties of weights stated in subsection 2.4.1. Consequently, we can use the linear maximal regularity Lemma 3.2 to obtain

$$\int_\Omega |\varepsilon(v_1 - v_2)|^2 \omega^j \leq C\mu^{-1} \int_\Omega \left|\mu\, \varepsilon(v_1) - \mathcal{S}(x, \varepsilon(v_1)) - \big(\mu\, \varepsilon(v_2) - \mathcal{S}(x, \varepsilon(v_2))\big)\right|^2 \omega^j \tag{5.16}$$

with finite right-hand side and j-independent C of, the latter due to $A_p(\omega_j) \leq \max(1, A_p(\omega_0))$. Next, using the estimate (2.1) of Lemma 2.2 in (5.16), we find that for any $\delta > 0$

$$\int_\Omega |\varepsilon(v_1 - v_2)|^2 \omega^j \leq C\mu^{-1}\delta \int_\Omega |\varepsilon(v_1 - v_2)|^2 \omega^j + C(\delta)\omega^j. \tag{5.17}$$

Thus, setting $\delta := \frac{\mu}{2C}$ yields

$$\int_\Omega |\varepsilon(v_1 - v_2)|^2 \omega^j \leq C(\delta) \int_\Omega \omega^j \leq C, \tag{5.18}$$

where the last inequality follows from the fact that Ω is bounded and $\omega^j \leq 1$. Hence, letting $j \to \infty$ in (5.18), together with $\omega^j \nearrow 1$ (which follows from the fact that $\omega_0 > 0$ almost everywhere) and the monotone convergence theorem, implies

$$\int_\Omega |\varepsilon(v_1 - v_2)|^2 \leq C.$$

Hence, via the Korn inequality, we see that $v_1 - v_2 \in W^{1,2}_0(\Omega)$. Consequently, using structural Assumption 1.2 on $\mathcal{S}$, we have that (for details, see [11])

$$\int_\Omega |\mathcal{S}(x, \varepsilon(v_1)) - \mathcal{S}(x, \varepsilon(v_2))|^2 \leq C.$$

Therefore, (5.14) holds for all $\varphi \in W^{1,2}_{0,\mathrm{div}}(\Omega)$, including $\varphi := v_1 - v_2$. The strict monotonicity finishes the proof of the uniqueness of v. Recalling that we have fixed the mean value of the pressure to 0, the uniqueness part of Theorem 1.5 is provided.

The entire Theorem 1.5 is now proved.

5.5. Proof of Corollary 1.7. The proof of Corollary 1.7 follows the lines of the proof of Theorem 1.5, with rather straightforward modifications related to involved inhomogeneities. More precisely, in Steps 1 and 2 of the proof of Theorem 1.5 we use now the inhomogeneous estimate of Theorem 3.1. It implies weak convergence in the respective spaces. To identify the limit (reconstruct the stress tensor) in Step 3, its arguments can be shown for $v - \gamma^{-1}(g) - \mathrm{Bog}(v - \gamma^{-1}(g))$, because the appearing extra terms are converging due to the weak-strong coupling. The subsequent inequalities can then be adapted immediately. The proof of the uniqueness is line by line the same.

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