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1 Vector-valued functions

[2, Section VII.1,2,3], [3, Chapter II, IV]

In the whole section X denotes a Banach space with norm $\|u\|_X$, X^* is dual of X, $\langle x^*, x\rangle_{X^*,X}$ is the duality between $x^* \in X^*$ and $x \in X$, $T > 0$, $I = [0, T]$. Some theorems hold also for unbounded or open interval I. Check it.

1.1 Vector-valued integrable functions – Bochner integral

Definition 1. *Function $u : I \to X$ is called*

1. *simple, if there is $N \in \mathbb{N}$, $A_j \subset I$, $j \in \{1, \dots, N\}$ Lebesgue measurable and $x_j \in X$, $j \in \{1, \dots, N\}$ such that $u(t) = \sum_{j=1}^N \chi_{A_j}(t) x_j$.*
2. *simple integrable on $\tilde{I} \subset I$ if for all $j \in \{1, \dots, N\}$: $|A_j \cap \tilde{I}| \leq +\infty$ or $x_j = \mathbf{o}$.*
3. *measurable (strongly measurable), if there are $u_n(t)$ simple such that $u_n(t) \to u(t)$ (strongly in X) for a.e. $t \in I$*

Definition 2. *Function $u : I \to X$ is called (Bochner) integrable, provided it is strongly measurable and there exist u_n simple integrable such that $\int_I \|u(t) - u_n(t)\|_X \, dt \to 0$ for $n \to \infty$.*

The (Bochner) integral of $u : I \to X$ is defined as follows:

1. *$\int_I u(t)\, dt = \sum_{j=1}^N x_j \lambda(A_j)$, if $u(t)$ is simple*
2. *$\int_I u(t)\, dt = \lim_{n\to\infty} \int_I u_n(t)\, dt$, if $u(t)$ is (Bochner) integrable*

Remark 1. *One has to check these definitions are correct (i.e. independent of x_j, A_j in the first part, and of $u_n(t)$ in the second part).*

One also proves that $\|\int_I u(t)\, dt\|_X \leq \int_I \|u(t)\|_X \, dt$ for any $u(t)$ integrable.

Theorem 1 (1-Bochner)**.** *Function $u : I \to X$ is Bochner integrable iff u is measurable and $\int_I \|u(t)\|_X \, dt < \infty$.*

Theorem 2 (2-Lebesgue)**.** *Let $u_n : I \to X$ be measurable, $u_n(t) \to u(t)$ for a.e. $t \in I$, and let there exist $g : I \to \mathbb{R}$ integrable such that $\|u_n(t)\| \leq g(t)$ for a.e. t and all n. Then u is Bochner integrable and $\int_I u_n(t)\, dt \to \int_I u(t)\, dt$; in fact one even has $\| \int_I u_n(t) - u(t)\, dt\|_X \leq \int_I \|u_n(t) - u(t)\|_X \, dt \to 0$, $n \to \infty$.*

Definition 3. *For $p \in [1, \infty)$, $u : I \to X$ we set*

$$L^p(I; X) = \left\{ u(t) : I \to X;\ u(t) \text{ is measurable and } \int_I \|u(t)\|_X^p \, dt < \infty \ \right\},$$

$$\|u\|_{L^p(I,X)} = \left(\int_I \|u(t)\|_X^p \, \mathrm{d}\, t \right)^{\frac{1}{p}}$$

For $p=\infty$ we set

$$L^\infty(I,X)=\{u(t):I\to X;\ u(t) \text{ is measurable and } t\mapsto \|u(t)\|_X \text{ is essentially bounded}\},$$
$$\|u\|=\operatorname{ess-sup}_{t\in I}\|u(t)\|_X.$$

Essential boundedness means: there is $c>0$ such that $\|u(t)\|_X\le c$ pro a.e. $t\in I$, $\operatorname{ess-sup}_{t\in I}\|u(t)\|_X=\inf\{M\in\mathbb{R};\lambda(\{t\in I;\|u(t)\|_X>M\})=0\}$.

Theorem 3 (3)**.** *[2, Section VII.3, Theorem 14] Let $p\in[1,+\infty]$. $(L^p(I,X),\|\cdot\|_{L^p(I,X)})$ is a Banach space (we identify the function equal a. e.).*

Let X be a Hilbert space with scalar product $\langle,\rangle$ then $L^2(I,X)$ with the scalar product $\langle u,v\rangle_{L^2(I,X)}=\int_I\langle u(t),v(t)\rangle\,\mathrm{d}\,t$ is a Hilbert space.

Let I bounded, $1\le p\le q\le+\infty$, then $L^q(I,X)\hookrightarrow L^p(I,X)$.

If $p\in[1,+\infty)$, the simple integrable functions are dense subspace of $L^p(I,X)$. [2, Section VII.3, Theorem 15]

Remark 2. *If $X=\mathbb{R}$ then the simple functions are dense in $L^\infty(I,\mathbb{R})$. This is not in general true if X is infinite dimensional.*

Definition 4. *Let $u\in L^1(\mathbb{R},X)$, $\varphi\in\mathcal{D}(\mathbb{R})$. We define for $t\in\mathbb{R}$ $u\star\varphi(t)=\int_{\mathbb{R}}u(t-s)\varphi(s)\,\mathrm{d}\,s$.*

Definition 5. *Let J be an interval. Let $u:J\to X$. We say that u is continuous ($u\in C(J,X)$) if for any $U\subset X$ open $u^{-1}(U)$ is open in J.*

We say that u is differentiable in $t\in J^\circ$ if the limit $\lim_{h\to0}(u(t+h)-u(t))/h$ exists in X. We denote it $u'(t)$. If u is differentiable at every $t\in J^\circ$ then $u':J^\circ\to X$. We say that $u\in C^1(J,X)$ if u,u' can be extended to functions in $C(J,X)$.

We define first derivative of u as $u^{(1)}=u'$ and for $k\in\mathbb{N}$, $k>1$ we define $u^{(k)}=(u^{(k-1)})'$. We say that $u\in C^k(J,X)$ if $u,u'\in C^{k-1}(J,X)$.

We say that $u\in C^\infty(J,X)$ if for all $k\in\mathbb{N}$ $u\in C^k(J,X)$.

By subscript $_c$ we always mean that the functions has compact support (in J).

Theorem 4 (4)**.** *Let $p\in[1,\infty)$.*

1. *The set $\{\varphi:I\to X|\exists N\in\mathbb{N},x_j\in X,\varphi_j\in C_c^\infty(I)$ for $j\in\{1,\dots,N\}$ such that $\varphi=\sum_{j=1}^N\varphi_jx_j\}$ is dense in $L^p(I,X)$.*
2. *If Y is dense subset of X, the set $\{\varphi:I\to X|\exists N\in\mathbb{N},y_j\in Y,\varphi_j\in C_c^\infty(I)$ for $j\in\{1,\dots,N\}$ such that $\varphi=\sum_{j=1}^N\varphi_jx_j\}$ is dense in $L^p(I,X)$. In particular $C_c^\infty(I,Y)$ is dense in $L^p(I,X)$.*
3. *If X is separable, then $L^p(I,X)$ is separable.*
4. *Let $\psi\in C^\infty(\mathbb{R})$, $\operatorname{spt}(\psi)\subset(-1,1)$, $\int_{\mathbb{R}}\psi=1$, $\psi\ge0$, $\psi_n(t)=n\psi(nt)$ for $n\in\mathbb{N}$. Let $u\in L^p(I,X)$ be extended by 0 outside I. Then $u\star\psi_n\to u$ in $L^p(I,X)$ as $n\to+\infty$.*

Remark 3. *If X is separable, then also $L^p(I;X)$ is separable for $p<\infty$. But none of these holds for $p=\infty$.*

Corollary 5 (5)**.** *If $p\in[1,+\infty)$, $\epsilon>0$ and $u\in L^p(U(I,\epsilon),X)$. Then $\int_I\|u(x+t)-u(x)\|_X^p\to0$ as $t\to0$.*

Lemma 6 (6)**.** *Let $p\in[1,+\infty]$, $u_n\to u$ in $L^p(I,X)$, then there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k}(t)\to u(t)$ in X for a.e. $t\in I$.*

Theorem 7 (7-Hölder's inequality.)**.** *Let $u \in L^p(I;X)$, $v \in L^{p'}(I;X^*)$, where p, p' are Hölder conjugate. Then $t \mapsto \langle v(t), u(t) \rangle$ is measurable and*

$$\int_I |\langle v(t), u(t) \rangle| \leq \left(\int_I \|u(t)\|_X^p dt \right)^{\frac{1}{p}} \left(\int_I \|v(t)\|_{X^*}^{p'} dt \right)^{\frac{1}{p'}}$$

Remark 4. *If $v \in L^{p'}(I, X^*)$, $p \in [1, +\infty]$ then $\Phi_v(u) = \int_I \langle v, u \rangle$ satisfies $\Phi_v \in (L^p(I, X))^*$.*

Theorem 8 (8-Dual space to $L^p(I;X)$.)**.** *Let X be reflexive, separable and $p \in [1, \infty)$. Denote $\mathscr{X} = L^p(I;X)$. Then for any $F \in \mathscr{X}^*$ there is $v \in L^{p'}(I, X^*)$ such that*

$$\langle F, u \rangle_{\mathscr{X}^*, \mathscr{X}} = \int_I \langle v(t), u(t) \rangle_{X^*, X} \, dt \qquad \forall u \in \mathscr{X}.$$

Moreover, v is uniquely defined, and its norm in $L^{p'}(I;X^*)$ equals to the norm of F in $\mathscr{X}^*$.

Remark 5. *If X is reflexive, separable, and $p \in (1, \infty)$, then $L^p(I;X)$ is also reflexive, separable. Any sequence bounded in $L^p(I;X)$ has a weakly convergent subsequence.*

1.2 Weakly differentiable function

Definition 6. *We say that $u : I \to X$ is weakly differentiable if $u \in L^1(I, X)$ and there is $g \in L^1(I, X)$ such that*

$$\forall \varphi \in \mathcal{D}(I) : \int_I u\varphi' = -\int_I g\varphi.$$

We call g a weak derivative of u and we write $\frac{\mathrm{d}}{\mathrm{d}t} u = g$ or $u_t = g$.

Theorem 9 (9)**.** *Let $u, g \in L^1(I;X)$. Then the following are equivalent:*

1. *$u_t = g$*
2. *$\frac{d}{dt} \langle x^*, u(t) \rangle = \langle x^*, g(t) \rangle$ in the sense of distributions on $(0, T)$, for every $x^* \in X^*$ fixed.*
3. *There exist $x_0 \in X$ such that $u(t) = x_0 + \int_0^t g(s)\, ds$ for a.e. $t \in I$.*

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Theorem 10 (10)**.** *Let $u : I \to X$ be weakly differentiable.*

1. *The weak derivative is linear.*
2. *If $\eta : I \to \mathbb{R}$ is C^1, then $u\eta : I \to X$ is weakly differentiable, and $\frac{d}{dt}\big(u(t)\eta(t)\big) = \frac{d}{dt}u(t)\eta(t) + u(t)\eta'(t)$ for a.e. $t \in I$.*

Theorem 11 (11)**.** *If $u : I \to X$ be in $L^1_{loc}(I, X)$, $\psi \in \mathcal{D}(I)$, $J = \{t \in I, t - \operatorname{spt} \psi \subset I\}$. Then $u \star \psi \in C^\infty(J^\circ)$ and $\forall t \in J^\circ : (u \star \psi)_t(t) = u \star \psi'$. If moreover u is weakly differentiable then $(u \star \psi)_t(t) = (u_t \star \psi)(t)$ for $t \in J^\circ$.*

Notation. Symbol $X \hookrightarrow Y$ means *embedding*: $X \subset Y$ and there is $c > 0$ such that $\|u\|_Y \leq c\|u\|_X$ for all $u \in X$. Symbol $X \hookrightarrow\hookrightarrow Y$ means *compact embedding*: $X \hookrightarrow Y$ and any sequence bounded in X has a subsequence converging strongly in Y.

Definition 7. *The claim $u \in L^p(I, X)$, $u_t \in L^q(I, Y)$ means that there is a Banach space Z such that $X \hookrightarrow Z$, $Y \hookrightarrow Z$, i.e. $u \in L^p(I, Z)$ and it has a weak derivative $u_t \in L^q(I, Z)$. Moreover $u \in L^p(I, X)$, $u_t \in L^q(I, Y)$. We will often have $X \hookrightarrow Y = Z$.*

Theorem 12 (12-Smooth approximation.)**.** *Let $u \in L^p(I; X)$ with $\frac{d}{dt}u(t) \in L^q(I; Y)$. Then there exist functions $u_n \in C^\infty(\overline{I}; X)$ with $u_t \in C^\infty(\overline{I}, Y)$ such that $u_n \to u$ in $L^p(I; X)$ and $u_n' \to \frac{d}{dt}u(t)$ in $L^q(I; Y)$.*

Theorem 13 (13-Extension operator.)**.** *Let $p, q \in [1, +\infty)$, $R > 0$, $\tilde{I} = U(I, R)$. We define*

$$\begin{aligned} \mathcal{Z} &= \{u : I \to X; u \in L^p(I, X), u_t \in L^q(I, Y)\}, \quad \|u\|_{\mathcal{Z}} = \|u\|_{L^p(I,X)} + \|u_t\|_{L^q(I,Z)} \\ \tilde{\mathcal{Z}} &= \{u : \tilde{I} \to X; u \in L^p(\tilde{I}, X), u_t \in L^q(\tilde{I}, Y)\}, \quad \|u\|_{\mathcal{Z}} = \|u\|_{L^p(\tilde{I},X)} + \|u_t\|_{L^q(\tilde{I},\mathcal{Z})} \end{aligned}$$

There is a linear, bounded mapping $E : \mathcal{Z} \to \tilde{\mathcal{Z}}$ such that for all $u \in \mathcal{Z} \cap C^\infty(\overline{I}, X)$, $u_t \in C^\infty(\overline{I}, Y)$ $Eu = u$ in I.

Remark 6. *The spaces $(\mathcal{Z}, \|\cdot\|_{\mathcal{Z}})$ and $(\mathcal{Z}, \|\cdot\|_{\mathcal{Z}})$ are Banach spaces.*

Remark 7. *We can require that Eu satisfies $Eu = 0$ in $U(I, R/2)^c$ but for this we need $X \hookrightarrow Y$, $q \le p$, I bounded, i.e. $L^p(I, X) \hookrightarrow L^q(I, Y)$.*

Definition 8. *We define the Sobolev-Bochner space for $p \in [1, +\infty]$*

$$W^{1,p}(I; X) = \big\{u : I \to X; u \in L^p(I; X);\ u_t \in L^p(I; X)\big\}, \quad \|u\|_{W^{1,p}(I;X)} = \|u\|_{L^p(I;X)} + \|u_t\|_{L^p(I;X)}.$$

Definition 9. *For $\alpha \in (0, 1]$ we denote $C^{0,\alpha}(\overline{I}, X) = \{u \in C(\overline{I}, X); [u]_{0,\alpha} < +\infty\}$ where*

$$[u]_{0,\alpha} = \sup_{s,t \in I, s \neq t} \frac{\|u(t) - u(s)\|_X}{|t - s|^\alpha}.$$

We denote $\|\cdot\|_{0,\alpha} = \|\cdot\|_{L^\infty(I,X)} + [\cdot]_{0,\alpha}$.

Remark 8. *The space $(C^{0,\alpha}(\overline{I}, X), \|\cdot\|_{0,\alpha})$ is a Banach space. They are called Hölder spaces. [1, p.254].*

Theorem 14 (13-embedding)**.** *Let $p \in (1, +\infty]$, $\alpha = 1 - 1/p$. Then $W^{1,p}(I, X) \hookrightarrow C^{0,\alpha}(\overline{I}, X)$.*

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In the definition (7) we claimed: $u \in L^p(I, X)$, $u_t \in L^q(I, Y)$ means that there is a Banach space Z such that $X \hookrightarrow Z$, $Y \hookrightarrow Z$, i.e. $u \in L^p(I, Z)$ and it has a weak derivative $u_t \in L^q(I, Z)$. Moreover $u \in L^p(I, X)$, $u_t \in L^q(I, Y)$. We will often have $X \hookrightarrow Y = Z$.

Can we say something more about relation of X and Y?

From (9) we see that for a.e. $t \in I$ we get $u(t) = u_0 + \int_0^t u_t(s)\,\mathrm{d}\,s$. We can find $u_1 \in X$ and $t_0 \in I$ so that $u(t) = u_1 + \int_{t_0}^t u_t(s)\,\mathrm{d}\,s$ so $u - u_1 \in C(\overline{I}, Y)$. Hypothesis: $u - u_1 \in L^p(I, X \cap Y)$ where $X \cap Y$ is equipped with the norm $\|\cdot\|_X + \|\cdot\|_Y$.

Conversely, we approximate u by $u_k \in C^\infty(\overline{I}, X)$ so that $u_k \to u$ in $L^p(I, X)$, $u_k)_t \to u_t$ in $L^q(I, Y)$ and $(u_k)_t \in C^\infty(\overline{I}, Y)$. Then $(u_k)_t \in C^\infty(\overline{I}, X \cap Y)$ and we get that $u_t \in cl(C^\infty(\overline{I}, X \cap Y))$ in $L^q(I, Y)$. Since $X \cap Y$ is dense in $cl(X \cap Y)$ in Y we get that $u_t = (u - u_1)_t \in L^q(I, cl(X \cap Y) in Y)$.

Consequently we define space $\tilde{X} = X \cap Y$ and $\tilde{Y} = cl(X \cap Y)$ in Y and we see that for any function u there is $u_1 \in X$ such that $u - u_1 \in L^p(I, \tilde{X})$, $(u - u_1)_t \in L^1(I, \tilde{Y})$ and $\overline{\tilde{X}}^{\tilde{Y}} = \tilde{Y}$.

Definition 10. *Let X be separable, reflexive, densely embedded into a Hilbert space H. By* Gelfand triple *we mean $X \hookrightarrow H \cong H^* \hookrightarrow X^*$.*

Remark 9. *Note that since $X \hookrightarrow H$ then the restriction of any functional $h \in H^*$ belongs to X^*. We denote this restriction by $R : H^* \to X^*$.*

Moreover, since X is dense in H, if $x^* \in R(H^*)$, i.e. $\exists C > 0, \forall x \in X : |x^*(x)| \leq C\|x\|_H$ there is a unique $h^* \in H^*$ such that $R(h^*) = x^*$.

Thanks to identification of H with H^* (via Riesz theorem), we have also "embedding" $\iota : X \to X^*$ defined by

$$\langle \iota u, v\rangle_{X^*,X} = (v, u)_H \qquad u,\ v \in X$$

where $(\cdot,\cdot)_H$ is the scalar product in H. In this sense, duality $\langle\cdot,\cdot\rangle_{X,X^*}$ can be seen as a generalization of $(\cdot,\cdot)_H$.

Theorem 15 (15-Continuous representative.)**.** *Let (X, H, X^*) be Gelfand triple, $p \in (1, +\infty)$, $\mathcal{X} = \{u : I \to X; u \in L^p(I, X), u_t \in L^q(I, Y)\}$ with a norm $\|\cdot\|_{\mathcal{X}} = \|\cdot\|_{L^p(I,X)} + \|(\cdot)_t\|_{L^q(I,Y)}$. Let $u \in L^p(I; X)$, $u_t \in L^{p'}(I; X^*)$, where p, p' are Hölder conjugate. Then:*

1. *$\mathcal{X}$ is a Banach space, $\mathcal{X} \hookrightarrow C(I; H)$ (in the sense of representative); in particular, there is $C > 0$ such that for any $u \in \mathcal{X}$ there is $\tilde{u}(t)$ such that*

$$\|\tilde{u}\|_{C(I;H)} \leq C\left(\|u(t)\|_{L^p(I;X)} + \|\frac{d}{dt}u(t)\|_{L^{p'}(I;X^*)}\right)$$

   and $u(t) = \tilde{u}(t)$ a.e. in I.

2. *function $t \mapsto \|u(t)\|_H^2$ is weakly differentiable with $\frac{d}{dt}\|u(t)\|_H^2 = 2\left\langle \frac{d}{dt}u(t), u(t)\right\rangle_{X^*,X}$ a.e. In particular*

$$\|\tilde{u}(t_2)\|_H^2 = \|\tilde{u}(t_1)\|_H^2 + 2\int_{t_1}^{t_2}\left\langle \frac{d}{dt}u(t), u(t)\right\rangle_{X^*,X} dt$$

   for any t_1, $t_2 \in I$, where $\tilde{u}(t)$ is the continuous representative.

   If $u, v \in \mathcal{X}$ we have for all $t, s \in I$, $t > s$

$$(u(t), v(t)) - (u(s), v(s)) = \int_s^t 2\langle u_t(\tau), v(\tau)\rangle + \langle v_t(\tau), u(\tau)\rangle \,\mathrm{d}\,\tau.$$

Remark 10. *Note that, by Theorem 8, u and u_t belong to mutually dual spaces.*

Lemma 16 (16-Ehrling.)**.** *Let $Y \hookrightarrow\hookrightarrow X \hookrightarrow Z$. Then for any $a > 0$ there is $C > 0$ such that*

$$\forall u \in Y : \|u\|_X \leq a\|u\|_Y + C\|u\|_Z$$

Theorem 17 (Aubin-Lions lemma.)**.** *Let $Y \hookrightarrow\hookrightarrow X \hookrightarrow Z$, where Y, Z are reflexive, separable. Let p, $q \in (1, \infty)$, I bounded. Define $\mathcal{X} = \{u : I \to X; u \in L^p(I; Y), u_t \in L^q(I; Z)\}$ with a norm $\|u\|_{\mathcal{X}} = \|u\|_{L^p(I,Y)} + \|u_t\|_{L^q(I,Z)}$. Then $\mathcal{X} \hookrightarrow\hookrightarrow L^p(I, X)$.*

Lemma 18 (18)**.** *Let $p \in [1, +\infty)$. Then*

- *$L^p(I, L^p(\Omega)) = L^p(I \times \Omega)$. More precisely, for $u \in L^p(I, L^p(\Omega))$ there is a representative $\tilde{u} \in L^p(I \times \Omega)$ and vice versa.*

- $W^{1,p}(I, L^p(\Omega)) = \{u : I \to L^p(\Omega); u, u_t \in L^p(I, L^p(\Omega))\} = \{u : I\times \to \Omega \to \mathbb{R}; u, u_t \in L^p(I \times \Omega)\}$.
- $L^p(I, W^{1,p}(\Omega) = \{u : I \to W^{1,p}(\Omega); u, \nabla u \in L^p(I, L^p(\Omega))\} = \{u : I\times \to \Omega \to \mathbb{R}; u, \nabla u \in L^p(I \times \Omega)\}$.

Remark 11. *By [John C. Oxtoby: Measure and Category, Theorem 14.4] there is a set $M \subset \mathbb{R}^2$ such that for every $x \in \mathbb{R}$ every section $M_x = \{y \in \mathbb{R}; (x, y) \in M\}$ is countable, i.e. null set, but the set M is of a nonzero measure. By Fubini theorem it follows that the set M cannot be measurable. Consequently, the identification in the last lemma cannot be done pointwisely for arbitrary representative of a class of the functions in $L^p(I, L^p(\Omega))$. A suitable representative must be chosen.*

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2 Linear parabolic PDE's of the second order

by [1, Chapter 7]

Assumption 1. *In this chapter we assume: $\Omega \subset \mathbb{R}^d$ open, bounded, with C^1 boundary, $I = (0, T)$, $Q = I \times \Omega$,*

- *$f \in L^2(I, W_0^{1,2}(\Omega)^*)$,*
- *$a^{ij}, b^i, c \in L^\infty(Q)$, $a^{ij} = a^{ji}$ for $i, j \in \{1, \dots, d\}$,*
- *there is $\theta > 0$ such that for all $\xi \in \mathbb{R}^d$ and a.e. $(t, x) \in Q$: $(A\xi, \xi) \geq \theta|\xi|^2$.*
- *$Lu = -\sum_{i,j=1}^d \partial_j(a^{ij}\partial_i u) + \sum_{i=1}^d b^i \partial_i u + cu$*
- *$g \in L^2(\Omega)$*

We will study the following initial-boundary value problem

$$\begin{aligned} u_t + Lu = f, &\qquad \text{in } Q, \\ u = 0, &\qquad \text{in } I \times \partial\Omega, \\ u = g, &\qquad \text{in } \{0\} \times \Omega, \end{aligned} \tag{1}$$

Definition 11. *We say that $u \in L^2(I, W_0^{1,2}(\Omega)) \cap C(\overline{I}, L^2(\Omega))$ with $u_t \in L^2(I, (W_0^{1,2}(\Omega))^*)$ is a weak solution to the problem (1) (with b.c. and i.c.) if $u(0) = g$ and*

$$\forall \varphi \in \mathcal{D}(I, W_0^{1,2}(\Omega)) : \int_Q (-u\varphi_t + \sum_{i,j=1}^d a^{ij}\partial_i u \partial_j \varphi + \sum_{i=1}^d b^i \partial_i u \varphi + cu\varphi - f\varphi) = 0 \tag{2}$$

Remark 12. *We define a mapping $\mathcal{L} : W_0^{1,2}(\Omega) \to (W_0^{1,2}(\Omega))^*$ by*

$$\forall v \in W_0^{1,2}(\Omega) : (\mathcal{L}u)(v) = \int_\Omega A\nabla u \cdot \nabla v + (b \cdot \nabla u + cu)v. \tag{3}$$

If $u \in L^2(I, W_0^{1,2}(\Omega)) \cap C(\overline{I}, L^2(\Omega))$ with $u_t \in L^2(I, (W_0^{1,2}(\Omega))^*)$ then the equation (2) is equivalent to the equation in $W_0^{1,2}(\Omega)^*$

$$u_t + \mathcal{L}u = f, \quad \text{a.e. in } I. \tag{4}$$

Remark 13. *The equation*

$$\forall v \in \mathcal{D}([0,T), W_0^{1,2}(\Omega)) : \int_Q -uv_t + A\nabla u \cdot \nabla v + (b \cdot \nabla u + cu)v = \int_Q fv + \int_\Omega gv(0) \tag{5}$$

is equivalent to (2) *with the initial condition* $u(0) = g$.

Theorem 19. *[20] Under Assumption 1 there is a weak solution to* (1) *(with b.c. and i.c.). This solution is unique and depends continuously on f and g. It satisfies the estimate*

$$\|u\|_{L^\infty(I,L^2(\Omega))} + \|u\|_{L^2(I,W_0^{1,2}(\Omega))} + \|u_t\|_{L^2(I,W_0^{1,2}(\Omega)^*)} \leq C(\|f\|_{L^2(I,W_0^{1,2}(\Omega)^*)} + \|g\|_{L^2(\Omega)},$$

C is independent of u, f and g.

Definition 12. *Let $\{w_k\}$ we a ON basis of $L^2(\Omega)$ and OG basis of $W_0^{1,2}(\Omega)$, e.g. normalized eigenfunctions of Dirichlet Laplacian, $m \in \mathbb{N}$. We introduce the approximation of the problem* (1) *as follows. We look for $d_m^k : [0,T) \to \mathbb{R}$ such that it satisfies for $k \in \{1, \dots, m\}$*

$$d_m^k(0) = (g, w_k) \tag{6}$$

$$(d_m^k)'(t) = \langle f, w_k \rangle - \int_\Omega A\nabla u_m \nabla w_k + (b \cdot \nabla u_m + cu_m)w_k. \tag{7}$$

The approximation of u is $u_m = \sum_{k=1}^m d_m^k w_k$.

Lemma 20. *There is a solution to the problem* (6) *and* (7) *on* $[0,T)$ *such that* $d_m^k \in W^{1,2}(I)$.

Lemma 21 (Gronwall lemma.)**.** *Let $y(t)$, $g(t)$ be nonnegative (scalar) functions, $y(t)$ continuous and $g(t)$ integrable, such that*

$$y(t) \leq K + \int_0^t g(s)y(s)\, ds \qquad \forall t \in I$$

Then

$$y(t) \leq K \exp\left(\int_0^t g(s)\, ds\right) \qquad \forall t \in I$$

Lemma 22 (22-Apriori estimates)**.** *There is $C > 0$ independent of u_m, m, f and g such that*

$$\|u_m\|_{L^\infty(I,L^2(\Omega))} + \|u_m\|_{L^2(I,W_0^{1,2}(\Omega))} + \|(u_m)_t\|_{L^2(I,W_0^{1,2}(\Omega)^*)} \leq C(\|f\|_{L^2(I,W_0^{1,2}(\Omega)^*)} + \|g\|_{L^2(\Omega)},$$

.... end of the fourth lecture

Lemma 23 (23)**.** *We can extract a subsequence $\{v_k\} = \{u_{m_k}\}$ of $\{u_m\}$ such that*

$$v_k \rightharpoonup u \qquad \text{in } L^2(I, W_0^{1,2}(\Omega))$$

$$(v_k)_t \rightharpoonup u_t \qquad \text{in } L^2(I, W_0^{1,2}(\Omega))^*$$

Remark 14. *We can also get $v_k \to u$ in $L^2(I, L^q(\Omega))$ for any $q \in [2, 2^*)$ by Aubin-Lions theorem.*

Remark 15. *To get regularity we want to use as a test function functions u, u_t, u_{tt}, Δu.*

Remark 16. *There are two approaches to regularity: 1) get uniqueness and then construct a regular solution. It follows that any (the unique) solution is regular. 2) take any weak solution and show that it is regular. Sometimes one can show that the regularity implies uniqueness.*

Theorem 24 (24)**.** *Let moreover to the assumptions of Theorem 19 it holds $g \in W_0^{1,2}(\Omega)$, $f \in L^2(I, L^2(\Omega))$, $\partial\Omega$ is C^∞, $A, b, c \in C^\infty(\Omega)$ depend only on $x \in \Omega$. Then the unique weak solution u of* (1) *satisfies*

$$u \in L^2(I, W^{2,2}(\Omega)) \cap L^\infty(I, W_0^{1,2}(\Omega)), \quad u_t \in L^2(I, L^2(\Omega))$$

together with the estimate

$$\|u\|_{L^2(I,W^{2,2}(\Omega))} + \|u\|_{L^\infty(I,W_0^{1,2}(\Omega))} + \|u_t\|_{L^2(I,L^2(\Omega))} \leq C(\|f\|_{L^2(I,L^2(\Omega))} + \|g\|_{W_0^{1,2}}).$$

The constant $C > 0$ is independent of u, f and g.

Remark 17. *Strategy of the proof: use the same approximation as in the proof of Theorem 19. Moreover test the approximated problem with u_t and then move u_t to the right hand side and use stationary theory on time levels.*

Lemma 25 (25-Apriori estimates)**.** *There is $C > 0$ independent of u_m, m, f and g such that*

$$\|u_m\|_{L^\infty(I,W^{1,2}(\Omega))} + \|(u_m)_t\|_{L^2(I,L^2(\Omega)^*)} \leq C(\|f\|_{L^2(I,L^2(\Omega)^*)} + \|g\|_{W_0^{1,2}(\Omega)}),$$

Two interesting Lemmas that were not presented in the lecture follow.

Lemma. *[3, Theorem II.2.9] Let $f : I \to X$ be Bochner integrable. The for a.e. $t \in I$*

$$\lim_{h\to 0} \fint_t^{t+h} \|f(s) - f(t)\|_X \,\mathrm{d}\, s = 0 \quad \textit{and} \quad \fint_t^{t+h} f(s) \,\mathrm{d}\, s = f(t).$$

. . . . end of the fifth lecture

Lemma. *[2, Corollary 28] (Banach-Alaoglu for normed spaces). Let X be a normed linear space. Then (B_{X^*}, w^*) is compact. If X is separable, (B_{X^*}, w^*) is moreover metrizable.*

Remark. *The previous lemma is used e.g. to extract weakly* convergent subsequences from the bounded sequences in $L^\infty(I, L^2(\Omega))$ and $L^\infty(I, W_0^{1,2}(\Omega))$ since $L^1(I, L^2(\Omega))^* = L^\infty(I, L^2(\Omega))$, $L^1(I, W_0^{1,2}(\Omega)^*)^* = L^\infty(I, W_0^{1,2}(\Omega))$, $L^1(I, L^2(\Omega))$ and $L^1(I, W_0^{1,2}(\Omega))$ are separable.*

Theorem 26. *If moreover to assumptions of Theorem 24 the assumptions $g \in W^{2,2}(\Omega)$, $f_t \in L^2(, L^2(\Omega))$ holds then the unique weak solution of the problem* (1) *u satisfies $u \in L^\infty(I, W^{2,2}(\Omega))$, $u_t \in L^\infty(I, L^2(\Omega)) \cap L^2(I, W_0^{1,2}(\Omega))$, $u_{tt} \in L^2(I, W_0^{1,2}(\Omega))^*$ and the estimate*

$$\|u\|_{L^\infty(I,W^{2,2}(\Omega))} + \|u_t\|_{L^\infty(I,L^2(\Omega))} + \|u_t\|_{L^2(I,W_0^{1,2}(\Omega))} + \|u_{tt}\|_{L^2(I,W_0^{1,2}(\Omega))^*} \leq C(\|g\|_{W^{2,2}(\Omega)} + \|f\|_{W^{1,2}(I,L^2(\Omega))})$$

holds.

Lemma 27. *[1, Section 7.5, Problem 6] If f_n is bounded in $L^\infty(I, L^2(\Omega))$ and $f_n \rightharpoonup f$ in $L^2(I, L^2(\Omega))$ then*

$$\|f\|_{L^\infty(I,L^2(\Omega))} \leq \sup_{m\in\mathbb{N}} \|f_m\|_{L^\infty(I,L^2(\Omega))}.$$

Lemma 28. *[1, Section 7.5, Problem 9] There is $\alpha, \beta > 0$ such that for all $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$*

$$\alpha\|u\|^2_{W^{2,2}(\Omega)} \leq \langle Lu, -\Delta u\rangle + \beta\|u\|^2_{L^2(\Omega)}.$$

Remark. *In Theorems 24 and 26 we needed that* $\mathrm{Tr}(g) = 0$. *It is so called compatibility condition of zeroth order. It causes that* $\mathrm{Tr}(u)$ *can be continuous. The complementarity condition of the first order reads* $f(0) - Lg \in W_0^{1,2}(\Omega)$ *causing that* $\mathrm{Tr}(u_t)$ *can be continuous up to* $t = 0$.

Remark. *If data are smooth and satisfy complementarity condition one can prove higher regularity than in Theorems 24 and 26, see [1].*

Remark. *For parabolic problems regularity is a local notion. If* $Q_{2R}(t_0, x_0) = (t_0 - (2R)^2, t_0) \times U(x_0, 2R) \subset Q$ *and data are regular in* $Q_{2R}(t_0, x_0)$ *then the weak solution is regular in* $Q_R(t_0, x_0)$.

. . . . end of the sixth lecture

3 Linear hyperbolic PDE's of the second order

In this section we follow [1, Chapter 7.2.1-7.2.4].

Assumption 2. *In this chapter we assume:* $\Omega \subset \mathbb{R}^d$ *open, bounded, with* C^1 *boundary,* $I = (0, T)$, $Q = I \times \Omega$,

- $f \in L^2(I, L^2(\Omega))$,
- $a^{ij}, b^i, c \in C^1(\overline{Q})$, $a^{ij} = a^{ji}$ *for* $i, j \in \{1, \dots, d\}$,
- *there is* $\theta > 0$ *such that for all* $\xi \in \mathbb{R}^d$ *and a.e.* $(t, x) \in Q$: $(A\xi, \xi) \geq \theta|\xi|^2$.
- $Lu = -\sum_{i,j=1}^d \partial_j(a^{ij}\partial_i u) + \sum_{i=1}^d b^i \partial_i u + cu$
- $g \in W_0^{1,2}(\Omega)$, $h \in L^2(\Omega)$

We will study the following initial-boundary value problem

$$\begin{aligned} u_{tt} + Lu &= f, && \text{in } Q, \\ u &= 0, && \text{in } I \times \partial\Omega, \\ u_t(0) = h, \quad u &= g, && \text{in } \{0\} \times \Omega, \end{aligned} \tag{8}$$

Definition 13. *We say that* $u \in L^\infty(I, W_0^{1,2}(\Omega))$ *with* $u_t \in L^\infty(I, L^2(\Omega))$ *and* $u_{tt} \in L^2(I, (W_0^{1,2}(\Omega))^*)$ *is a weak solution to the problem* (8) *if* $u(0) = g$, $u_t(0) = h$ *and*

$$\forall \varphi \in \mathcal{D}(I, W_0^{1,2}(\Omega)) : \int_Q (u\varphi_{tt} + \sum_{i,j=1}^d a^{ij}\partial_i u \partial_j \varphi + \sum_{i=1}^d b^i \partial_i u \varphi + cu\varphi - f\varphi) = 0 \tag{9}$$

Remark 18. *We recall the mapping* $\mathcal{L} : W_0^{1,2}(\Omega) \to (W_0^{1,2}(\Omega))^*$ *defined by*

$$\forall v \in W_0^{1,2}(\Omega) : (\mathcal{L}u)(v) = \int_\Omega A\nabla u \cdot \nabla v + (b \cdot \nabla u + cu)v \tag{10}$$

(it depends on $t \in I$ *through mappings* A, b *and* c*). If* $u \in L^\infty(I, W_0^{1,2}(\Omega))$ *with* $u_t \in L^\infty(I, L^2(\Omega))$ *and* $u_{tt} \in L^2(I, (W^{1,2}(\Omega)^*)$ *then the equation* (9) *is equivalent to the equation in* $W_0^{1,2}(\Omega)^*$

$$u_{tt} + \mathcal{L}u = f, \quad \text{a.e. in } I. \tag{11}$$

Remark 19. *If $u \in L^\infty(I, W_0^{1,2}(\Omega))$ with $u_t \in L^\infty(I, L^2(\Omega))$ and $u_{tt} \in L^2(I, (W^{1,2}(\Omega)^*)$, the equation*

$$\forall v \in \mathcal{D}([0,T), W_0^{1,2}(\Omega)) : \int_Q u v_{tt} + A\nabla u \cdot \nabla v + (b \cdot \nabla u + cu)v = \int_Q fv + \int_\Omega hv(0) - \int_\Omega g v_t(0) \quad (12)$$

is equivalent to (9) *with the initial condition $u(0) = g$ and $u_t(0) = h$.*

Theorem 29. *[29] Under Assumption 2 there is a weak solution to* (8)*. It satisfies the estimate*

$$\|u\|_{L^\infty(I,W_0^{1,2}(\Omega))} + \|u_t\|_{L^\infty(I,L^2(\Omega))} + \|u_{tt}\|_{L^2(I,W_0^{1,2}(\Omega)^*)} \le C(\|f\|_{L^2(I,L^2(\Omega))} + \|g\|_{W^{1,2}(\Omega)} + \|h\|_{L^2(\Omega)}),$$

C is independent of u, f, g and h.

Theorem. *[Correction of Theorem 19] Any weak solution u of* (1) *satisfies*

$$\|u\|_{L^\infty(I,L^2(\Omega))} + \|u\|_{L^2(I,W_0^{1,2}(\Omega))} + \|u_t\|_{L^2(I,W_0^{1,2}(\Omega)^*)} \le C(\|f\|_{L^2(I,W_0^{1,2}(\Omega)^*)} + \|g\|_{L^2(\Omega)},$$

C is independent of u, f and g.

Consequently, the weak solution of (1) *is unique.*

Theorem 30. *[30] The weak solution of* (8) *is unique.*

Remark 20. *The equation* (9) *cannot be tested with u_t since it does not belong to the correct regularity space.*

.... end of the eighth lecture

Theorem 31 (31)**.** *Let moreover to the assumptions of Theorem 29 A, b, c be smooth and independent of t, $g \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, $h \in W_0^{1,2}(\Omega)$, $f \in W^{1,2}(I, L^2(\Omega))$. Let u be a unique weak solution of the problem* (8)*. Then $u \in L^\infty(I, W^{2,2}(\Omega))$, $u_t \in L^\infty(I, W_0^{1,2}(\Omega))$, $u_{tt} \in L^\infty(I, L^2(\Omega))$ and $u_{ttt} \in L^2(I, (W^{1,2}(\Omega)^*)$ and it satisfies the estimate*

$$\begin{aligned} \|u\|_{L^\infty(I,W^{2,2}(\Omega))} + \|u_t\|_{L^\infty(I,W^{1,2}(\Omega))} + &\|u_{tt}\|_{L^\infty(I,L^2(\Omega))} + \|u_{ttt}\|_{L^2(I,W_0^{1,2}(\Omega)^*)} \\ &\le C(\|f\|_{W^{1,2}(I,L^2(\Omega))} + \|g\|_{W^{2,2}(\Omega)} + \|h\|_{W^{1,2}(\Omega)}). \end{aligned} \quad (13)$$

Theorem 32 (32)**.** *[4, Theorem 7.2.6] Assume $m \in \mathbb{N}$, $g \in W^{m+1,2}(\Omega)$, $h \in W^{m,2}(\Omega)$, $(\partial/\partial t)^k f \in L^2(I, W^{m-k,2}(\Omega))$ for $k \in \{1, \dots, m\}$. Let compatibility conditions hold*

$$\begin{gathered} g_0 = g \in W_0^{1,2}(\Omega), \quad h_0 = h \in W_0^{1,2}(\Omega) \\ \forall l \in \mathbb{N}, 2l \le m : g_{2l} = (\frac{\partial^{2l-2}}{\partial t^{2l-2}} f)(0) - L g_{2l-2} \in W_0^{1,2}(\Omega) \\ \forall l \in \mathbb{N}, 2l+1 \le m : g_{2l+1} = (\frac{\partial^{2l-1}}{\partial t^{2l-1}} f)(0) - L h_{2l-1} \in W_0^{1,2}(\Omega). \end{gathered} \quad (14)$$

Then $(\partial/\partial t)^k u \in L^\infty(I, W^{m+1-k,2}(\Omega))$ for $k \in \{1, \dots, m+1\}$.

Corollary. *Assume $g, h \in C^\infty(\overline{\Omega})$, $f \in C^\infty(\overline{Q})$ and the compatibility conditions* (14) *hold for every $l \in \mathbb{N}$. Then the unique weak solution of* (8) *satisfies $u \in C^\infty(\overline{Q})$.*

Further we assume $b = 0$, $c = 0$ in Q and A is smooth independent of t. We fix $x_0 \in \Omega$ and assume existence of a function $q : \Omega \to [0, +\infty)$ such that

- $q \in C^\infty(\Omega \setminus \{x_0\})$,
- $q(x_0) = 0$,
- $A\nabla q \cdot \nabla q = 1$ in $\Omega \setminus \{x_0\}$.

Example 1. *If $A = Id$ we define $q = |x - x_0|$.*

Further we fix $t_0 \in I$ and define

$$\begin{aligned} C &= \{(x,t) \in Q, q(x) < t_0 - t\}, \\ C_t &= \{x \in \Omega, q(x) < t_0 - t\} \quad \text{for } t \in [0, t_0]. \end{aligned}$$

Remark 21. *For $t \in [0, t_0)$ the set C_t has smooth boundary.*

Theorem 33 (33-Coarea formula)**.** *[4, par.C.3] Let $u : \mathbb{R}^d \to \mathbb{R}$ be Lipschitz, for a.e. $r \in \mathbb{R}$ the set $\{x \in \mathbb{R}^d; u(x) = r\}$ be a smooth, $d-1$ dimensional hypersurface in $\mathbb{R}^d$, $f : \mathbb{R}^d \to \mathbb{R}$, $f \in C(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. Then*

$$\int_{\mathbb{R}^d} f = \int_{-\infty}^{\infty} \int_{\{x \in \mathbb{R}^d; u(x) = r\}} \frac{f}{|\nabla u|} dS dr, \quad \frac{\partial}{\partial r} \int_{x \in \mathbb{R}^d; u(x) < r\}} f = \int_{\{x \in \mathbb{R}^d; u(x) = r\}} \frac{f}{|\nabla u|} dS.$$

Corollary. *A particular case of Theorem 33 is a sferical Fubini theorem. It corresponds to the case $u(x) = |x|$.*

Theorem 34 (34)**.** *Let u be a smooth weak solution of the problem* (8)*, $(t_0, x_0) \in Q$, $C \subset Q$. If $u = u_t = 0$ on C_0, $f = 0$ on C then $u = 0$ on C.*

Remark 22. *To prove Theorem 34 we do not need to know boundary values.*

Corollary. *If u_1 and u_2 are two smooth solutions of* (8) *that coincide in C_0 then $u_1 = u_2$ in C.*

Homework 10. *Show that Theorem 34 holds for weak solutions. Find minimal assumptions on regularity of A. Hypothesis: It is enough to assume $A \in C(\mathbb{R}^d)$.*

.... end of the ninth lecture (28.4.2017)

4 Semigroup theory

4.1 Lecture by doc. Pražák

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Up to now we considered evolution PDEs: $\frac{d}{dt}u - \Delta u + \cdots$. The rest of lectures is more close to functional analysis.

Motivation:

Let us have the equation $x' = ax$, where $x(0) = 1$ ... the solution is e^{at}, an exponential function. Generalization: $a \leftarrow A \in \mathbb{R}^{n \times n}$: $x' = Ax$, $x(0) = x_0$ the solution is $e^{tA}x_0$, a matrix exponential function. Goal: generalization to general Banach space, the study of equations of the type

$$\text{(4.1)} \quad \frac{d}{dt}x = Ax,\ x(0) = x_0,\ x \in X,$$

where X is a Banach space, $A : X \to X$ is a linear operator, e. g., $A = \Delta$. How to define a general exponential function e^{tA}? The power series is suitable only for bounded operators. Problem: Δ is unbounded operator, $\sum_{n=0}^{\infty} \frac{t^n A^n}{n!}$ in general does not make sense. Remark: $-\Delta : W_0^{1,2} \to W^{-1,2}$ is bounded, but in different spaces.

Idea:

A is "well unbounded" (i. e., bounded from above), then e^{tA} will be possible to define for $t > 0$.

Notation: [Unbounded operator]

- X ... Banach space with respect to $\|\cdot\|$.
- $\mathcal{L}(X) = \{L : X \to X$ is linear continuous operator$\}$ is a Banach space, $\|L\|_{\mathcal{L}(X)} = \sup_{\substack{x\in X\\ \|x\|=1}} \|Lx\|$,
- Unbounded operator is the couple $(A, D(A))$, where $D(A) \subset\subset X$ is a subspace (domain of definition of A), $A : D(A) \to X$ is linear.

Def.: [Semigroup, c_0-semigroup]

The function $S(t) : [0; \infty) \to \mathcal{L}(X)$ is called a semigroup, iff

1. $S(0)$ is identity
2. $S(t)\,S(s) = S(t + s)$, $\forall t, s \geq 0$
3. If moreover $S(t)\,x \to x$, $t \to 0^+$ for $\forall x \in X$ fixed, we call $S\ (t)$ a c_0-semigroup.

Remark:

- c_0-semigroup ... abstract exponential. Possible definitions of standard exponential: either a solution of $x' = ax$, $x(0) = 1$ or a power series $e^{at} = \sum_{n=0}^{\infty} \frac{(at)^n}{n!} = \lim_{n\to\infty} \left(1 + \frac{at}{n}\right)^n$, or a solution of functional equation: $f(x + y) = f(x)\,f(y)$ + continuity and $f(\cdot)$ is nonzero. Then „$S(t) = e^{ta}$“, c_0-semigroup is a suitable candidate for exponential.
- stronger assumption $(3')$ $\|S(t) - I\|_{\mathcal{L}(X)} \to 0$, $t \to 0^+$ (so called uniform continuity) implies $S(t) = e^{tA}$ for some linear continuous operator A, see ex. 5.1.

Lemma 35. *[Exponential estimates, continuity in time of c_0-semigroup]*
Let $S(t)$ be a c_0-semigroup in X. Then

1. $\exists M \geq 1$, $\omega \geq 0$ *s. t.* $\|S(t)\|_{\mathcal{L}(X)} \leq M \cdot e^{\omega t}$ *for* $\forall t \geq 0$.
2. $t \mapsto S(t)\,x$ *is continuous* $[0, \infty) \to X$ *for* $\forall x \in X$ *fixed.*

Proof:

1. we claim: $\exists M \geq 1$, $\exists \delta > 0$ s. t. $\|S(t)\|_{\mathcal{L}(X)} \leq M$, $\forall t \in [0, \delta]$: by contradiction: if not, then $\exists t_n \to 0^+$ s. t. $\|S(t_n)\|_{\mathcal{L}(X)} \to +\infty$, but $S(t_n)\,x \to x$ for $\forall x \in X$ fixed due to the part (3) of semigroup definition and so $\|S(t_n)\,x\|$ is bounded. That is a contradiction to the principle of uniform boundednes, see functional analysis (a set of operators is bounded in operator norm iff $\|S(t_n)\,x\|$ is bounded for $\forall x$).
Set $\omega = \frac{1}{\delta} \ln M$, i. e.. $M = e^{\omega\delta}$, then for $t \geq 0$ arbitrary it holds that $t = n\delta + \varepsilon$, $\varepsilon \in [0, \delta)$, $n \in \mathbb{N}$. Then $\|S(t)\|_{\mathcal{L}(X)} = \big\|S\big(\underbrace{\delta + \delta + \cdots + \delta}_{n\times} + \varepsilon\big)\big\|_{\mathcal{L}(X)} = \|S(\delta) \cdots S(\delta)\,S(\varepsilon)\|_{\mathcal{L}(X)} \leq$
$\|S(\delta)\|^n_{\mathcal{L}(X)} \|S(\varepsilon)\|_{\mathcal{L}(X)} \leq M \cdot \underbrace{M^n}_{= e^{\omega n\delta}} \leq M \cdot e^{\omega t}$.

2. continuity: in 0^+ we have due to part (3) of definiton of semigroup. Continuity (from the right and from the left) in $t > 0$ remains:
   Continuity from the right: $S(t+h)\,x = S(t)\underbrace{S(h)\,x}_{\to x} \to S(t)\,x,\ h \to 0^+$ due to the property (3), $S(t) \in \mathcal{L}(X)$.
   Continuity from the left: (WLOG $h < t$) $S(t-h)\,x - \underbrace{S(t)\,x}_{S(t-h)S(h)x} = S(t-h)\,[x - S(h)\,x]$. Estimate:
   $$\|S(t-h)\,x - S(t)\,x\| \le \underbrace{\|S(t-h)\|_{\mathcal{L}(X)}}_{\le Me^{\omega t},\text{ independent of } h \text{ due to the first part}} \overbrace{\|x - S(h)\,x\|}^{\to 0 \text{ due to (3)}},\ h \to 0^+.$$

Def.: [Generator of a semigroup]

An unbounded operator $(A, D(A))$ is called a generator of semigroup $S(t)$ iff

$$Ax = \lim_{h\to 0^+} \frac{1}{h}\left(S(h)\,x - x\right),\ \ D(A) = \left\{ x \in X,\ \lim_{h\to 0^+} \frac{1}{h}\left(S(h)\,x - x\right) \text{ exists v } X \right\}.$$

Remark:

it is easy to show that the operator defined by this formula is linear and $D(A) \subset X$ is a linear subspace.

Theorem 36. *[Basic properties of a generator]*
Let $(A, D(A))$ be a generator of $S(t)$, a c_0-semigroup in X. Then:

1. *$x \in D(A) \implies S(t)\,x \in D(A)$ for $\forall t \ge 0$,*
2. *$x \in D(A) \implies AS(t)\,x = S(t)\,Ax = \frac{d}{dt}S(t)\,x$ for $\forall t \ge 0$ (in $t = 0$ only from the right),*
3. *$x \in X,\ t \ge 0 \implies \int_0^t S(s)\,x ds \in D(A),\ A\left(\int_0^t S(s)\,x ds\right) = S(t)\,x - x$.*

Proof:

1. $x \in D(A)$, $t \ge 0$ given, $\underbrace{\frac{1}{h}\big(\overbrace{S(h)\,S(t)}^{= S(h+t) = S(t)\,S(h) \text{due to (2)}}x - S(t)\,x\big)}_{(*)} \overset{?}{\to} y \implies S(t)\,x \in D(A)\,, AS(t)\,x = y$
   $$(*) = \frac{1}{h}\left(S(t)\,S(h)\,x - S(t)\,x\right) = S(t)\underbrace{\left(\frac{1}{h}\left(S(h)\,x - x\right)\right)}_{\to Ax} \to S(t)\,Ax$$
2. $x \in D(A) \ldots\ AS(t)\,x = S(t)\,Ax$ see part 1, $\frac{d}{dt}S(t)\,x = S(t)\,Ax$ from the right for $\forall t \ge 0$, see first part $\left(\frac{1}{h}\left(S(t+h)\,x - S(t)\,x\right) \to S(t)\,Ax\right)$, $h \to 0^+$.
   From the left? $\frac{S(t-h)x - S(t)x}{-h} \to S(t)\,Ax$ as $h \to 0^+$ for $t > 0$ fixed? $\frac{S(t-h)x - S(t)x}{-h} = S(t-h)\left[\frac{x - S(h)x}{-h}\right] - S(t)\,Ax = S(t-h)\left[\frac{x - S(h)x}{-h}\right] - S(t-h)\,S(h)\,Ax \overset{?}{\to} 0$: $\underbrace{S(t-h)}_{\text{L. 4.1, 1: bounded in } \|\cdot\|_{\mathcal{L}(X)}}\left\{ \underbrace{\left[\frac{S(h)\,x - x}{h}\right]}_{\to Ax} - \underbrace{S(h)\,Ax}_{\to Ax \text{ due to (3)}} \right\} \to 0$
   as in L. 4.1, 2.

3. Denote $y = \int_0^t S(s)\,x ds$, $x \in X$ for $t > 0$ fixed.

$$\begin{aligned}
\frac{1}{h}\left(S(h)\,y - y\right) &= \frac{1}{h}\Bigg(\underbrace{S(h)\int_0^t S(s)\,x ds}_{\int_0^t S(s+h)\,x ds,\ \text{subst. -- a shift by } h} - \int_0^t S(s)\,x ds\Bigg) \\
&= \frac{1}{h}\left(\int_h^{t+h} S(s)\,x ds - \int_0^t S(s)\,x ds\right) \\
&= \frac{1}{h}\int_t^{t+h} S(s)\,x ds - \frac{1}{h}\int_0^h S(s)\,x ds \underbrace{\overset{h\to 0^+}{\to}}_{\text{derivative of continuous integrand (L. 4.1) w. r. t. upper bound}} S(t)\,x - \overbrace{S(0)\,x}^{=\text{x}}
\end{aligned}$$

Therefore $y \in D(A)$, $Ay = S(t)\,x - x$, which was to be proven.

Remark:

The theorem states:

1. $D(A)$ is invariant w. r. t. $S(t)$.
2. $S(t)$, A commute in v $D(A)$, moreover $t \mapsto S(t)\,x$ is a classical solution of $\frac{d}{dt}x = Ax$, $x(0) = x_0$, if $x \in D(A)$.

Def.: [Closed operator]

We say that an unbounded operator $(A, D(A))$ is closed, iff: $u_n \in D(A)$, $u_n \to u$, $Au_n \to v \implies u \in D(A)$ and $Au = v$.

Remark:

it is easy to show that $(A, D(A))$ is closed $\iff$ $D(A)$ is complete (i. e., Banach) with respect to the norm $\|u\| + \|Au\|$, the so-called graph norm.

Remark:

unbounded, but closed operators: natural property of derivative in different function spaces, examples:

1. $\mathscr{X} = L^1(I, X)$, $A : u(t) \mapsto \frac{d}{dt}u(t)$, $D(A) = W^{1,1}(I, X)$ ... see chap. 1: statement (see ex. 2.1): $u_n(t) \in W^{1,1}(I, X)$, $u_n(t) \to u(t)$ v $L^1(I, X)$, $\frac{d}{dt}u_n(t) \to g(t)$ v $L^1(I, X) \implies u(t) \in W^{1,1}(I, X)$, $\frac{d}{dt}u(t) = g(t)$. This is equivalent to closedness of $(A, D(A))$.
2. $X = C^1([0,1])$ ... theorem from analysis: $f_n(t) \in C^1([0,1])$, $f_n(t) \rightrightarrows f(t)$ v $[0,1]$, $\frac{d}{dt}f_n(t) \rightrightarrows \frac{d}{dt}g(t)$ v $[0,1] \implies f(t) \in C^1([0,1])$, $\frac{d}{dt}f(t) = g(t)$. That is equivalent to closedness of "$\frac{d}{dt}$" in $C([0,1]) = X$ with the definition domain $C^1([0,1])$.

Theorem 37. *[Density and closedness of generator] Let $(A, D(A))$ be a generator of a c_0-semigroup $S(t)$ in X. Then $D(A)$ is dense in X and $(A, D(A))$ is closed.*

Proof:

Density ... $x \in X$ given. $x = \lim_{h\to 0^+} \underbrace{\frac{1}{h}\int_0^h S(s)\,xds}_{\in D(A) \text{ dle V.4.1.3}}$ (continuity of integrand), i. e., we have elements from the definition domain which approximate given element.

Closedness ... $x_n \in D(A)$ given, $x_n \to x$, $Ax_n \to y \overset{?}{\Longrightarrow} x \in D(A)$, $Ax = y$. Observe: $s \mapsto S(s)\,x_n$ is C^1 since $\frac{d}{dt}S(s)\,x_n = S(s)\,Ax_n$ due to Th.4.1, 2 and due to Newton-Leibnitz we have $S(h)\,x_n - S(0)\,x_n = \int_0^h \frac{d}{ds}S(s)\,x_n ds = \int_0^h S(s)\,Ax_n ds$. Take a limit $n \to \infty$. $LHS \to S(h)\,x - x$, RHS ... exchange of lim and $\int$: $Ax_n \to y$, therefore $\|S(s)\,Ax - S(s)\,y\| \le \underbrace{\|S(s)\|_{\mathcal{L}(X)}}_{\text{bounded independently of } s\in[0,h]} \|Ax_n - y\|$, uniform convergence. Therefore by the limit we obtain $\frac{1}{h}(S(h)\,x - x) = \frac{1}{h}\int_0^h S(s)\,yds$, take $h \to 0^+$. $RHS \to y$ (continuity of integrand), i. e., $LHS \to y$ or in other words $x \in D(A)$, $Ax = y$, which was to be proven.

Remark:

Theorem 4.2, proof of closedness: $S(h)\,x_n - x_n = \int_0^h S(s)\,Ax_n ds$, $x_n \in D(A)$ is needed. In Th. 4.1, 3 we already have $S(h)\,x - x = A\left(\int_0^h S(s)\,xds\right)$, $x \in X$. Wouldn't it be possible to shift A into the integral straight away?

Problem: does it hold that $A\left(\int_I f(s)\,ds\right) = \int_I Af(s)\,ds$? For continuous operators A it holds, see ex. 1.1. For closed operators A it is possible to prove, see ex. 5.3. It is not possible to use this argument above as we are just proving the closedness of generator A.

Remark:

- Th. 4.1:$(A, D(A))$ is a generator of a semigroup $S(t) \Longrightarrow \forall x_0 \in D(A)$ is $x(t) = S(t)\,x_0$ a classical solution of (4.1).
- Key problem: $(A, D(A))$ given, $\overset{?}{\Longrightarrow} \exists c_0$-sg. $S(t)$ s. t. A is a generator of $S(t)$.

Lemma 38. *[Uniqueness of a semigroup] Let $S(t)$, $\tilde{S}(t)$ be c_0-semigroups which have the same generator. Then $S(t) = \tilde{S}(t)$ for $\forall t > 0$.*

Proof:

Trick: $y(t) = S(T-t)\,\widetilde{S}(t)\,x$, $x \in D(A)$, check $y(t) \in C([0,T], X)$, $y'(t) = 0$ $\forall t \in (0,T) \Longrightarrow y(T) = \widetilde{S}(T)\,y(0) = S(T)\,x$. $D(A)$ is dense in X (Th. 4.2)

Def.: [Resolvent, resolvent set, spectrum]

Let $(A, D(A))$ be an unbounded operator. We define

resolvent set $\rho(A) = \{\lambda; \lambda I - A \to X \text{ is one-to-one}\} \subset \mathbb{R}$ (generally can be considered a subset of $\mathbb{C}$),

resolvent $R(\lambda, A) = (\lambda I - A)_{-1} : X \to D(A), \lambda \in \rho(A)$,

spectrum $\sigma(A) = \{\lambda \in \mathbb{C}, \lambda I - A \text{ is not invertible}\}$. Equivalently $\sigma(A) = \mathbb{C}\backslash\rho(A)$.

Remark:

- $(A, D(A))$ is closed $\implies R(\lambda, A) \in \mathcal{L}(X)$, since A is closed $\iff$ $D(A)$ is Banach with graph norm $\|x\| + \|Ax\|$. Moreover by the closedness is t $A : D(A) \to X$ continuous. Banach theorem on open mapping: inversion is continuous, i. e., $R(\lambda, A) : X \to D(A)$ is continuous.

- the following relations hold:

$$\begin{aligned}
(i) \quad AR(\lambda, A)x &= \lambda R(\lambda, A)x - x \ \forall x \in X, \\
(ii) \quad R(\lambda, A)Ax &= \lambda R(\lambda, A)x - x \ \forall x \in D(A), \\
(iii) \quad R(\lambda, A)x - R(\mu A)x &= (\mu - \lambda) R(\lambda, A) R(\mu, A)x \ \forall x \in X,
\end{aligned}$$

where (iii) is so-called resolvent identity.

Proof of (i):

$AR(\lambda, A)x = [(A - \lambda I) + \lambda I] R(\lambda, A)x = \underbrace{-(\lambda I - A) R(\lambda, A) X}_{-x} + \lambda R(\lambda, A)x.$

The other relations are proven similarly, (i)-(ii) $\implies AR(\lambda, A)x = RA(\lambda, A)x, \ x \in D(A)$. Heuristics: $R(\lambda, A) = \frac{1}{\lambda - A}$.

Lemma 39. *[Formula for the resolvent by Laplace transform]*

Let $(A, D(A))$ be a generator of c_0-semigroup $S(t)$; let $\|S(t)\|_{\mathcal{L}(X)} \le Me^{\omega t}$. Then $\lambda \in \rho(A)$ for $\forall \lambda > \omega$ and the resolvent can be expressed as $R(\lambda, A)x = \int_0^\infty e^{-\lambda t} S(t)\, x dt$, $\|R(\lambda, A)\|_{\mathcal{L}(X)} \le \frac{M}{\lambda - \omega}$.

Proof:

WLOG: $\omega = 0$, (see ex. 6.2) since c_0-sg. generated by $(A, D(A)) \iff \widetilde{S}(t) = e^{-\omega t} S(t)$ is c_0-sg. generated by $\left(\widetilde{A}, D\left(\widetilde{A}\right)\right)$ where $\widetilde{A} = A - \omega I$, $D\left(\widetilde{A}\right) = D(A)$. Moreover $R\left(\lambda, \widetilde{A}\right) = R(\lambda + \omega, A)$.

Therefore $\|S(t)\|_{\mathcal{L}(X)} \le M$, $\lambda > 0 \overset{?}{\implies} \lambda \in \rho(A)$. Denote $\widetilde{R}x = \int_0^\infty e^{-\lambda t} S(t)\, x dt$ (Laplace transform of semigroup $S(t)$), $x \in X$, $\lambda > 0$ fixed. Integral defined: integrand continuous (L. 4.1), $\|\text{integrand}\| \le e^{-\lambda t} M \|x\| \in L^1(0, \infty)$,
$\left\|\widetilde{R}x\right\| \le \int_0^\infty e^{-\lambda t} M \|x\| \, dt = \frac{M}{\lambda} \|x\|$, i. e., $\widetilde{R} \in L(X)$, $\left\|\widetilde{R}\right\|_{\mathcal{L}(X)} \le \frac{M}{\lambda}$.

We will show that $\widetilde{R}x \in D(A)$:

$$\frac{1}{h}[S(h) - I]\widetilde{R}x = \frac{1}{h}\left[\int_0^\infty e^{-\lambda t} \underbrace{S(h)\, S(t)}_{S(h+t)} x - e^{-\lambda t} S(t)\, x \, dt\right],$$

Substution in first integral: $\int_h^\infty e^{-\lambda(t-h)} S(t)\, x$, $\pm \int_0^h e^{-\lambda(t-h)} S(t)\, x$, together:

$$= \frac{e^{\lambda h} - 1}{h} \underbrace{\int_0^\infty e^{-\lambda t} S(t)\, x \, dt}_{\widetilde{R}x} - \frac{e^{\lambda h}}{h} \int_0^h e^{-\lambda t} S(t)\, x \, dt$$

Take $h \to 0^+$: $\to \lambda \widetilde{R}x - x$.

I. e., $\widetilde{R}x \in D(A)$, $A\widetilde{R}x = \lambda\widetilde{R}x - x$, $\forall x \in X$, in other words $(\lambda I - A)\widetilde{R}x = x$, i. e. $\lambda I - A : D(A) \to X$ is onto.

Is injective? Let $x \in D(A)$ be fixed,

$$A\widetilde{R}x = A\left(\int_0^\infty e^{-\lambda t} S(t)\, x\right) \overset{\text{Ex. 5.4, } A \text{ closed (Th. 4.2.)}}{=} \int_0^\infty A\left(e^{-\lambda t} S(t)\, x\right) dt$$

(exchange op. and sg: Th. 4.1, 2) $= \displaystyle\int_0^\infty e^{-\lambda t} S(t)\, Axdt = \widetilde{R}Ax$

, i. e. $A\widetilde{R} = \widetilde{R}A$ in $D(A) \implies \widetilde{R}(\lambda I - A)x = \lambda\widetilde{R}x - \widetilde{R}AX \overset{A\widetilde{R}x=\lambda\widetilde{R}x-x}{=} x$, i. e., $\forall x \in D(A)$: $\widetilde{R}(\lambda I - A)x = x \ldots$ $\lambda I - A$ is injective! I. e., $\widetilde{R} = R(\lambda, A)$, the proof is done.

Def.: [Semigroup of contractions]

We say that $S(t)$ is a semigroup of contractions, if $\|S(t)\|_{\mathcal{L}(X)} \leq 1$, $\forall t \geq 0$

Theorem 40. *[Hille–Yosida (for contractions)]*
Let $(A, D(A))$ be an unbounded operator. Then it is equivalent:

1. *$\exists c_0$-semigroup of contractions, which is generated by $(A, D(A))$.*

2. *$(A, D(A))$ is closed, $D(A)$ is dense in X, $\lambda \in \rho(A)$ for $\forall \lambda > 0$ and it holds that*

$$\|R(\lambda, A)\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda}.$$

Homework 11. 1. *Prove Lemma 38: If c_0-semigroups $S(t)$ and $\tilde{S}(t)$ have the same generator, then $S(t) = \tilde{S}(t)$ for all $t \geq 0$.*

Hint. For a fixed $\tau > 0$ and $x \in \mathcal{D}(A)$, show that the function $y(t) = S(\tau - t)\tilde{S}(t)x$ is constant on $[0, \tau]$.

2. *Fill the gap in proof of Lemma 39: show that $(A, \mathcal{D}(A))$ is the generator of $S(t)$ if and only if $(\tilde{A}, \mathcal{D}(\tilde{A}))$ is the generator of $\tilde{S}(t)$, where $\tilde{S}(t) = e^{-\omega t}S(t)$ and $\tilde{A} = A - \omega I$ with $\mathcal{D}(\tilde{A}) = \mathcal{D}(A)$.*

   Also show that in this situation $R(\lambda, \tilde{A}) = R(\lambda + \omega, A)$, whenever $\lambda \in \rho(\tilde{A}) \iff \lambda + \omega \in \rho(A)$.

   Hint. It is enough to prove just one implication.

3. *Verify the resolvent identity*

$$R(\lambda, A)x - R(\mu, A)x = (\mu - \lambda)R(\lambda, A)R(\mu, A)x$$

   for all $x \in X$, and λ, $\mu \in \rho(A)$.

   Hint. Deduce first formally using $R(\lambda, A) = \frac{1}{\lambda - A}$.

4.2 Follow-up of the course by PK

Remark 23 (Generalized Hille-Yosida theorem). *Let $M > 0$, $\omega \in \mathbb{R}$. Then $(A, \mathcal{D}(A))$ generates a c_0 semigroup satisfying an estimate $\|S(t)\|\mathcal{L}(X) \leq Me^{\Omega t}$ for all $t > 0$ iff $(A, \mathcal{D}(A))$ is closed, densely defined, for all $\lambda > \omega$: $\lambda \in \rho(A)$ and for all $n \in \mathbb{N}$ $\|R^n(\lambda, A)\|\mathcal{L}(X) \leq M/(\lambda - \omega)^n$.*

Remark 24 (Lumer-Philips theorem). *[5, Theorem 4.3,Section 1.4] If for all $x \in \mathcal{D}(A)$, $\lambda > 0$ the inequality $\|\lambda x - Ax\| \geq \lambda\|x\|$ holds and there is $\lambda_0 > 0$ such that $\lambda_0 I - A : \mathcal{D}(X) \to X$ is onto, then $(A, \mathcal{D}(A))$ generates a c_0 semigroup of contractions.*

If $(A, \mathcal{D}(A))$ generates a c_0 semigroup of contractions on X, then for all $x \in \mathcal{D}(A)$, $\lambda > 0$ the inequality $\|\lambda x - Ax\| \geq \lambda\|x\|$ holds and $\lambda \in \rho(A)$.

Remark 25. *If $(A, \mathcal{D}(A))$ generates a c_0 semigroup S of contractions on X, $x_0 \in \mathcal{D}(A)$, then $u : t \to S(t)x_0$ solves the PDE $u_t = Au$ in $(0, +\infty)$, $u(0) = x_0$. Moreover, $u \in C([0, +\infty), X) \cap C^1([0, +\infty), X)$, $Au \in C([0, +\infty), X)$.*

If $x_0 \in X \setminus \mathcal{D}(A)$ we in general do not get the same statement. There are 2 possibilities

1. introduce a new weaker notion of solution, i.e. solution of the problem is $t \to S(t)x_0$.

   Definition 14. *(it was not mentioned in the lecture) Let $(A, \mathcal{D}(A))$ generates a c_0 semigroup S of contractions on X, $x_0 \in X$. We call $u : t \to S(t)x_0$ a mild solution of the problem $u_t = Au$ in $(0, +\infty)$ with the initial condition $x_0 \in X$.*

2. introduce a semigroup with better properties—differentiable semigroups, see [5, Section 2.4, Theorem 4.7].

Assumption 3. *We assume:*

- *$\Omega \subset \mathbb{R}^d$ open, bounded, with smooth boundary.*
- *$a^{ij}, b^i, c \in C^\infty(\overline{\Omega})$ and are independent of t for $i, j \in \{1, \ldots, d\}$,*
- *there is $\theta > 0$ such that for all $\xi \in \mathbb{R}^d$ and a.e. $(t, x) \in Q$: $(A\xi, \xi) \geq \theta|\xi|^2$.*
- *$Lu = -\sum_{i,j=1}^d \partial_j(a^{ij}\partial_i u) + \sum_{i=1}^d b^i \partial_i u + cu$*

Theorem 41. *Let Assumption 3 hold. Define $X = L^2(\Omega)$, $\mathcal{D}(-L) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. Then $(-L, \mathcal{D}(-L))$ generates a c_0-semigroup.*

Remark 26. *Remark 25 together with Theorem 41 gives existence of a mild solution of the problem $u_t = -Lu$ in $(0, +\infty)$ with the initial condition $u(0) \in L^2(\Omega)$.*

Homework 12. *Let $X = L^2(\Omega)$. We set $S(0) = I$ and for $t > 0$ we define $S(t) : u \in X \to S(t)u \in X$ by*

$$\forall x \in \mathbb{R}^d : S(t)u(x) = \big(\frac{1}{4\pi t}\big)^{\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} u(y) \,\mathrm{d}\, y.$$

Show that S is a c_0 semigroup on X and that its generator is the Laplace operator Δ.

.... end of the lecture (12.5.2017)

Further we want to study the problem

$$\begin{aligned} u_t &= Au + f \quad \text{in } (0, T), \\ u(0) &= u_0 \in X, \end{aligned} \tag{15}$$

where $f : (0, T) \to X$ is integrable, $(A, \mathcal{D}(A))$ an unbounded operator, generator of a c_0 semigroup S.

Definition 15. *[5, Definition 2.1, Section 4.2] A function $u : [0,T] \to X$ is a classical solution of the problem* (15) *if $u \in C([0,T), X) \cap C^1((0,T), X)$ and for all $t \in (0,T)$ $u(t) \in \mathcal{D}(A)$ and* (15) *holds pointwisely.*

Remark 27. *Any classical solution u of* (15) *satisfies*

$$u(t) = S(t)u_0 + \int_0^T S(t-s)f(s)ds. \tag{16}$$

Definition 16. *Let $(A, \mathcal{D}(A))$ be a generator of a c_0 semigroup S, $u_0 \in X$, $f \in L^1(0,T,X)$. Then $u : [0,T] \to X$, $u \in C([0,T], X)$ satisfying for all $t \in (0,T)$ the equality* (16) *is called a mild solution of the problem* (15).

Theorem 42 (42)**.** *Let X be a c_0 semigroup and $(A, \mathcal{D}(A))$ be its generator. If $u_0 \in \mathcal{D}(A)$, $f \in C^1([0,T], X)$ then the mild solution of* (15) *is the classical one.*

Corollary (of Theorems 41 and 42)**.** *Under Assumption 3 if $f \in L^1(0,T,L^2(\Omega))$, $u_0 \in L^2(\Omega)$, S is the semigroup constructed in Theorem 41, there is a mild solution of the problem $u_t + Lu = f$ in $(0,T)$ with $u(0) = u_0$. If moreover $f \in C^1([0,T], L^2(\Omega))$ and $u_0 \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ then the mild solution is the classical one.*

Now we want to concentrate to the wave equation

$$\begin{gathered} u_{tt} + Lu = f \text{in } (0,T) \times \Omega, \\ u = 0 \quad \text{on } (0,T) \times \partial\Omega, \\ u = g, u_t = h \quad \text{in } \{0\} \times \Omega, \end{gathered}$$

which can be reformulated to

$$\begin{gathered} u_t = v, \quad v_t = -Lu + f \quad \text{in } (0,T) \times \Omega, \\ u = 0, v = 0 \quad \text{on } (0,T) \times \partial\Omega, \\ u = g, v = h \quad \text{in } \{0\} \times \Omega. \end{gathered} \tag{17}$$

Theorem 43. *Let Assumption 3 hold with A symmetric, $b = 0$, $c \geq 0$ on Ω. We set $X = W_0^{1,2}(\Omega) \times L^2(\Omega)$, with the scalar product $\langle\langle (u,v), (f,g) \rangle\rangle = \int_\Omega A\nabla u \nabla f + cuf + vg$ and the corresponding norm $\|\cdot\|_X$. We set $\mathcal{D}(\tilde{A}) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$ and for all $(u,v) \in \mathcal{D}(\tilde{A})$ we set $\tilde{A}(u,v) = (v, -Lu)$. Then $(\tilde{A}, \mathcal{D}(\tilde{A}))$ is a generator of a c_0 semigroup of contraction on X.*

Corollary. *Under Assumption 3 if $f \in L^1(0,T,L^2(\Omega))$, $g \in W_0^{1,2}(\Omega)$, $h \in L^2(\Omega)$, S is the semigroup constructed in Theorem 43, there is a mild solution of the problem* (17)*. If moreover $f \in C^1([0,T], L^2(\Omega))$ and $g \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, $h \in W_0^{1,2}(\Omega)$ then the mild solution is the classical one.*

5 Nonlinear parabolic equations of second order

In this section we want to consider a nonlinear parabolic problem

$$\begin{gathered} u_t - \operatorname{div} a(\nabla u) + f(u) = h \quad \text{in } I \times \Omega, \\ u = u_0 \quad \text{in } \{0\} \times \Omega, \\ u = 0 \quad \text{on } I \times \partial\Omega. \end{gathered} \tag{18}$$

As usually $I = (0,T)$, $T > 0$, $\Omega \subset \mathbb{R}^d$, $Q = I \times \Omega$. Given are functions $a : \mathbb{R}^d \to \mathbb{R}^d$, $f : \mathbb{R} \to \mathbb{R}$ and data $u_0 : \Omega \to \mathbb{R}$, $h : Q \to \mathbb{R}$. We search for the unknown function $u : Q \to \mathbb{R}$.

Assumption 4. *Let us assume*

- $I = (0, T)$, $T > 0$, $\Omega \subset \mathbb{R}^d$ *with a smooth* $\partial\Omega$, $p \in (1, +\infty)$,
- $a : \mathbb{R}^d \to \mathbb{R}^d$ *satisfies* $a \in C(\mathbb{R}^d)$, *a(0)=0,*

$$\begin{aligned}
&\exists \alpha > 0, \forall \xi \in \mathbb{R}^d : \alpha|\xi|^p \leq a(\xi) \cdot \xi \\
&\exists \beta > 0, \forall \xi \in \mathbb{R}^d : |a(\xi)| \leq \beta(|\xi|^{p-1} + 1) \\
&\forall \xi_1, \xi_2 \in \mathbb{R}^d : 0 \leq (a(\xi_1) - a(\xi_2)) \cdot (\xi_1 - \xi_2)
\end{aligned}$$

- $f : \mathbb{R} \to \mathbb{R}$ *satisfies*

$$\exists \gamma > 0, q \in [0, p-1], \forall w \in \mathbb{R} : |f(w)| \leq \gamma(|w|^q + 1)$$

- $h \in (L^p(I, W_0^{1,p}(\Omega)))^*$, $u_0 \in L^2(\Omega)$.

Definition 17. *We say that* $u \in L^p(I, W_0^{1,p}(\Omega)$ *si a weak solution of the problem* (18) *if* $u(0) = u_0$ *and*

$$\forall \varphi \mathcal{D}(I, W^{1,p}(\Omega)) : \int_Q -u\varphi_t + a(\nabla u) \cdot \nabla \varphi + f(u)\varphi = \int_I \langle h, \varphi \rangle.$$

Homework 13. *Show that if u is a weak solution of the problem* (18)*, the initial condition is well defined.*

Alternatively show that is u is a weak solution of the problem (18) *then* $u \in C([0, T], L^2(\Omega))$.

Example 2. *Let* $S \in N$*.Show that there is a ON basis of* $L^2(\Omega)$ *such that it is also a OG basis of* $W_0^{s,2}(\Omega)$*. It consists e.g. from the solutions* $u_\lambda \in W_0^{s,2}(\Omega)$ *of the problem*

$$\forall \varphi \in W_0^{s,2}(\Omega) : \int_\Omega \nabla^s u : \nabla^s \varphi = \lambda \int_\Omega u\varphi.$$

The problem corresponds to the eigenvalue problem $(-1)^s \Delta^s u = \lambda u$ *in* Ω *with suitable boundary conditions.*

Recall Minty Browder's trick from PDE's 1. See, e.g., [4, Section 9.1].

. . . . end of the lecture (19.5.2017)

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